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# Fundamental Operations on Truncated $L$ Fuzzy Numbers and a Parametric Total Order on $L$ Fuzzy Numbers

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## Part I. Truncated $L$ Fuzzy Numbers

### 1. Truncated $L$ Fuzzy Numbers

**Definition 1.1 :** A *fuzzy number* $A$ is defined as any fuzzy set on the space of real numbers $\mathbf{R}$, whose membership function $\mu_A$ satisfies the following conditions:

(i) $\mu_A$ is a mapping from $\mathbf{R}$ to the closed interval $[0, 1]$,

(ii) there exists a unique real number $m$ such that

(a) $\mu_A(m) = 1$,

(b) $\mu_A$ is nondecreasing on $(-\infty, m]$,

(c) $\mu_A$ is nonincreasing on $[m, +\infty)$.

We call the number $m$ in (ii) the *center* of $A$, and denote the center of $A$ by $m_A$ similarly the center of $B$ by $m_B$ etc.

We denote the set of all fuzzy numbers defined as above by $\mathbf{F}$. Since the membership function of the real number, i. e. the characteristic function, satis-fies the conditions of Definition 1.1, it holds that $\mathbf{R} \subset \mathbf{F}$.

**Definition 1.2 :** Let $L$ be a function from $\mathbf{R}$ to $\mathbf{R}$ satisfying the following conditions :

(i) $L(x) = L(-x) \quad \forall x \in \mathbf{R}$,

(ii) $L(x) = 1$ iff $x = 0$,

(iii) $L(\cdot)$ is strictly decreasing and continuous on $[0, +\infty)$,

(iv) $L(x) > 0$ for $\forall x \in \mathbf{R}$ ,

(v) $\lim_{x \to +\infty} L(x) = 0$ .

Then the function $L$ is called a *nonvanishing shape function.*

**Definition 1.3 :** Let $m$ be an arbitrary real number, and let $\alpha$ and $a$ be arbitrary positive numbers. Let $L$ be any shape function. Then a fuzzy number $A$ whose membership function $\mu_A$ is expressed by the formula :

$$\mu_A(x) = \begin{cases} L\left(\dfrac{x-m}{\alpha}\right) & \text{on} \quad [m-a,\ m+a] \\ 0 & \text{on} \quad (-\infty,\ m-a)\cup(m+a,\ +\infty) \end{cases} \tag{1.1}$$

is called a *truncated L fuzzy number*. The points $m-a$ and $m+a$ are called the *truncation points* of A.

By the definition, $m$ in (1.1) is the center of $A$. We call the numbers $\alpha$ and $a$ the *deviation parameter* and the *truncation parameter* of $A$, respectively.

Notice that

$$\mu_A(m-a) = \mu_A(m+a) = L\left(\frac{a}{\alpha}\right). \tag{1.2}$$

By virtue of (v) in Definition 1.2 and (1.2), the graph of the membership function (1.1) concentrates around the vertical line $x=m$ as $\alpha$ and $a$ both tend to $+0$. For $\alpha=0$ and $a=0$, hence, we interpret the formula (1.1) as

$$\mu_A(x) = \begin{cases} 1 & \text{if } x=m, \\ 0 & \text{if } x\neq m. \end{cases} \tag{1.3}$$

The formula (1.3) is no other than the characteristic function of the real number m. Then we define anew a truncated $L$ fuzzy number as a fuzzy number whose membership function is given by either (1.1) or (1.3). We introduce the parameter expression of the truncated $L$ fuzzy number by

$$A=(m,\alpha\,;a)_L\,. \tag{1.4}$$

By the interpretation stated above, we have

$$(m,0\,;0)_L = m\,.$$

Given a shape function $L$, let us denote the set of all truncated $L$ fuzzy numbers by $\mathbf{TF}_L$. Then we have $\mathbf{R} \subset \mathbf{TF}_L$ for every shape function $L$.

## 2. Fundamental Operations among Truncated $L$ Fuzzy Numbers

For two fuzzy numbers $A$, $B \in$ F, the fundamental operation $\oplus$ by the so-called " extension principle " is given by

$$\mu_{A\oplus B}(z) = \max_{x+y=z} \min(\mu_A(x), \mu_B(y)), \qquad z \in \mathbf{R}. \qquad (2.1)$$

Similarly, the fundamental operation $\ominus$ is given by

$$\mu_{A\ominus B}(z) = \max_{x-y=z} \min(\mu_A(x), \mu_B(y)), \qquad z \in \mathbf{R}. \qquad (2.2)$$

We apply the fundamental operations (2.1) and (2.2) to the members of the family $\mathbf{TF}_L$.

Let $A=(m, \alpha; a)_L$ and $B=(n, \beta; b)_L$ be two truncated $L$ fuzzy numbers. Then the values of the membership functions of $A$ and $B$ at their truncation points are given by $L(a/\alpha)$ and $L(b/\beta)$, from (1.2).

