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Multi-dimensional localized behavior of electrostatic ion wave in a magnetized plasma

東大工　西成 活裕　(Katsuhiro Nishinari)
東大数理科学　薩摩 順吉　(Junkichi Satsuma)

§1. Introduction

Over the last few decades a considerable number of studies have been made on nonlinear dynamics of (1+1) dimension by means of the soliton theory. Behaviors of nonlinear phenomena in multi-dimension, however, are still not well understood. Several articles have been devoted to the studies in multi-dimension, e.g., the resonant interaction of solitons[1)] and the coherent structures. The coherent structures are usually unstable in more than two spacial dimension. For example, Zkaharov showed that the wave described by the three-dimensional nonlinear Schrödinger(NLS) equation collapses in a finite time[2)]. Whether there are stable localized structures or not has been an object of study for a long time. Zkaharov and Kuznetsov found that in a longitudinal magnetic field a localized ion sound wave of low frequency mode can propagate parallel to the magnetic field without deformation. They also showed that the coherent structure is stable by Lyapunov theorem[3)]. The equation which they derived to describe an ion wave of low frequency mode in a three-dimensional magnetized plasma is of the KdV type. As for the NLS type, particular solutions with two-dimensionally localized structures called "dromion"[4] were recently found in the Davey-Stewartson(DS) equations[5] by the soliton theory. There are two types of the DS equations[6]; DS1 and DS2. It should be mentioned that the DS1 equations admit the dromion solutions, but DS2 do not.

In our previous paper[7], we have shown that a multi-dimensional ion wave packet *without* magnetic field is described by the DS2 equations. Hence we do not expect to have two-dimensionally localized structures in such a system. As far as we know, the DS1 equations have only been derived physically in the long wave limit of the Benny-Roskes equations[8] which describe the evolution of three-dimensional packets of surface water waves.

In this paper, we study three-dimensional electrostatic ion wave in a magnetized plasma and show that such a wave is approximately described by the DS equations which can be DS1 in some cases. The system which we consider is the same system as Zkaharov and Kuznetsov, but we focus on high frequency mode of the wave instead of low frequency mode.

§2. Dynamics of ion wave packet in a magnetic field

Let us consider a plasma in an applied magnetic field $\mathbf{B_0}$. We assume that the plasma is collisionless and described by its fluid behavior. We also assume that the temperature of the electrons is so high that the temperature of the ions can be neglected. Moreover, the charge of the ions is assumed to be compensated by the electrons, and the electrons are considered to be in thermal equilibilium as long as we focus upon movement of the ions. The basic equations of a nondimensional form governing such plasma dynamics are

$$\frac{\partial n}{\partial t} + \mathrm{div}(n\mathbf{v}) = 0, \tag{1}$$

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot \mathrm{grad})\mathbf{v} = -\mathrm{grad}\phi + a(\mathbf{v}\times\mathbf{b}), \tag{2}$$

$$\Delta\phi = \exp\phi - n, \tag{3}$$

where $n, \mathbf{v} = (v_x, v_y, v_z)$ are the density and the velocity of ions, respectively, ϕ is the electrostatic potential, $\mathbf{b} = (1,0,0)$ and $\Delta = \partial^2/\partial x^2 + \partial/\partial y^2 + \partial/\partial z^2$. The nondimensional parameter a is given by $a = \frac{\omega_i}{\Omega_i}$, where $\omega_i = \frac{ZeB_0}{Mc}$ and $\Omega_i^2 = \frac{4\pi n_0 Ze^2}{M}$. Here, n_0 is the initial unperturbed ion density, M and Z is the mass and the charge number of an ion, respectively, $-e$ is the charge of an electron, c is the speed of light, and B_0 is the magnitude of the applied magnetic field.

The linear dispersion relation of this system is given by

$$\omega^4 - \left(a^2 + \frac{|\mathbf{k}|^2}{1+|\mathbf{k}|^2}\right)\omega^2 + a^2(\mathbf{b}\cdot\mathbf{k})^2\left(\frac{1}{1+|\mathbf{k}|^2}\right) = 0. \tag{4}$$

This dispersion relation has two modes, which we call higher mode and lower mode in this paper. In the following, we focus upon higher mode. We note that Zkaharov and Kuznetov consider the lower mode of this system[3).

