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BRS Symmetry on Background-Field and Renormalizability Problem of Quantum Gravity [1]

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Abstract

The two dimensional gravity is newly formulated in the backgroud field method. We take a new way of splitting the metric field in order to clarify the role of the Weyl mode. The gauge-fixing and ghost lagrangians are introduced follwowing the standard BRS procedure. The BRS transformation of all fields is obtained. Both the Weyl symmetry and the BRS symmetry can be looked transperently in this formalism.

§1. Introduction

BRS symmetry has been playing the central role in gage theories[1,2,3,4]. This article aims at clarifying its role in the background formalism (effective action formalism) which is one popular approach to gauge theories. In the workshop, I talked about the 4D gauge and gravitational theories , and about the 2D gravitational theory. In particular, as for the 4D gravitational theory, the connection with the renormalizability problem was stressed. Because the content about the 4D theories wil soon been published elsewhere[5], I report here only the 2D gravitational theory.

New points of the present formulation of 2D quantum gravity are as follows.

1. 2D quantum gravity is newly formulated in the general framework of the background field formalism. The 2D general covariance and invariance are completely kept in the background effective action.

[1] US-93-1

2. We take a new way of splitting the metric field. As the quantum field, the traceless modes and the Weyl (trace) mode are separately treated. The new splitting makes us clarify the role of the Weyl mode and the Weyl symmetry so much.

3. Because the quantization procedure is based on the standard BRS procedure, the BRS symmetry is respected manifestly.

4. We can find a new aspect of the background quantization in connection with the compatibility of the BRS symmetry with the Weyl symmetry in defining the physical quantities.

§2. Q-type,C-type and Weyl Symmetries

The background formalism is exploiting the technique of splitting fields into the bacground ones and the quantum ones. Generally there are some ways of splitting. We must adopt the best choice of splitting by taking account of the symmetry of the system. For example, as for Yang-Mills theory (in 3+1 Dim) ,we should take the following linear splitting.

$$\bar{A}_\mu \quad = \quad A^{cl}_\mu + \ A_\mu \quad . \qquad\qquad (2.1)$$

This is because the symmetry ,$\delta A_\mu = \partial_\mu \Lambda + \Lambda \times A_\mu$, is linear with respect to A_μ. As for the non-linear sigma-model, we should take the nonlinear splitting [ref. 6,7] as

$$\bar{g} \quad = \quad g_{cl} \cdot exp(i\tau^a \pi_a) \quad , \qquad\qquad (2.2)$$

where $\bar{g}$ and g_{cl} are elements of the group G ,and τ^a is an element of the corresponding Lie algebra. As for the (2D) gravitaional theory, the symmetry looks linear and nonlinear depending on the choice of independent fields. In the present case it seems best to take the following one. [2]

$$\bar{g}_{ab} = g_{ab}e^{\varphi} + h^T_{ab} \quad , \; g^{ab}h^T_{ab} = 0 \quad . \qquad\qquad (2.3)$$

[2]The 2D induced gravity was analysed in the way of linear-splitting in ref.8 and ref.9.

This is because the Weyl mode plays the essential role in 2D quantum gravity. The symmety of the general covariance can be written in terms of the full field as

$$\begin{aligned}\delta\bar{g}_{ab} &= -\bar{\nabla}_a\epsilon_b - \bar{\nabla}_b\epsilon_a \\ &= -\bar{g}_{ac}\partial_b\epsilon^c - \bar{g}_{bc}\partial_a\epsilon^c - \partial_c\bar{g}_{ab}\cdot\epsilon^c \qquad (2.4) \\ &= -\bar{g}_{ac}\nabla_b\epsilon^c - \bar{g}_{bc}\nabla_a\epsilon^c - \nabla_c\bar{g}_{ab}\cdot\epsilon^c \quad ,\end{aligned}$$

where $\bar{\nabla}_a\epsilon_b = \bar{g}_{bc}\bar{\nabla}_a\epsilon^c = \bar{g}_{bc}(\partial_a\epsilon^c + \Gamma^c_{ad}(\bar{g})\epsilon^d)$ $\quad ,\nabla_a\epsilon^b = \partial_a\epsilon^b + \Gamma^b_{ac}(g)\epsilon^c$ and $\nabla_c\bar{g}_{ab} = \partial_c\bar{g}_{ab} - \Gamma^d_{ca}(g)\bar{g}_{db} - \Gamma^d_{cb}(g)\bar{g}_{da}$. This symmetry can be eqivalently expressed by the various combination of two transformations: the transformation of the quantum field and that of the background field. Particularly the following two characteristic ways are useful.