In the case where $L(a/\alpha) = L(b/\beta)$, the fundamental operations on $A$ and $B$ follow the usual manners. Then we treat the case where $L(a/\alpha) \neq L(b/\beta)$. Without loss of generality we may assume that

$$L(a/\alpha) < L(b/\beta) \qquad (2.3)$$

Let $x = c_A$ be the uniqe solution of the system :

$$\begin{cases} \mu_A(x) = L(b/\beta), \\ x < m, \end{cases} \qquad (2.4)$$

and let $x = d_A$ be the unique solution of the system :

$$\begin{cases} \mu_A(x) = L(b/\beta), \\ x > m. \end{cases} \qquad (2.5)$$

Then we have

$$m+n-a-b < c_A+n-b < d_A+n+b < m+n+a+b. \qquad (2.6)$$

**Theorem 2.1 :** Let $L$ be an arbitrary nonvanishing shape function. For two truncated $L$ fuzzy numbers $A=(m, \alpha; a)_L$ and $B=(n, \beta; b)_L$, we assume (2.3). Then the membership function of the sum $A \oplus B$ computed from the extension principle is given by

$$\mu_{A\oplus B}(z) = \begin{cases} L\left(\dfrac{m+n-b-z}{\alpha}\right) & \text{if} \quad m+n-a-b \le z < c_A+n-b, \\ L\left(\dfrac{m+n-z}{\alpha+\beta}\right) & \text{if} \quad c_A+n-b \le z \le d_A+n+b, \\ L\left(\dfrac{m+n+b-z}{\alpha}\right) & \text{if} \quad \mathrm{d_A}+n+b < z \le m+n+a+b, \\ 0 & \text{if} \quad z < m+n-a-b \ \text{ or } \ z > m+n+a+b. \end{cases}$$

**Theorem 2.2 :** Let $L$ be an arbitrary nonvanishing shape function. For two truncated $L$ fuzzy numbers $A=(m,\alpha\,;a)_L$ and $B=(n,\beta\,;b)_L$, we assume (2.3). Then the membership function of the difference $A\ominus B$ computed from the extension principle is given by

$$\mu_{A\ominus B}(z) = \begin{cases} L\left(\dfrac{m-n-b-z}{\alpha}\right) & \text{if} \quad m-n-a-b \le z < c_A-n-b, \\ L\left(\dfrac{m-n-z}{\alpha+\beta}\right) & \text{if} \quad c_A-n-b \le z \le d_A-n+b, \\ L\left(\dfrac{m-n+b-z}{\alpha}\right) & \text{if} \quad \mathrm{d_A}-n+b < z \le m-n+a+b, \\ 0 & \text{if} \quad z < m-n-a-b \ \text{ or } \ z > m-n+a+b. \end{cases}$$

The proofs of Theorems 2.1 and 2.2 are owing to the results given by Dubois and Prade ([ 1 ]) and the formulae (2.1) and (2.2).

## 3. Fuzzy Max Order on Truncated $L$ Fuzzy Numbers

**Definition 3.1 :** ( Fuzzy Max order ) For two fuzzy numbers $A$, $B$ $\in$ **F**, $A \preceq B$ iff it holds that

(i) $m_A \le m_B$ ,

(ii) there exists a real number $c$ such that

(a) $m_A \le c \le m_B$ ,

(b) $\mu_A(x) \ge \mu_B(x) \qquad \forall x < c,$

(c) $\mu_A(x) \le \mu_B(x) \qquad \forall x > c.$

**Theorem 3.1 :** The set **F** of all fuzzy numbers is a partially ordered set with respect to the Fuzzy Max order.

We apply the Fuzzy Max order to the family $\mathbf{TF}_L$ to get the following characterization theorem of the order in terms of the parameters.

**Theorem 3.2 :** Let $L$ be an arbitrary nonvanishing shape function. For two truncated $L$ fuzzy numbers $A=(m,\alpha\,;a)_L$ and $B=(n,\beta\,;b)_L$, then we have

$$A \preceq B \iff \begin{cases} |a-b| \le n-m, \\ a(\alpha-\beta) \le \alpha(n-m), \\ b(\beta-\alpha) \le \beta(n-m). \end{cases}$$

## Part II. A Parametric Total Order on $L$ Fuzzy Numbers

### 4. Fuzzy Numbers Generated by Vanishing Shape Functions

In Part II we treat the class of fuzzy numbers generated by vanishing shape functions, whereas we have considered fuzzy numbers generated by nonvanishing ones in Part I. Our aim of this part is to introduce a new concept of a total order relation on fuzzy numbers. For this purpose we restrict the shape function to the vanishing one.

**Definition 4.1 :** Let $L$ be a function from **R** to **R** satisfying the following conditions :

(i) $L(x) = L(-x) \quad \forall x \in \mathbf{R}$,

(ii) $L(x) = 1$ iff $x=0$,

(iii) $L(\,\cdot\,)$ is nonincreasing on $[0, +\infty)$,

(iv) let $x_0 = \inf\{\, x>0 \mid L(x) = 0 \,\}$ then $0<x_0<+\infty$ .

