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Fresh Look at BRS Procedure

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Abstract: BRS procedure is examined from modern viewpoint with emphasis on its general utility as a book-keeping device for the introduction of new variables in Lagrangian field theories. Simple BRS gauge-fixing method is also extended in a form with its mathematical structure transparent.

1. INTRODUCTION

FP ghosts are needed for covariant gauge-fixing in (nonabelian) gauge theories. They were historically introduced in a perturbative consideration and then by means of path integral. Afterwards BRS symmetry of the gauge-fixed action was revealed and its cohomology was considered.[1]

However, it is too restricted a viewpoint to regard BRS cohomology as a structure only of the gauge-fixed version of gauge theories. In fact, the BRS procedure itself possesses much larger applicability beyond what it is as the usual gauge-fixing method.

In this note, according to Ref.[2], we point out that the BRS procedure provides a generic and useful framework for introducing new fields into a Lagrangian which are absent at the beginning, without altering the dynamical content of the system. We mention such examples as the equivalence theorem[3] and generalized field transformations including spectrum-changing ones.[4,5]

We also reconsider the gauge-fixing in the light of our viewpoint, which leads to a direct way of BRS gauge-fixing for general gauge theories. This extension yields simple BRS gauge-fixing method in a form with its mathematical structure[6] transparent. It enables us to perform simple BRS gauge-fixing of open and reducible gauge theories in principle.

Let us first review the simple BRS gauge-fixing procedure,[7] which is an elegant way of introducing gauge-fixing and FP ghost terms in a Lagrangian. It is widely used in order to perform Lagrangian gauge-fixing of gauge-invariant systems (with closed gauge algebras).

For definiteness, we take Yang-Mills theory as an example. Let G be a compact Lie group, $\mathcal{G}$ its Lie algebra. We consider the Yang-Mills Lagrangian $\mathcal{L}_A$ of a gauge field A_μ taking values in $\mathcal{G}$.

The BRS transformation law for the field A_μ is given by

$$\delta A_\mu = D_\mu c = \partial_\mu c + i[A_\mu, c] \tag{1}$$

based on its gauge transformation, where c denotes an FP ghost which takes values in $\mathcal{G}$.

With the aid of the nilpotency $\delta^2 A_\mu = 0$, the BRS transformation of the FP ghost is so determined as

$$\delta c = -\frac{i}{2}[c, c], \tag{2}$$

which also satisfies the nilpotency $\delta^2 c = 0$.

We then introduce an FP anti-ghost, dual to the FP ghost, with the BRS transformation

$$\delta \bar{c} = ib. \tag{3}$$

Here b denotes an NL field, which obeys $\delta b = 0$ due to the nilpotency of δ.

By means of a Grassmann-odd gauge-fixing function Ω (whose ghost number equals -1), we obtain a gauge-fixed Lagrangian as follows:

$$\mathcal{L} = \mathcal{L}_A - i\delta\Omega. \tag{4}$$

This is BRS invariant by construction. For example, a choice

$$\Omega = \bar{c}\partial^\mu A_\mu \tag{5}$$

leads to the Lagrangian for the Landau gauge

$$\mathcal{L} = \mathcal{L}_A + b\partial^\mu A_\mu + i\bar{c}\partial^\mu D_\mu c. \tag{6}$$

The above procedure is extremely simple and more general than the historical path-integral approach: we can obtain Lagrangians containing quartic ghost terms, for instance, straightforwardly by means of a gauge-fixing function Ω containing cubic ghost terms.

However, the procedure has one unsatisfactory aspect from a fundamental standpoint. It is not always possible for the transformation law (2) of the FP ghost to be obtained by requiring the nilpotency of the 'gauge' transformation (1). The closure of the gauge algebra is indispensable for the above method to work directly. Theories with open algebras make it impossible[8] for us to construct (off-shell) nilpotent BRS transformations. Even when (2) is all right, it also seems rather nontrivial whether the nilpotency $\delta^2 c = 0$ holds or not. Therefore some improvement seems desirable for the simple BRS gauge-fixing method itself, which will be performed in section 4.

