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'INVERSES' OF VIRASORO OPERATORS

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The definition of vertex operator algebras is now essentially fixed due primarily to the book written by Frenkel, Lepowsky and Meurman [3] (see [1] also). We mostly follow their definition in this notes. In particular we assume the existence of the vacuum vector 1 and the conformal vector w. The components w_n of the vertex operator $Y(w,z)$ of w form the Virasoro algebra Vir spanned by $L(n)$'s if we set $L(n) = w_{n+1}$. The $L(n)$'s satistfy the famous commutation relation :

$$[L(m), L(n)] = (m-n)L(m+n) + \frac{1}{12}(m^3 - m)\delta_{m+n,0}c,$$

where c is called the central charge of the Virasoro algebra Vir or the rank of the vertex operator algebra V. The central charge c is assumed to be a rational number in [3], but we do not need it in this paper. If V_k is the eigenspace of the Virasoro operator $L(0)$ with eigenvalue k, then it is assumed that k is an integer and V is the direct sum of V_k's :

$$V = \coprod_{k=k_0}^{\infty} V_k.$$

V_k is the subspace of the homogeneous elements of V and the elements of V_k are said to have weight k. The dimension of V_k is assumed to be finite in [3]. We, however, do not need it. We, as in [3], assume that the weight of V is bounded below and so $V_k = 0$ if $k < k_0$. That $L(-1)$ is injective is noted by Li in [4] and that $L(1)$ is surjective is shown by Dong, Lin, and Mason [2]. In this note, we shall obtain, as a corollary, an 'extension' of their result : for all $k > 0$ and $n \geq 0$, $L(-n)$ is injective on V_k and $L(n)$ is surjective on V_{k+n} provided that the central charge c of the Virasoro algebra Vir is nonnegative and the negative weight states do not occur, i.e. $V_k = 0$ for $k < 0$. These conditions are not assumed in [2] or [4]. What we actually prove is the existence of certain operators $U_{k,n}$ and $D_{k,n}$ composed of the Virasoro operators $L(n), L(-n)$ such that $L(n)U_{k,n}|_{V_k} = Id|_{V_k}$ and $D_{k,n}L(-n)|_{V_k} = Id|_{V_k}$. See Theorem 4 below for the precise statement.The injectivity

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itself of $L(-n)$ for $n \geq 0$ under our assumption is easy to show, hence so will be the surjectivity of $L(n)$ if the duality is used. We, however, believe that the explicit operators $U_{k,n}$, and $D_{k,n}$ (i.e. 'inverses' of $L(n)$ and $L(-n)$) are of some interest for studying the Monster module, for example. The operators $U_{k,n}$ and $D_{k,n}$ can not be defined unless some conditions are met. The conditions on the central charge c and the negative states mentioned above are the simplest. For more general cases, see Theorem 7. See Corollary 8 also, where we prove, under some assumption,

$$V_{k+n} = Ker(L(n)|_{V_{k+n}}) \oplus Im(L(-n)|_{V_k}),$$

for all $n > 0$.

We start with an elementay lemma :

Lemma 1. *For $k \in \{1, 2, 3, ...\}$, we have :*

(a). $[L(0), L(n)^k] = -knL(n)$; *and,*

(b). $[L(n), L(-n)^k] = 2knL(-n)^{k-1}L(0) + kn((k-1)n + \frac{1}{12}(n^2-1)c)L(-n)^{k-1}$.

Proof. (a) is an easy exercise by induction. (b) is also shown by induction as follows. Set

$$f(k) = 2kn,$$

and

$$g(k) = kn((k-1)n + \frac{1}{12}(n^2-1)c).$$

If $k = 1$, then $f(1) = 2n$ and $g(1) = \frac{1}{12}(n^3-n)c$ and so (b) is just a defining relation of the Virasoro algebra. Suppose that (b) holds for k. We have, by a property of derivations,

$$\begin{aligned}
[L(n), L(-n)^{k+1}] &= [L(n), L(-n)]L(-n)^k + L(n)[L(n), L(-n)^k] \\
&= (2nL(0) + \frac{1}{12}(n^3-n)c)L(-n)^k + L(-n)(f(k)L(-n)^{k-1}L(0) \\
&+ g(k)L(-n)^{k-1}) \\
&= 2n(L(-n)^kL(0) + knL(-n)^k) + \frac{1}{12}(n^3-n)cL(-n)^k \\
&+ f(k)L(-n)^kL(0) + g(k)L(-n)^k \\
&= (2n + f(k))L(-n)^kL(0) + (2kn^2 + \frac{1}{12}(n^3-n)c + g(k))L(-n)^k.
\end{aligned}$$