We begin with considering a time evolution of a perturbation $\sim \exp[\imath(\mathbf{k}\cdot\mathbf{x}-\omega t)]$ on this system. Without loss of generality, we take $\mathbf{k}=(kx, ky, 0)$. Initial ion wave packets of perturbation are modulated by nonlinear effect. If we see the packet from the coordinate moving at the group velocity which is determined by the linear dispersion relation (4), then the time variation of the wave packets looks slow. Hence we can introduce the stretched variables, $\xi=\varepsilon(x-V_{gx}t)$, $\eta=\varepsilon(y-V_{gy}t)$, $\zeta=\varepsilon z$, $\tau=\varepsilon^2 t$, where V_{gx} and V_{gy} is the x and y components of the group velocity, respectively. Next we expand the physical quantities around their stationary values as

$$n=1+\sum_{n=1}^{\infty}\varepsilon^n\sum_{l=-\infty}^{\infty}n_l^{(n)}(\xi,\eta,\zeta,\tau)\exp[\imath l(\mathbf{k}\cdot\mathbf{x}-\omega t)], \tag{5}$$

$$\phi=\sum_{n=1}^{\infty}\varepsilon^n\sum_{l=-\infty}^{\infty}\phi_l^{(n)}(\xi,\eta,\zeta,\tau)\exp[\imath l(\mathbf{k}\cdot\mathbf{x}-\omega t)], \tag{6}$$

$$\mathbf{v}=\sum_{n=1}^{\infty}\varepsilon^n\sum_{l=-\infty}^{\infty}\begin{pmatrix}v_{xl}^{(n)}(\xi,\eta,\zeta,\tau)\\ v_{yl}^{(n)}(\xi,\eta,\zeta,\tau)\\ v_{zl}^{(n)}(\xi,\eta,\zeta,\tau)\end{pmatrix}\exp[\imath l(\mathbf{k}\cdot\mathbf{x}-\omega t)], \tag{7}$$

where the relations $n_l^{(n)*}=n_{-l}^{(n)}$, $\phi_l^{(n)*}=\phi_{-l}^{(n)}$, $\mathbf{v}_l^{(n)*}=\mathbf{v}_{-l}^{(n)}$ should be satisfied because of the reality condition of physical variables. We then substitute eqs.(5)-(7) into the basic eqs. (1)-(3) and equate each coefficient of ε. In the following calculation, we shall closely follow the procedure in our previous paper[7].

We calculate up to the third order of ε to obtain the coupled equations of $n_1^{(1)}$ and $n_0^{(2)}$. For simplicity, we introduce the notations A and

Q by $A = n_1^{(1)}$ and $Q = n_0^{(2)} + cv_{x0}^{(2)}$, where c is a constant, and the resultant equations are the followings:

$$\imath A_\tau + \alpha_1 A_{\xi\xi} + \alpha_2 A_{\xi\eta} + \alpha_3' A_{\eta\eta} + \alpha_4 A_{\zeta\zeta} + \alpha_5 |A|^2 A + \alpha_6 QA = 0 , \quad (8)$$

and

$$(1 - V_{gx}^2) Q_{\xi\xi} - 2V_{gx} V_{gy} Q_{\xi\eta} - V_{gy}^2 Q_{\eta\eta}$$
$$+\alpha_7 (|A|^2)_{\xi\xi} + \alpha_8 (|A|^2)_{\xi\eta} + \alpha_9 (|A|^2)_{\eta\eta} = 0. \quad (9)$$

Here and hereafter, the subscripts denote the partial differentiations with respect to the indicated variables. Coefficients are function of a and $\mathbf{k}$. They are, however, so complicated that we shall omit the explicit form of them in this paper. The coupled equations (8) and (9) describe the nonlinear evolution of the ion wave packet in this system. These equations are the well-known DS equations except the terms which have cross derivative of ξ and η. The relation between eqs. (8) - (9) and DS equations will be discussed in Sec.4.

§3. The wave pallarel to the magnetic field

We consider limiting cases of foregoing results in this and next sections. First, we study an ion wave propagating parallel to the applied magnetic field. If we put $k_y = 0$, then from eq.(4) we have the dispersion relations of this case is $\omega = a$ and $\omega = \frac{k_x}{\sqrt{1+k_x^2}}$. In the following, we consider the wave which has the dispersion relation $\omega = \frac{k_x}{\sqrt{1+k_x^2}}$.

The equation is given by taking $k_y = 0$. We shall omit the details of the calculation and finally obtain

$$\imath\frac{\partial}{\partial\tau}A + c_1\frac{\partial^2}{\partial\xi^2}A + c_2(\frac{\partial^2}{\partial\eta^2}A + \frac{\partial^2}{\partial\zeta^2}A) + c_3|A|^2A = 0, \tag{10}$$

where

$$\left.\begin{aligned} c_1 &= \frac{1}{2}\frac{\partial^2\omega}{\partial k_x^2}, \quad c_2 = \frac{k_x^2}{2\omega(1+k_x^2)^2}(\frac{1}{\omega^2-a^2} - 1), \\ c_3 &= -\frac{k_x^3}{V_{gx}(1+k_x^2)^2} \\ &\quad - \frac{4+34k_x^2+33k_x^4+5k_x^6}{12\omega(1+k_x^2)^2} - \frac{k_x^2(2+k_x^2)(2+k_x^2V_{gx}^2)}{2\omega V_{gx}^2(V_{gx}^2-1)(1+k_x^2)^4}. \end{aligned}\right\} \tag{11}$$

This is a three dimensional generalized NLS equation. In the case $k_y = 0$, we obtain the single equation (10), while we have obtained coupled equations (8) and (9) for $k_y \neq 0$.

