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A Graph Medial Axis Transform

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1. Introduction

The graph structure is a strong formalism for representing pictures in syntactic pattern recognition. Many models for graph grammars have been proposed as a kind of hyper-dimensional generating systems (see e.g., [1], [2], and [3]). As one of such graph grammars, we introduced node-replacement path-controlled embedding graph grammars (nPCE graph grammars) in [4] for describing uniform structures.

On the other hand, region representation on digital spaces is an important issue in image processing and computer graphics. The quadtree representation of a digital image provides a variable resolution encoding of a region according to the sizes and number of maximal nonoverlapping blocks. Samet [5], [6] provides good tutorial and bibliography of the researches on quadtrees as well as their applications. In [7], we introduced an optimal image compression algorithm using the concept of graph rewriting rules and showed that its time complexity is same as the case of the best quadtree representation.

The development of the quadtree was motivated to a large degree by a desire to save storage. A quadtree medial axis transform (QMAT), which is introduced in [8], is more compact than the quadtree and has a decreased shift sensitivity.

In this paper, we will introduce a medial axis transform based on our graph expression for a given image, rather than quadtree. Since our image compression algorithm uses the concept of graph rewriting rules, the medial axis transform for graph expression can be defined as an extension of graph rewriting.

2. Basic definitions

In this section, we review the definitions of nPCE graph grammars [9]. At first, A *directed node- and edge-labelled graph* (*EDG-graph*) over Σ and Γ is a quintuple H = $<V, E, \Sigma, \Gamma, \varphi>$, where V is the finite, nonempty set of nodes, Σ is the finite, nonempty set of node labels, Γ is the finite nonempty set of edge labels, E is the set of edges of the form $<v, \lambda, w>$, where u, $w \in V$, $\lambda \in \Gamma$, φ: $V \rightarrow \Sigma$ is the node labelling function. An EDG-graph H is called an *OS-graph* if (1) for each $\lambda \in \Gamma$ there exists an inverse edge label $\lambda^{-1} \in \Gamma$, (2) Γ is simply ordered by a relation $\leq$, and (3) for each $v \in V$, if there exists $<v, \lambda, w> \in E$ then there does not exist <v, γ, z> such that $\lambda = \gamma$ or <z, β, v>$\in$E such that $\lambda^{-1}=\beta$. Let us take a set of edge labels for our OS-graphs as {h, -h, v, -v}. Each of its elements represents "EAST", "WEST", "NORTH", and "SOUTH", respectively, and are ordered -h $\leq$ -v $\leq$ h $\leq$ v. The inverse edge labels of h, -h, v, and -v are -h, h, -v, and v, respectively.

Now we review the definitions of nPCE graph grammars. At first we review the definitions of the path groups describing the square grid [10].

Definition 2.1. A *discrete space* is a finitely presented abelian path group Γ = (X/D), where X has 2n generators $s_1, s_2, \ldots, s_n, s_1^{-1}, s_2^{-1}, \ldots, s_n^{-1}$, and D contains all relations other than the commutativity ($s_is_js_i^{-1}s_j^{-1}$= 1) and the inverse iterations ($s_is_i^{-1}$= 1). The *square grid* is a discrete space described by a four generators s_1= (north), s_2= (east), s_1^{-1}= (south), s_2^{-1}= (west), and D=$\varnothing$.

Note that the path groups defined above can also be defined on a graph generated by a graph grammar by regarding the edge labels of the generated graph as the generators.

Definition 2.2. *A node-replacement graph grammar using path controlled embedding with 4 generators abelian path groups*, (*nPCE$_4$ grammar*), is a construction G=<Σ_N, P, ψ, Z, Δ_N, Δ_E>, where Σ_N is a finite nonempty set of node labels, Δ_N is a finite nonempty subset of Σ_N, called terminal node labels, Δ_E={h', v', h, v}, called terminal edge labels, P is a finite set of productions of form (v_a, β), where v_a is a graph consisting of only one node labelled with a$\in\Sigma_N$, β is a connected OS-graph, ψ is a mapping from Δ_E^+ into Δ_E provided that for any $\pi\in\Delta_E^+$, ψ maps π into c, the first label of π, i.e., there exists a $\sigma\in\Delta_E^*$ such that π=cσ, Z is a connected OS-graph over (Σ_N, Δ_E) called *the axiom.*

A direct derivation step of a nPCE$_4$ grammar G, $\Rightarrow_G$, is performed as follows:

Let H be an OS-graph. Let p = (v_a, β) be a production in P. Let β' be isomorphic to β (with h, an isomorphism from β' into β), where β' and H–v_a have no common nodes. Then the result of the application of p to H (by using h) is obtained by replacing v_a with β' and adding edges <u, λ, w> between every nodes u in β' and every w in H-v_a such that if the path from node 1 to node u on β' is σ then w is the node of H defined by v_acσ or its equivalent path under abelian path group with four generators and if ψ(cσ)=h or v then the added edges are <u, ψ(cσ), w>, otherwise <w, $\psi(c\sigma)^{-1}$, u>, where c is an element of Δ_E.