2. BRS Cohomology

In this section, we expose a simple model which shows the mathematical structure[6] of the BRS cohomology in a transparent manner.

We consider a point particle on a connected Riemannian manifold M. Let $x^\mu(t)$ denote the position of the particle at the time t.

We define a BRS transformation

$$\delta x^\mu = c^\mu, \quad \delta \bar{c}_\mu = i b_\mu \tag{7}$$

with the requirement of the nilpotency $\delta^2 = 0$, and investigate the theory given by a BRS-invariant Lagrangian

$$\begin{aligned} \mathcal{L} &= -i\delta[\bar{c}_\mu(\dot{x}^\mu - \frac{1}{2}b^\mu)] \\ &= b_\mu \dot{x}^\mu + i\bar{c}_\mu \dot{c}^\mu - \frac{1}{2}b_\mu b^\mu, \end{aligned} \tag{8}$$

where the overdot denotes a time derivative.

The form of the Lagrangian (8) indicates that $b_\mu, x^\mu, \bar{c}_\mu$, and c^μ constitute a set of canonical variables for this theory. Hence the mechanical functions on the phase space are given by

$$\mathcal{C} = \{F(b_\mu, x^\mu, \bar{c}_\mu, c^\mu)\}, \tag{9}$$

which can be written as

$$\mathcal{C} = \oplus\, \mathcal{C}^n_m, \tag{10}$$

since $\bar{c}_\mu$ and c^μ are Grassmann odd. Here we have introduced subsets

$$\mathcal{C}_m^n = \{\Sigma \bar{c}_{i_1} \cdots \bar{c}_{i_m} f_j^i(b_\mu, x^\mu) c^{j_1} \cdots c^{j_n}\} \tag{11}$$

with its grading given by the numbers of ghosts and anti-ghosts.

The BRS coboundary operator δ defined in (7) is trivially decomposed into two independent nilpotent derivations as follows:

$$\begin{aligned} &\delta = \delta_D + \delta_K; \\ &\delta_D x^\mu = c^\mu, \quad \delta_K \bar{c}_\mu = i b_\mu. \end{aligned} \tag{12}$$

Then $\{\mathcal{C}_m^n; \delta_D, \delta_K\}$ constitutes a double complex, whose total complex is none other than the BRS complex we are investigating.

Let us look at a complex $\{\mathcal{C}_m^*; \delta_K\}$. This is a Koszul complex graded by the number of anti-ghosts, which is acyclic: the sequence with the bordered complex $\mathcal{A}^* = \{\mathcal{A}^n; \delta_D\}$ augmented

$$\cdots \to \mathcal{C}_2^* \to \mathcal{C}_1^* \to \mathcal{C}_0^* \to \mathcal{A}^* \to 0 \tag{13}$$

is exact, and it gives a Koszul resolution to $\mathcal{A}^*$. Owing to the transformation law (12), we can identify $\mathcal{A}^n$ as the coefficient module

$$\{\Sigma f_j(x^\mu) c^{j_1} \cdots c^{j_n}\} \tag{14}$$

to the complex $\{\mathcal{C}_m^n; \delta_K\}$.

Furthermore, the complex $\mathcal{A}^*$ is recognized as the de Rham complex for the manifold M when we regard the coboundary operator δ_D and FP ghosts as the exterior derivative and one-forms on M, respectively. Therefore the BRS cohomology $H^*(\mathcal{C}; \delta)$ turns out to obey

$$H^*(\mathcal{C}; \delta) = H^*(A^*) = H_d^*(M), \tag{15}$$

where $H_d^*(M)$ denotes the de Rham cohomology for M.

In particular, the zeroth cohomology group $H^0(\mathcal{C}; \delta)$ yields constant functions on M, which constitute trivial observables. This fact enables us to construct a theory (8) which has nontrivial degrees of freedom in a kinematical sense, and which is trivial in its dynamical content. This should be the mathematical essence behind the BRS procedure.