Equation (10) is transformed into

$$\imath\frac{\partial}{\partial\tau}A + d_1\frac{\partial^2}{\partial\xi^2}A + d_2(\frac{\partial^2}{\partial\eta^2}A + \frac{\partial^2}{\partial\zeta^2}A) + d_3|A|^2A = 0, \tag{12}$$

by suitable scaling transformations of variables. In eq.(12), d_1, d_2 and d_3 are normalized to ±1. It is easily shown that eq.(12) has the following two conserved quantities, $I_1 = \int |A|^2 dv$ and

$$I_2 = \int (d_1|A_\xi|^2 + d_1d_2(|A_\eta|^2 + |A_\zeta|^2) - \frac{d_1d_2}{2}|A|^4)dv, \tag{13}$$

where $dv = d\xi d\eta d\zeta$. Following Gibbon and Mcguinness[9)], we consider the possibility of collapse of ion wave by means of these conserved quantities. From the virial theorem, we obtain the identity

$$\frac{\partial^2}{\partial t^2}\int r^2|A|^2 dv =$$
$$8\int (d_1^2|A_x|^2 + d_2^2(|A_y|^2 + |A_z|^2))dv - 2d_3(d_1 + 2d_2)\int |A|^4 dv, \quad (14)$$

where $r^2 = \xi^2 + \eta^2 + \zeta^2$. We find from eq.(14) that there exist two types of behaviors for the solutions of eq.(12).

First, in the case $(d_1, d_2, d_3) = (\pm 1, \pm 1, \pm 1)$, eq.(14) becomes

$$\frac{\partial^2}{\partial t^2}\int r^2|A|^2 dv = 8I_2 - 2\int |A|^4 dv \leq 8I_2. \quad (15)$$

Then we obtain

$$\int r^2|A|^2 dv \leq 4I_2 t^2 + l_1 t + l_2. \quad (16)$$

Thus, if $I_2 < 0$, the complex amplitude A has a singularity within a finite time, namely, the wave collapses.

Second, in the case $(d_1, d_2, d_3) = (\pm 1, \pm 1, \mp 1)$, eq.(36) becomes

$$\frac{\partial^2}{\partial t^2}\int r^2|A|^2 dv = 8I_2 + 2\int |A|^4 dv \geq 8I_2. \quad (17)$$

It is clear that $I_2 > 0$ in this case and we recognize that any localized structures will disperse away.

Let as go back to eq.(10). From the above consideration and the values of c_i's, we obtain the following results. When $a > 1$、localized structures will disperse if $k_x < k_{cr}$, and will collapse if $k_x \geq k_{cr}$. When $1 > a > a_{cr}$, they will disperse if $k_x < k_{cr}$, and will collapse if $k_{cr} \leq$

$k_x \le k_a$. Finally, when $a_{cr} > a$, they will disperse if $k_x \le k_a$. Here $k_{cr} = 1.47$ is the root of $c_3 = 0$, k_a is the root of $k_x/\sqrt{1+k_x^2} = a$ and $a_{cr} = 1.47/\sqrt{1+1.47^2} \simeq 0.827$.

This shows that if $a > a_{cr}$, the ion wave collapses for $k_x > 1.47$. This wave number is the same as that of the modulational instability of ion sound wave[10].

§4. The wave perpendicular to the magnetic field

In the case of perpendicular propagation, we must take into account the fact that electron inertia should not be neglected. Then the basic eq.(3) is not valid in an exact sense. Electrons form the Boltzmann distribution along the magnetic line. On the other hand, they can not keep up with ion motion perpendicular to the magnetic field due to its small ramour radius. Thus eq.(3) is valid in the range that the angle between the vector normal to the direction of the magnetic field and the wave number vector is larger than $vi_{th}/ve_{th} \sim \sqrt{(mT_i)/(MT_e)}$, where vi_{th} is ion thermal velocity and so on[11]. The magnitude of $\sqrt{(mT_i)/(MT_e)}$ is, however, so small usually that we may consider "perpendicular" propagation in this sense. Then we can take $k_x \to 0$ in Sec.2 and assume that all the variable is independent of ζ to obtain