Q-type transformation

$$\begin{aligned}\delta g_{ab} &= 0 \quad , \\ \delta e^{\varphi} &= -e^{\varphi}(\nabla_a\epsilon^a + \partial_a\varphi\cdot\epsilon^a) - h^T_{ab}\nabla^a\epsilon^b \quad , \qquad (2.5a) \\ \delta h^T_{ab} &= -e^{\varphi}D_{ab}{}^c\epsilon_c - h^T_{dc}D_{ab}{}^d\epsilon^c - \nabla_c h^T_{ab}\cdot\epsilon^c \quad .\end{aligned}$$

$D_{ab}{}^c$ above is defined as

$$\begin{aligned}&D_{ab}{}^c \equiv \delta^c_a\nabla_b + \delta^c_b\nabla_a - g_{ab}\nabla^c \quad , \\ &D_{ab}{}^b = n\nabla_a \quad , g^{ab}D_{ab}{}^c = (2-n)\nabla^c \quad , \qquad (2.5b)\end{aligned}$$

where $n(=2)$ is the space-tome dimension. $D_{ab}{}^c$ is essentially the same as that operator which appeared in ref.10 and ref.11 as the jacobian factor transforming from the metric field to Weyl mode in the conformal gauge. The Q-type transformation is characterized by the condition that the quantum fields (φ, h^T_{ab}) undertake full variation of the symmetry transformation and the background fields g_{ab} are fixed. This condition makes the transformation (2.5a) unique.

C-type transformation

$$\begin{aligned}\delta g_{ab} &= -\nabla_a\epsilon_b - \nabla_b\epsilon_a \quad , \\ \delta e^{\varphi} &= -\partial_c e^{\varphi}\cdot\epsilon^c \quad , \qquad (2.6) \\ \delta h^T_{ab} &= -h^T_{ac}\epsilon^c{}_{,b} - h^T_{bc}\epsilon^c{}_{,a} - h^T_{ab,c}\epsilon^c \quad .\end{aligned}$$

The C-type transformation is characterized by the condition that the background field transforms in the same way as the full field. Of course, the reason why we can equivalently express the transformation (2.4) in so many ways is that the number of fields are doubled: from 3 ($\bar{g}_{ab}$) to 6 ($3(g_{ab})+1(\varphi)+2(h^T_{ab})$). Note further e^{φ} and h^T_{ab} transform as the scalar and tensor respectively ,in C-type transformation. There is an additional symmetry which originates not from the symmetry of the lagrangian but from the choice of the splitting form (2.3).

$$\begin{aligned} {g_{ab}}' = &\quad g_{ab}\, e^{\tau(\sigma)} \quad , \\ \varphi' = &\quad \varphi - \tau(\sigma) \quad , \\ {h^T_{ab}}' = &\quad h^T_{ab}\ (\text{fixed}) \quad , \end{aligned} \qquad (2.7)$$

where $\tau(\sigma)$ is the arbitray local parameter (Weyl freedom). This is called (local) Weyl symmetry. This freedom remains even when the cosmological term is added in the theory.

§3. Background Quantization of 2D Quantum Gravity

Let us consider the 2D quantum gravity interacting with a scalar field $X(\sigma)$.

$$\begin{aligned} \mathcal{L}[X, g_{ab}] &= \mathcal{L}_G[g_{ab}] + \mathcal{L}_M[X, g_{ab}] \quad , \\ \mathcal{L}_G[g_{ab}] &= \sqrt{g}(\omega R - \lambda) \quad , \\ \mathcal{L}_M[X, g_{ab}] &= \tfrac{1}{2}\sqrt{g}\nabla_a X \cdot \nabla^a X \quad , \end{aligned} \qquad (3.1)$$

where $1/\omega$ and λ are the gravitational constant and the cosmological constant respectively. As for the matter field, we adopt the the linear splitting.

$$\bar{X} = \quad X + x \quad , \qquad (3.2)$$

where X is the background field and x is the quantum one. From the starting symmetry:

Original Symmetry

$$\delta\bar{X} = -\epsilon^a \nabla_a(\bar{g})\bar{X} = -\epsilon^a \partial_a \bar{X} \quad , \qquad (3.3)$$

satisfied by the full field, we can obatin the Q-type symmetry and C-type one following the procedure presented in Sec.2.