We next come to use the above structure in the context of field theory.

3. Field Transformation

This section deals with change of variables in field theory in terms of BRS cohomology. For simplicity, let us adopt a Lagrangian (in n-dimensional spacetime)

$$\mathcal{L}(\phi) = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi), \tag{16}$$

where ϕ is a real bosonic field. We can make a local point transformation $\phi = f(\phi')$ in the Lagrangian as follows: We first introduce a new field ϕ' and its BRS transformation

$$\delta\phi' = c, \quad \delta\bar{c} = ib; \quad \delta\phi = 0, \tag{17}$$

which corresponds to (7) in the previous section. Then we add, to the Lagrangian (16), a BRS-coboundary term

$$\mathcal{L}_B = -i\delta[\bar{c}\{\phi - f(\phi')\}] = b\{\phi - f(\phi')\} - i\bar{c}\frac{\delta f}{\delta\phi'}c, \tag{18}$$

which corresponds to (8). Note that this causes no change in the dynamical content of the theory when we consider the BRS cohomology just as in the previous section. Integrating out the fields b and ϕ successively, we are left with the desired Lagrangian

$$\mathcal{L}_T = \mathcal{L}(f(\phi')) - i\bar{c}\frac{\delta f}{\delta\phi'}c \tag{19}$$

endowed with the reduced BRS transformation

$$\delta\phi' = c, \quad \delta\bar{c} = -i\frac{\delta\mathcal{L}}{\delta\phi}(f(\phi')). \tag{20}$$

This transformation law is obtained from the BRS transformation (17) by eliminating the fields b and ϕ with the help of their equations of motion $b = -\delta\mathcal{L}/\delta\phi$ and $\phi = f(\phi')$. The theory (19) is clearly equivalent to the original one (16) by construction.

On the other hand, we can start from the naively transformed Lagrangian

$$\mathcal{L}'(\phi') = \mathcal{L}(f(\phi')) = \frac{1}{2}f'^2\partial_\mu\phi'\partial^\mu\phi' - V(f(\phi')); \quad f' = \frac{\delta f}{\delta\phi'}. \tag{21}$$

Then unitarity of the theory necessitates[9] adding a Lee-Yang term[10]

$$\mathcal{L}_{LY} = -i\delta^n(0)\ln\det\frac{\delta f}{\delta\phi'} \tag{22}$$

to the Lagrangian (21). The resultant Lagrangian (21) + (22) coincides with the Lagrangian (19) with the ghosts $\bar{c}$ and c integrated out. This is in accord with the equivalence theorem[3] which states that two Lagrangians which are naively transformed to each other by a point transformation yield the same theory after canonical quantization.

The BRS procedure for field transformations stated above can be applied not only to point transformations but also to spectrum-changing ones[4] including spacetime derivatives of the field ϕ'. For example, let us consider the transformation $\phi = f(\phi') = (\partial_\mu\partial^\mu + m^2)\phi'$. This is not a one-to-one correspondence, and the spectrum of the naively transformed theory $\mathcal{L}(f(\phi'))$ is different from that of the original one (16). Hence they are not equivalent to each other on the contrary to the case of point transformations. Nevertheless, the equivalence between the two theories (16) and (19) still holds (if the latter theory exists at all), because the ghosts compensate the extra modes introduced by the spectrum-changing transformation. Moreover, such generalized transformations[5] as $\phi = f(\phi', \phi'')$ can also be treated in a similar manner. We consider an example of generalized transformations in the next section.

4. Gauge-Fixing

In the preceding section, we concern ourselves with the way of performing field transformations through BRS procedure. In this section on the other hand, we reconsider gauge-fixing from the viewpoint of field transformation.