It now remains to show that

$$f(k+1) = f(k) + 2n,$$

and

$$g(k+1) = 2kn^2 + \frac{1}{12}(n^3-n)c + g(k).$$

The first is trivial. To show the second we compute :

$$\begin{aligned} 2kn^2 + \frac{1}{12}(n^3 - n)c + g(k) \\ &= 2kn^2 + \frac{1}{12}(n^3 - n)c + kn((k-1)n + \frac{1}{12}(n^2 - 1))c \\ &= kn(2n + kn - n) + (kn + n)\frac{1}{12}(n^2 - 1)c \\ &= kn(k+1)n + (k+1)n\frac{1}{12}(n^2 - 1)c \\ &= (k+1)n(kn + \frac{1}{12}(n^2 - 1)c) \\ &= g(k+1), \end{aligned}$$

as required.

Definition. For each pair (k,n) of integers we define the 'up' operator

$$U_{k,n} : V_k \longrightarrow V_{k+n},$$

and the 'down' operator

$$D_{k,n} : V_{k+n} \longrightarrow V_k,$$

as follows:

$$U_{k,n} = \sum_{j=1}^{\infty} a_j L(-n)^j L(n)^{j-1},$$

and

$$D_{k,n} = \sum_{j=1}^{\infty} a_j L(-n)^{j-1} L(n)^j,$$

where

$$a_j = -(\frac{-12}{n})^j \prod_{i=1}^{j} (\frac{1}{i((n^2 - 1)c - 12(i-1)n + 24k)}),$$

or

$$a_j = 0$$

if

$$(n^2 - 1)c - 12(i-1)n + 24k = 0$$

for some i, where $j \geq i \geq 1$.

Remark. If $(n^2 - 1)c - 12(i-1)n + 24k = 0$ for some i, then

$$i = \frac{(n^2 - 1)c + 24k}{12n} + 1.$$

In particular, such an $i = i_0$ is uniquely determined for a given pair (k, n). Obviously $a_j = 0$ for all $j \geq i_0$, and $a_j \neq 0$ for $j < i_0$. Since the weight of V is bounded below, both operators are well defined on V. Note also that the coefficient a_j involves k.

Lemma 2. *Suppose* $c \geq 0$ *and the weight of* V *is bounded below by 0, i.e.* $V_k = 0$ *for* $k < 0$. *Then for* $k > 0$, $L(n)^{j-1}V_k = L(n)^j V_{k+n} = 0$ *holds for all* $n \geq 1$ *and for all* $j \geq \frac{(n^2-1)c+24k}{12n} + 1$.

Proof. Suppose

$$j \geq \frac{(n^2-1)c+24k}{12n} + 1.$$

Then

$$(j-1)n \geq \frac{(n^2-1)c}{12} + 2k,$$

and so

$$k-(j-1)n \leq -k - \frac{(n^2-1)c}{12}.$$

On the other hand, we have :

$$L(n)^{j-1}V_k \subseteq V_{k-(j-1)n}.$$

Suppose

$$-k - \frac{(n^2-1)c}{12} \geq 0.$$

Then, since $n \geq 1$, we get $k = 0$, against our assumption. Since

$$L(n)^j V_{k+n} \subseteq V_{k-(j-1)n},$$

also, we obtain the lemma.

Corollary 3. *Suppose* $n \geq 1$ *and*

$$(n^2-1)c - 12n(i_0 - 1) + 24k = 0,$$

for an integer i_0, *then*

$$L(n)^{i_0-1}V_k = L(n)^{i_0}V_{k+n} = 0,$$

if $k > 0$.

Proof. Immediate from the previous lemma.

Theorem 4. *Suppose the central charge* c *of the Virasoro algebra* Vir *of the Vertex operator algebra* V *is nonnegative ; i.e.* $c \geq 0$ *and the negative states do not occur ; i.e.* $V_k = 0$ *for all* $k < 0$. *Then*

$$L(n)U_{k,n}|_{V_k} = Id|_{V_k},$$

and,

$$D_{k,n}L(-n)|_{V_k} = Id|_{V_k},$$

for all $k > 0$, $n > 0$.