Q-type Symmetry

$$\delta X = 0 \quad , \delta x = -\epsilon^a \partial_a (X + x) \quad , \qquad (3.4a)$$

C-type Symmetry

$$\delta X = -\epsilon^a \partial_a X \quad , \delta x = -\epsilon^a \partial_a x \quad . \qquad (3.4b)$$

We consider first the conformal gauge.

$$h^T_{ab} \quad = \quad 0 \qquad (3.5)$$

Note that this gauge is C-type invariant. For convenience we prefer to take the more general gauge.

$$\begin{aligned} \mathcal{L}_{gauge}[b^T_{ab}, h^T_{ab}; \beta] = & \sqrt{g}(\tfrac{\beta}{2} b^{ab}_T b^T_{ab} + b^{ab}_T h^T_{ab}) \\ = & \sqrt{g}(\tfrac{\beta}{2}(b^{ab}_T + \tfrac{1}{\beta} h^{ab}_T)(b^T_{ab} + \tfrac{1}{\beta} h^T_{ab}) - \tfrac{1}{2\beta} h^{ab}_T h^T_{ab}) \quad , \qquad (3.6) \\ g^{ab} b^T_{ab} = & \quad 0 \quad , \end{aligned}$$

where b^T_{ab} is the auxiliary field (so-called B-field) which is traceless, and β is the gauge parameter. The case (3.5) is obtained by the limit $\beta \to 0$. We can obtain C-type transformation of b^T_{ab} by requiring the C-type invariance of (3.6).

$$\delta b^{ab}_T = b^{ac}_T \epsilon^b{}_{,c} + b^{bc}_T \epsilon^a{}_{,c} - \epsilon^c b^{ab}_T{}_{,c} \quad . \qquad (3.7a)$$

The corresponding ghost lagrangian is obtained, by Q-type transformation of h^T_{ab} ,as

$$\mathcal{L}_{ghost} = \sqrt{g}\ \bar{\chi}^{ab}_T [-e^{\varphi} D_{ab}{}^c \chi_c - h^T_{dc} \cdot D_{ab}{}^d \chi^c - (\nabla_c h^T_{ab}) \cdot \chi^c] \quad ,$$

$$g_{ab}\ \bar{\chi}^{ab}_T \quad = \quad 0 \quad . \qquad (3.8)$$

Noting the fact that e^{φ} and h^T_{ab} transform, in the C-type transformation (2.6), as the scalar and tensor respectively, C-type symmetry of the ghost lagrangian is gauranteed by taking the ordinary C-type symmetry of tensor for $\bar{\chi}^{ab}_T$ and that of vector for χ_a.

$$\begin{aligned} \delta \chi_a = & \quad -\chi_b \epsilon^b{}_{,a} - \epsilon^b \chi_{a,b} \quad , \\ \delta \bar{\chi}^{ab}_T = & \quad \bar{\chi}^{ac}_T \epsilon^b{}_{,c} + \bar{\chi}^{bc}_T \epsilon^a{}_{,c} - \epsilon^c \bar{\chi}^{ab}_T{}_{,c} \quad . \qquad (3.7b) \end{aligned}$$

The total lagrangian is given by

$$\mathcal{L}^T[e^{\varphi}, h^T, x, \bar{\chi}_T^{ab}, \chi_a, b_{ab}^T; \beta, \omega, \lambda] \equiv \mathcal{L}_G[ge^{\varphi} + h^T] + \mathcal{L}_M[X + x, ge^{\varphi} + h^T]$$
$$+\mathcal{L}_{gauge}[b_{ab}^T, h_{ab}^T; \beta] + \mathcal{L}_{ghost}[\bar{\chi}_T^{ab}, \chi_a, e^{\varphi}, h^T] \quad . \qquad (3.9)$$

We can show this lagrangian has the BRS symmetry. The final result is that the total lagrangian $\mathcal{L}^T$ is invariant under the following BRS transformation.

$$\delta g_{ab} = 0 \quad , \quad \delta X = 0 \quad ,$$
$$\delta e^{\varphi} = e^{\varphi}\delta\varphi = \xi[-e^{\varphi}({\chi^a}_{,a} + \varphi_{,a}\chi^a) - h_{ab}^T\chi^{b,a}] \equiv \xi\ se^{\varphi} \ ,$$
$$\delta h_{ab}^T = \xi[-e^{\varphi}{D_{ab}}^c\chi_c - h_{cd}^T{D_{ab}}^c\chi^d - h_{ab,c}^T\chi^c] \equiv \xi\ sh_{ab}^T \ ,$$
$$\delta x = -\xi\ \chi^a(X_{,a} + x_{,a}) \equiv \xi\ sx \quad , \qquad (3.10)$$
$$\delta\chi_a = -\xi\ \chi^b\chi_{a,b} \equiv \xi\ s\chi_a \quad ,$$
$$\delta\bar{\chi}_T^{ab} = -\xi\ b_T^{ab} \equiv \xi\ s\bar{\chi}_T^{ab} \quad , \qquad \delta b_T^{ab} = 0 \quad ,$$