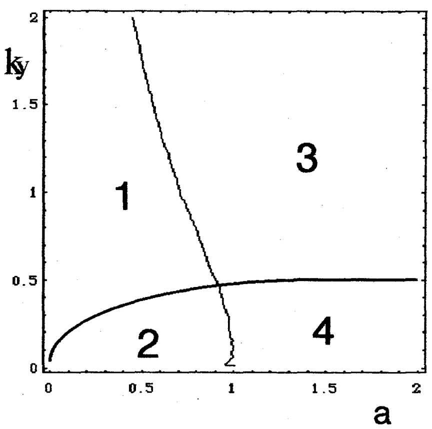
$$\imath A_\tau + \alpha_1 A_{\xi\xi} + \alpha_3 A_{\eta\eta} + \alpha_5 |A|^2 A + \alpha_6 Q A = 0, \tag{18}$$

$$Q_{\xi\xi} - v_{gy}^2 Q_{\eta\eta} + \alpha_7(|A|^2)_{\xi\xi} = 0, \tag{19}$$

from eqs.(8) and (9). It is natural that the terms of cross derivatives vanish, because this system has the spacial symmetry in this case.

Equations (18) and (19) are the DS equations , and become DS1 if $\alpha_1\alpha_3 > 0$.

The regions of parameters a and k_y, where eqs.(18) and (19) become DS1, are given in the following figure 1.



[Figure 1]

The regions of parameters a and k_y where eqs.(18) and (19) satisfies the DS1 condition, self-focusing occurs and modulational instability occurs.

In the regions 2 and 3, the DS1 condition is satisfied.

We next consider the modulational instability and self-focusing for eqs. (18) and (19). The plane wave solution with modulation is given by

$$A = (A_0 + A') \exp \imath(p_1\xi + p_2\eta + p_3\zeta - \Omega\tau + \theta_0 + \theta'), \quad Q = Q_0 + Q', \tag{20}$$

where

$$\begin{pmatrix} A' \\ \theta' \\ Q' \end{pmatrix} = \begin{pmatrix} \Delta A \\ \Delta\theta \\ \Delta Q \end{pmatrix} \mathrm{Re}[\exp \imath(\mu_1\xi + \mu_2\eta + \mu_3\zeta - \nu\tau)], \tag{21}$$

and $\mu_1, \mu_2, \mu_3, \Delta A, \Delta\theta$ and ΔQ are real constants. Substituting eqs.(20) and (21) into eqs.(18) and (19), we obtain a stability condition that ν does not have imaginary part:

$$-(\mu_1^2\alpha_1+\mu_2^2\alpha_3)(-\mu_1^2\alpha_1-\mu_2^2\alpha_3+2\alpha_5A_0^2+2A_0^2\alpha_6\frac{\mu_1^2\alpha_7}{V_{gy}^2\mu_2^2-\mu_1^2})>0. \quad (22)$$

Let us consider two particular cases of long wave modulation.

Case1) $\mu_2=0, \mu_1\to 0$.

The condition (22) is approximated by $-\alpha_1(\alpha_5-\alpha_6h_7)>0$, which is illustrated also in fig.1. In the regions 1 and 2, the system is stable. We see that if the magnetic field becomes strong, the critical wave number decreases. Moreover, if $a>1$, then plane wave is always unstable.

Case 2) $\mu_1=0, \mu_2\to 0$.

The condition (22) is approximated by $-\alpha_3\alpha_5>0$, which is also illustrated in fig.1. The system is stable in the regions 2 and 4. Regardless of a, there is region in which the system becomes unstable for any value of k_y. Instabilities of the cases 1 and 2 correspond to self-focusing and modulational instability, respectively. It is to be noted that both of them are possible to occur in this system.

§5. Concluding discussions

In this paper we have derived by means of RPM the DS-like eqs.(8) and (9) which describe dynamics of an ion wave packet propagating arbitrary direction in a magnetized plasma. In the particular case of pallarel propagation, the equations are reduced into single equation, which is the three dimensionally generalized NLS equation. From the virial theorem, we have shown the possibility of collapse of ion wave. We have also given the critical wavenumber and the magnitude of magnetic field. The critical wave number of collapse is the same as the modulational instability of ion sound wave, $k_{cr} = 1.47$. At such high wave numbers ion Landau damping will occur, and as a result the observation of the modulational instability and hence collapse might be difficult. The effect of finite ion temperature, however, causes a significant lowering of the critical wavenumber k_{cr}, which will make the observation less difficult[12]. In Sec.4 , we have shown the DS1 condition of eqs.(18) and (19). In some region of a and $\mathbf{k}$ shown in Sec.4, there is the possibility of observing dromion. We consider a formation of dromion as follows: In the case of wave propagation perpendicular to a magnetic field, we have shown that both of self-focusing and modulational instability are possible to occur. If both of them grow simultaneously and then nonlinear saturation occur, the formation of a localized structure can be expected in this system. In the region of 3 in fig.2, the DS1 condition is satisfied and also both of the instabilities can occur. Thus we believe that the localized structure are nothing but dromion when an appropriate boundary condition[4] is set in this system.

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