Then the function $L$ is called a *vanishing shape function*, and we call the point $x_0$ in (iv) the *zero point* of $L$.

Notice, in the above definition, that any continuity is not assumed to the shape function.

**Definition 4.2 :** Let $m$ be any real number and let $\alpha$ any positive number. Let $L$ be any shape function. A fuzzy number $A$ whose membership function $\mu_A$ is expressed by the formula :

$$\mu_A(x) = L\left(\frac{x-m}{\alpha}\right) \vee 0, \qquad x \in \mathbf{R}, \tag{4.1}$$

is called a ( *nontruncated* ) $L$ *fuzzy number*, where $a \vee b = \max(a, b)$ .

By the definition, $m$ in (4.1) is equal to the center of $A$ . We call the number $\alpha$ the *deviation parameter* of $A$ .

Given a vanishing shape function $L$, we denote the family of all $L$ fuzzy numbers by $\mathbf{F}_L$. By the same reason as for the truncated $L$ fuzzy numbers we can assume that $\mathbf{R} \subset \mathbf{F}_L$.

For the $L$ fuzzy number $A$ whose membership function is expressed by (4.1), we use a parameter expression as follows :

$$A = (m, \alpha)_L.$$

## 5. A Parametric Total Order on $L$ Fuzzy Numbers

Let $\preceq$ denote the Fuzzy Max order introduced in Definition 3.1 of Part I. As stated in Theorem 3.1, the Fuzzy Max order is not necessarily a total order, but usually a partial order. Therefore, for two given fuzzy numbers A and B, it may happen that neither $A \preceq B$ nor $B \preceq A$ holds.

**Definition 5.1 :** Let $L$ be an arbitrary shape function. Let $A = (m, \alpha)_L$ and $B = (n, \beta)_L$ be two $L$ fuzzy numbers. We denote the relation that neither $A \preceq B$ nor $B \preceq A$ holds by $A \succ\!\prec B$ .

**Proposition 5.1 :** For $A = (m, \alpha)_L$ and $B = (n, \beta)_L$ , it holds that

$$A \succ\!\prec B \iff x_0 \mid \alpha - \beta \mid > \mid m - n \mid . \tag{5.2}$$

We introduce a new concept of an order relation on $\mathbf{F}_L$ by the following

**Definition 5.2 :** Let $0 \leq \lambda \leq 1$ be fixed arbitrarily. For two $L$ fuzzy numbers $A = (m, \alpha)_L$ and $B = (n, \beta)_L$ , we define an order relation with the parameter $\lambda$ by

$$A \leq_\lambda B \underset{def}{\Leftrightarrow} \begin{cases} \text{(i)} \quad A \preceq B, \\ \text{or} \\ \text{(ii)} \quad \lambda x_0 |\alpha - \beta| < n - m < x_0 |\alpha - \beta|, \\ \text{or} \\ \text{(iii)} \quad |n - m| \leq \lambda x_0 |\alpha - \beta| \quad \text{and} \quad \beta > \alpha. \end{cases} \tag{5.3}$$

**Proposition 5.2 :** (a) When $\lambda = 0$, we have

$$A \leq_0 B \Leftrightarrow m \leq n. \tag{5.4}$$

(b) When $\lambda = 1$, we have

$$A \leq_1 B \Leftrightarrow \begin{cases} \text{(i)} \quad A \preceq B, \\ \text{or} \\ \text{(ii)} \quad A \succ\prec B \text{ and } \beta > \alpha. \end{cases} \tag{5.5}$$

Proposition 5.2 states that the relation $\leq_0$ is equal to the order among the centers of $L$ fuzzy numbers, and the relation $\leq_1$ is the order relation that makes fuzzy numbers, unordered with respect to the partial order $\preceq$, arrange according to their sizes of ambiguity. These two orders are the extreme ones, and situated on opposite sides each other.

When $0 < \lambda < 1$, the relation $\leq_\lambda$ determines an intermediate order as follows. For two $L$ fuzzy numbers $A = (m, \alpha)_L$ and $B = (n, \beta)_L$ such that $A \succ\prec B$, they are ordered according to their values of center when $|m - n|$ is comparatively large ( (ii) of (5.3) ), and ordered according to their sizes of ambiguity when $|m - n|$ is comparatively small ( (iii) of (5.3) ). In this contexst, an index which gives a criterion of the judgment whether $|m - n|$ is comparatively small or not is the parameter $\lambda$. The smaller $\lambda$ is, the larger the possibility of ordering by the value of center is, and on the contrary the larger $\lambda$ is, the larger the possibility of ordering by the size of ambiguity is.

Finally we can prove the following theorem.

**Theorem 5.2 :** For every vanishing shape function $L$, the order $\leq_\lambda$ is a total order relation on $\mathbf{F}_L$.

We call the order relation $\leq_\lambda$ a *parametric total order* with the parameter $\lambda$.

**References**

[ 1 ] Dubois, D. ; H. Prade : Operations on fuzzy numbers. Int. J. Systems Sci., 9 (1978), 613–626.

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