Gauge-fixing in a gauge theory is a gauge transformation in nature. Let us deal with the Yang-Mills Lagrangian $\mathcal{L}_A$ of a gauge field A_μ, which was also considered in the Introduction. The choice of covariant gauge means that one performs a field transformation from A_μ into A'_μ and g such that

$$A_\mu = gA'_\mu g^{-1} + i(\partial_\mu g)g^{-1}, \quad \partial_\mu A'^\mu = 0, \tag{23}$$

where g is a field taking values in the gauge group G, and that one uses the transformed field A'_μ instead of the original one A_μ.

We can make this transformation along the lines of the BRS procedure considered in the previous section. We first introduce the following BRS transformation:

$$\begin{aligned} &\delta A'_\mu = g^{-1} c_\mu g, \quad \delta \bar{c}^\mu = i b^\mu; \\ &-ig^{-1}\delta g = c, \quad \delta \bar{c} = ib. \end{aligned} \tag{24}$$

Here we have written the ghost corresponding to g as igc so as to make c be $\mathcal{G}$-valued. Then, to impose the relation (23), we add a term

$$\begin{aligned} \mathcal{L}_B &= -i\delta[\bar{c}^\mu(A_\mu - gA'_\mu g^{-1} - i(\partial_\mu g)g^{-1}) + \bar{c}\partial_\mu A'^\mu] \\ &= b^\mu(A_\mu - gA'_\mu g^{-1} - i(\partial_\mu g)g^{-1}) - i\bar{c}^\mu(c_\mu - g(D_\mu c)g^{-1}) - i\delta[\bar{c}\partial_\mu A'^\mu] \end{aligned} \tag{25}$$

to the original Lagrangian $\mathcal{L}_A$, where $D_\mu c = \partial_\mu c + i[A'_\mu, c]$.

Integrating out the fields $b^\mu, A_\mu, \bar{c}^\mu$, and c_μ successively, we are left with

$$\mathcal{L}_T = \mathcal{L}_{A'} - i\delta(\bar{c}\partial_\mu A'^\mu) = \mathcal{L}_{A'} + b\partial_\mu A'^\mu + i\bar{c}\partial_\mu D^\mu c, \tag{26}$$

where $\mathcal{L}_{A'}$ is the same as $\mathcal{L}_A$ except for A_μ replaced by A'_μ. The reduced BRS transformation (obtained in a similar fashion to (20)) is given by

$$\delta A'_\mu = D_\mu c, \quad -ig^{-1}\delta g = c, \quad \delta \bar{c} = ib. \tag{27}$$

The expression (26) is nothing other than the total Lagrangian of Yang-Mills theory in the Landau gauge (coinciding with (6) in the Introduction). We note that the form of the BRS transformation $-ig^{-1}\delta g = c$ clearly indicates that the FP ghost c is a Maurer-Cartan form on the group of gauge transformations.[6] It immediately leads to the following BRS transformation law: $\delta c = -\frac{i}{2}[c, c]$. This is automatically consistent with the nilpotency of δ and the transformation law $\delta A'_\mu = D_\mu c$.

It is remarkable that we have never fixed the original gauge degree of freedom of the field A_μ in the above procedure of 'gauge-fixing', as is clear from $\delta A_\mu = 0$. The Lagrangian (26) still has the field g as one of its variables, though g does not appear in it explicitly because of the original gauge invariance of $\mathcal{L}_A$. We only make a change of field variables A_μ into A'_μ and separate the gauge degree of freedom g.

The gauge-fixing procedure exhibited above can be applied to open and reducible gauge theories in an analogous manner, as is explained in Ref.[2].

5. CONCLUSION

We argued that the BRS procedure provides a useful book-keeping device for introducing new fields into a Lagrangian without changing dynamics. Gauge-fixing was also reconsidered from this point of view, and the simple BRS gauge-fixing method was generalized so as to be applicable to generic gauge theories including open and reducible ones.

We note that introduction of new fields (by means of generalized field transformation) is quite a generic tool to investigate Lagrangian field theories in a kinematical manner. Kinematical reformulation frequently makes the investigation of dynamics tractable. Thus we hope that the BRS framework offers a basic setting for investigation of the dynamical content of various field theories.

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