where ξ is the global grassman parameter, s is the BRS operator. The operator s satisfies the nilpotency property

$$s^2e^{\varphi} = s^2h_{ab}^T = s^2x = s^2\chi_a = s^2\bar{\chi}_T^{ab} = s^2b_T^{ab} = 0 \quad , \qquad (3.11a)$$

where s^2 is defined by (3.11b)

$$\delta\mathcal{O} = \xi\ s\mathcal{O} \quad , \quad \delta(s\mathcal{O}) \equiv \xi\ s^2\mathcal{O} \quad , \qquad (3.11b)$$

for an arbitrary operator $\mathcal{O}$. Furthermore we can write $\mathcal{L}_{gauge} + \mathcal{L}_{ghost}$ as the BRS-exact form.

$$\mathcal{L}_{gauge} + \mathcal{L}_{ghost} = -s(\bar{\chi}_T^{ab}h_{ab}^T + \frac{\beta}{2}\bar{\chi}_T^{ab}b_{ab}^T) \ . \quad (3.12)$$

We summerize here the physical dimension and the ghost number for every field, prameter and operator.

Ghost # \ Dim	m^{-2}	m^{-1}	m^0	m^1	m^2
+1			ξ		$\bar{\chi}_T^{ab}$
0	β	ϵ^a	X , g_{ab} x , φ , h^T_{ab} ω , λ	∇_a , $D_{ab}{}^c$	b_T^{ab} $\mathcal{L}$
-1		χ^a			

Table.1

§4. Weyl Invariance and New Treatment of Weyl Mode in Background Formalism

In the background formalism, the effective action $\Gamma[A]$ is conventionally defined as (see ref.5)

$$e^{\Gamma[A]} = \int \mathcal{D}a\mathcal{D}(ghost)\mathcal{D}(b-field)\ exp\int d^2\sigma\{\ \mathcal{L}[A+a] - \mathcal{L}_{,i}[A]a^i + \mathcal{L}_{gauge} + \mathcal{L}_{ghost}\ \}\ . \quad (4.1)$$

We notice the following points.

1. The subtraction of the linear term (with respect to the quantum field), $-\mathcal{L}_{,i}[A]a^i$,is important to define the stable perturbation around a minimum of the potential.
2. $\mathcal{L}^T \equiv \mathcal{L}[A+a] + \mathcal{L}_{gauge} + \mathcal{L}_{ghost}$ is BRS invariant, but the subtraction term is not.
3. BRS invariance of physical quantities are guaranteed by the on-shell condition $\mathcal{L}_{,i}[A] = 0$.

On the other hand, in the present case, the subtraction procedure is valid for (h^T, x), but not for the Weyl mode φ. This is because the system contains the Liouville potential and it does not seem to have a minimun in the φ-direction. The best way to define the effective action is to make the whole system 'inert' with respect to the Weyl mode: make the theory Weyl invariant (taking account even of the Weyl anomaly) at the cost of BRS symmetry at this intermediate stage. Note that it is enough to require BRS symmetry for the physical quantites which are obtained by imposing the on-shell condition in the background approach.

In the total lagrangian (3.9), $\mathcal{L}_G$ and $\mathcal{L}_M$ are Weyl invariat; invariant under the transformation (2.7),

$$
\begin{aligned}
g_{ab}{}' &= g_{ab}\ e^{\tau(\sigma)} \quad , \\
\varphi' &= \varphi - \tau(\sigma) \quad , \qquad\qquad (4.2a) \\
h^{T}_{ab}{}' &= h^{T}_{ab}\ (\text{fixed}) \quad .
\end{aligned}
$$

However the gauge and ghost parts ($\mathcal{L}_{gauge}, \mathcal{L}_{ghost}$) are not. Let us modify $\mathcal{L}_{gauge}$ and $\mathcal{L}_{ghost}$ and make them Weyl invariant. The simplest choice of Weyl transformation for the auxiliary field b^T_{ab} is

$$
b^{T}_{ab}{}' = b^{T}_{ab}\ (\text{fixed}) \quad . \qquad\qquad (4.2b)
$$
$$
(b^{ab}_T = g^{ac}g^{bd}b^T_{cd} \rightarrow b^{ab\prime}_T = e^{-2\tau}b^{ab}_T)
$$