Corollary 5. *Under the same assumption as in Theoem 4, we have : $L(n)$ is surjective on V_{k+n} and $L(-n)$ is injective on V_k for all $n > 0, k > 0$.*

Remark. Alternatively, the injectivity of $L(-n)|_{V_k}$ can easily be established as follows (under a slightly weaker condition). Suppose $L(-n)v = 0$, where $v \in V_k$. Since the weight of V is bounded below, we have $L(m)v = 0$, for a large $m > 0$. We may assume $n|m$. Using the Virasoro relation :

$$[L(m), L(-n)]v = (m+n)L(m-n)v + \frac{1}{12}(m^3 - m)\delta_{m-n,0}cv = 0,$$

repeatedly we obtain, with m=n,

$$(2nL(0) + \frac{1}{12}(n^3 - n)c)v = 0.$$

Therefore

$$(2nk + \frac{1}{12}(n^3 - n)c)v = 0.$$

Now suppose

$$24nk + (n^2 - 1)c \neq 0,$$

which obviously holds if $n > 0, k > 0, c \geq 0$. Then $v = 0$, as desired.

Proof of Theorem 4. We directly compute :

$$\begin{aligned} L(n)U_{k,n} &= \sum_{j=1}^{\infty} a_j L(n)L(-n)^j L(n)^{j-1} \\ &= \sum_{j=1}^{\infty} a_j \{L(-n)^j L(n) + f(j)L(-n)^{j-1}L(0) + g(j)L(-n)^{j-1}\}L(n)^{j-1}, \end{aligned}$$

where as in Lemma 1,

$$f(j) = 2jn,$$

and

$$g(j) = jn((j-1)n + \frac{1}{12}(n^2 - 1)c).$$

Since $L(n)^{j-1}V_k \subseteq V_{k-(j-1)n}$ and $L(0)|_{V_{k-(j-1)n}} = k - (j-1)n$, a scalar mulptiple, we have:

$$\begin{aligned} L(n)U_{k,n}|_{V_k} &= \sum_{j=1}^{\infty} a_j L(-n)^j L(n)^j \\ &+ \sum_{j=1}^{\infty} a_j \{f(j)(k-(j-1)n) + g(j)\}L(-n)^{j-1}L(n)^{j-1} \\ &= a_1\{f(1)k + g(1)\}Id|_{V_k} \\ &+ \sum_{j=2}^{\infty} \{a_{j-1} + a_j(f(j)(k-(j-1)n) + g(j))\}L(-n)^{j-1}L(n)^{j-1}. \end{aligned}$$

It now suffices to show:

$$a_1\{f(1)k+g(1)\}=1,$$

and, for $j>1$,

$$\{a_{j-1}+a_j(f(j)(k-(j-1)n)+g(j))\}L(-n)^{j-1}L(n)^{j-1}|_{V_k}=0.$$

We have :

$$f(1)=2n, g(1)=\frac{1}{12}(n^3-n)c,$$

and so

$$f(1)k+g(1)=2kn+\frac{1}{12}(n^3-n)c\neq 0,$$

and

$$a_1=-(\frac{-12}{n})\frac{1}{(n^2-1)c+24k}=\frac{1}{f(1)k+g(1)}$$

and so

$$a_1(f(1)k+g(1))=1.$$

We will next show, for $j\geq 2$,

$$a_{j-1}+a_j(f(j)(k-(j-1)n)+g(j))=0,$$

if $a_j\neq 0$ (and hence $a_{j-1}\neq 0$ also).

Recall

$$a_j=-(\frac{-12}{n})^j\prod_{i=1}^{j}(\frac{1}{i((n^2-1)c-12(i-1)n+24k)}),$$

Replacing $f(j)$ and $g(j)$ with their respective expressions given above, we obtain

$$f(j)(k-(j-1)n)+g(j)=jn\{\frac{1}{12}(n^2-1)c-(j-1)n+2k\}=\frac{jn}{12}\{(n^2-1)c-12(j-1)n+24k\}.$$

By the definition of a_j, we obtain

$$\frac{a_j}{a_{j-1}}=-\frac{12}{jn}\frac{1}{(n^2-1)c-12(j-1)n+24k}$$

Therefore

$$a_{j-1}+a_j(f(j)(k-(j-1)n)+g(j))=0,$$

as desired.

Finally we treat the cases where $a_j=0$ for some j. To this case to occur, there must exit an integer i_0 such that (see Remark)

$$i_0=\frac{(n^2-1)c+24k}{12n}+1.$$

In this case, we have $a_j = 0$ for all $j \geq i_0$, and $a_j \neq 0$ for $j < i_0$. It then suffices to show :

$$\{a_{i_0-1} + a_{i_0}(f(i_0)(k-(i_0-1)n) + g(i_0)\}L(-n)^{i_0-1}L(n)^{i_0-1}|V_k = 0.$$

This, however, has been shown in Corollary 3.