In this choice $\mathcal{L}_{gauge}$ transforms to $e^{-\tau}\mathcal{L}_{gauge}$, therefore the Weyl invariant modification is obtained by

$$
\begin{aligned}
\mathcal{L}_{gauge}{}' = &\quad e^{-\varphi}\mathcal{L}_{gauge} \\
= &\quad \sqrt{g}\ e^{-\varphi}(\ \tfrac{\beta}{2}b^{ab}_T b^T_{ab} + b^{ab}_T h^T_{ab}\ ) \quad . \qquad\qquad (4.3)
\end{aligned}
$$

As for the ghost fields $(\bar{\chi}^{ab}_T, \chi_a)$, the simplest choice is

$$
\bar{\chi}^{ab}_T{}' = \bar{\chi}^{ab}_T\ (\text{fixed}) \quad , \qquad \chi_a{}' = \chi_a\ (\text{fixed}) \quad . \qquad\qquad (4.2c)
$$
$$
(\chi^a = g^{ab}\chi_b \rightarrow \chi^{a\prime} = e^{-\tau}\chi^a)
$$

Then ,by substituting $\nabla_c h^T_{ab}$ and $D_{ab}{}^c\chi_d$ in (3.8) by their Weyl-invariant counterparts,

$$
\begin{aligned}
\overline{\nabla_c h^T_{ab}} \equiv &\quad \nabla_c h^T_{ab} - \tfrac{1}{2}(h^T_{ad}D_{bc}{}^{d}\varphi + a \leftrightarrow b) \quad , \\
\overline{D_{ab}{}^c\chi_d} \equiv &\ D_{ab}{}^c\chi_d - \tfrac{1}{2}(\ \delta^c_a\chi_e D_{bd}{}^e\varphi + \delta^c_b\chi_e D_{ad}{}^e\varphi - g_{ab}\chi_e D^c{}_d{}^e\varphi\ ) \quad , \qquad (4.4)
\end{aligned}
$$

respectively, the Weyl invariant version of ghost lagrangian $\mathcal{L}_{ghost}{}'$ is obtained as

$$\mathcal{L}_{ghost}{}' = \sqrt{g}\bar{\chi}_T^{ab}[-e^{\varphi}\overline{D_{ab}{}^c\chi_c} - h_{dc}^T g^{ce}\overline{D_{ab}{}^d\chi_e} - \overline{\nabla_c h_{ab}^T}\cdot\chi^c\,] \quad . \qquad (4.5)$$

Note that, in (4.4), $D_{ab}{}^c\varphi$ plays the role of Weyl gauge field.

In conclusion , the Weyl invariant version of $\mathcal{L}^T$,(3.9), is obtained by substituting $\mathcal{L}_{gauge}{}'$ and $\mathcal{L}_{ghost}{}'$ for $\mathcal{L}_{gauge}$ and $\mathcal{L}_{ghost}$ respectively. As for the treatment of Weyl anomaly, we comment on the next section.

§5. New Effective Action and Discussions

Now we propose a new effective action $\Gamma[g, X]$ defined by

$$\begin{aligned} exp\ \Gamma[g,X] = \quad & \int \mathcal{D}h_{ab}^T\mathcal{D}\varphi\mathcal{D}x\mathcal{D}b_{ab}^T\mathcal{D}\bar{\chi}_T^{ab}\mathcal{D}\chi_a\ exp\ \{ \\ & \mathcal{L}_G[ge^{\varphi}+h^T] + \mathcal{L}_M[X+x, ge^{\varphi}+h^T] \\ & -h^T\frac{\delta\mathcal{L}_G[ge^{\varphi}]}{\delta(ge^{\varphi})} - x\frac{\delta\mathcal{L}_M[X,ge^{\varphi}]}{\delta X} - h^T\frac{\delta\mathcal{L}_M[X,ge^{\varphi}]}{\delta(ge^{\varphi})} \\ & +\mathcal{L}_{gauge}{}' + \mathcal{L}_{ghost}{}' + \sqrt{g}(\ a_1 g^{ab}\partial_a\varphi\ \partial_b\varphi + a_2 R\varphi + \lambda' e^{a_3\varphi})\ \ \} \quad . \qquad (5.1) \end{aligned}$$

We notice the following points.

1. The last three terms are added to absorb the Weyl anomaly induced by the transformation of the integration measure[12,13]. They are C-type gauge invariant,but are ,by themselves ,neither BRS invariant nor Weyl invariat.

2. We notice here the kinetic term of φ appears in the lagragian through the cancelling procedure of the Weyl anomaly.

3. The BRS non-invariat part gives us the on-shell condition for the bacground field. In particular φ-mode part gives us a new type of condition.

4. This formalism can naturally be generalized to the string theory by introducing the external fields such as the graviton field, $G_{\mu\nu}(X)$,and the dilaton field $\Phi(X)$.

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