The corresponding statement for the down operator :

$$D_{k,n}L(-n)|_{V_k} = Id|_{V_k}$$

can be proved by making the following observation :

Lemma 6. *The following relation holds :*

$$\begin{aligned} L(n)L(-n)^jL(n)^{j-1}|_{V_k} &= L(-n)^{j-1}L(n)^jL(-n)|_{V_k} \\ &= L(-n)^jL(n)^j \\ &+ \{2jn(k-(j-1)n + jn((j-1)n + \frac{1}{12}(n^2-1)c))\}L(-n)^{j-1}L(n)^{j-1}. \end{aligned}$$

Proof. We set $a_j = 1$ and $a_i = 0$ for $i \neq j$ in the calculation of $L(n)U_{k,n}|_{V_k}$. Then its proof reads :

$$L(n)L(-n)^jL(n)^{j-1}|_{V_k} = L(-n)^jL(n)^j + \{f(j)(k-(j-1)n) + g(j)\}L(-n)^{j-1}L(n)^{j-1}.$$

The equality of the first and the third quantity in the lemma is now obvious. To show the remaining equality, let us write

$$f(j) = f(j,n) = 2jn$$

and

$$g(j) = g(j,n) = jn((j-1)n + \frac{1}{12}(n^2-1)c),$$

as f and g are functions of two variables j and n. Then by Lemma 1, we obtain :

$$L(n)^jL(-n) = L(-n)L(n)^j - f(j,-n)L(n)^{j-1}L(0) - g(j,-n)L(n)^{j-1}.$$

Therefore

$$L(-n)^{j-1}L(n)^jL(-n)|_{V_k} = L(-n)^jL(n)^j + \{-f(j,-n)k - g(j,-n)\}L(-n)^{j-1}L(n)^{j-1}.$$

It now sufices to show :

$$-f(j,-n)k - g(j,-n) = f(j,n)(k-(j-1)n) + g(j,n)),$$

or

$$-g(j,-n) = -2jn(j-1)n + g(j,n),$$

which can be established easily.

It is now immediate from Lemma 6 that

$$D_{k,n}L(-n)|_{V_k} = L(n)U_{k,n}|_{V_k} = Id|_{V_k}.$$

This completes the proof of the theorem.

We do not see an immediate application of it, but what we actually proved in Theorem 5 was :

Theorem 7. *Suppose :*

(a). $(n^2-1)c-12(i-1)n+24k \neq 0$ for any $i \geq 1$; or,

(b). $L(n)^{j-1}V_k = 0$ for all $j \geq i_0$, where

$$i_0 = \frac{(n^2-1)c+24k}{12n}+1.$$

Then

$$L(n)U_{k,n}|_{V_k} = Id|_{V_k},$$

and,

$$D_{k,n}L(-n)|_{V_k} = Id|_{V_k},$$

for all $n > 0$.

Corollary 8. *Under the same assumption as in Theorem 7 (in particular if $c \geq 0$, $k_0 = 0$, and $k > 0$), we have*

$$V_{k+n} = Ker(L(n)|_{V_{k+n}}) \oplus Im(L(-n)|_{V_k}),$$

for all $n > 0$.

Proof. Consider the exact sequence :

$$0 \longrightarrow Ker(L(n)|_{V_{k+n}}) \xrightarrow{i} V_{k+n} \xrightarrow{L(n)} V_k \longrightarrow 0,$$

where i is the natural injection. Since the 'up' operator $U_{k,n}$ splits the exact sequence, we have

$$V_{k+n} = Ker(L(n)|_{V_{k+n}}) \oplus Im(U_{k,n}).$$

Since

$$U_{k,n} = \sum_{j=1}^{\infty} a_j L(-n)^j L(n)^{j-1},$$

we obtain :

$$Im(U_{k,n}) \subseteq Im(L(-n)).$$

Hence

$$V_{k+n} = Ker(L(n)|_{V_{k+n}}) + Im(L(-n)|_{V_k}).$$

To show the sum is direct, let

$$v \in Ker(L(n)|_{V_{k+n}}) \cap Im(L(-n)|_{V_k}).$$

Then $v = L(-n)v'$, where $v' \in V_k$, and $L(n)v = 0$. We can now apply the 'down' operator

$$D_{k,n} = \sum_{j=1}^{\infty} a_j L(-n)^{j-1} L(n)^j,$$

to the both sides of the relation $v = L(-n)v'$ to obtain $v' = 0$. This completes the proof of the corollary.

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