The language of G, denoted as L(G), is defined by L(G) = {H | H is an OS-graph over (Δ_V, Δ_E) and Z $\Rightarrow_G^*$ H}.

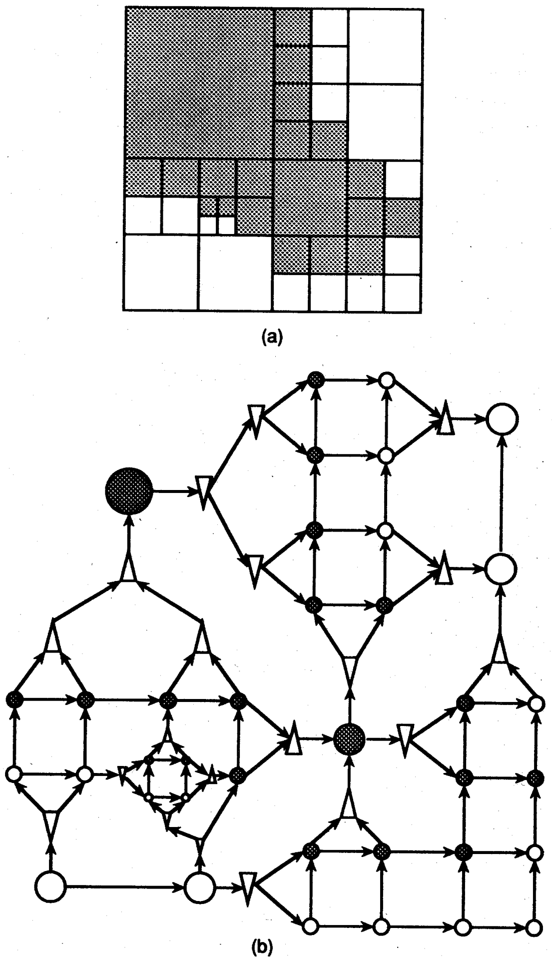


Fig. 1. An input image and its graph representstion.

The graph compression rules (see [7]) which work on the OS-graphs representing two-dimensional rectangular arrays rewrite four nodes having same label and forming a unit square into a node with the same label to get hierarchical graph representation. Such rules are almost equal to the compression law in the quadtrees. These graph rewriting rules, however, need some intermediate nodes to preserve the neighborhood relations of the original graph and to restrict both of the indegree and outdegree within two, respectively. These intermediate nodes are Δ or ∇, which mean the ascending and descending compression levels according to the direction of edges attached to the intermediate nodes and labelled with "h" or "v". If we traverse such edges against their direction, the meaning of the intermediate nodes will be inverted. The image reconstruction process on a given compressed graph is done by using graph expansion rules. Roughly speaking, the graph expansion rules for the OS-graphs are defined as the inverse rules of the graph compression rules.

3. A distance transform on graphs

The quadtree medial axis transform (QMAT) of an image is the quadtree whose BLACK nodes correspond to the BLACK blocks and each BLACK node has an associated distance transform value. For the QMAT, the Maximum Value distance (also known as the chessboard distance): $d_M(p, q) = \max\{|p_x - q_x|, |p_y - q_y|\}$ is used since its maximal blocks are squares. On the other hand, our graph medial axis transform (GMAT) uses the path-length distance which is introduced in [11]. It is based on the chessboard distance, but depends also on paths through the input image. The distance transform T for a given graph expression is defined as a function that gives, for each BLACK node in the given graph expression, the distance from the center of the block represented by the node to the border of the nearest WHITE block. A *path* is defined as a sequence of blocks which are adjacent along their sides to their previous and next blocks. There are no diagonal steps in a path. A path P_{XZ} through a sequence of blocks $b_1, b_2, ..., b_n$, where x is the center of b_1 and z is on the border of b_n. Assume the centers of blocks $b_1, ..., b_n$ are points $x_1, ..., x_n$, the *length* of the path $L(P_{xz})$ is defined as:

$$L(P_{xz}) = \sum_{i=1}^{n-2} d_M(x_{i+1}\, x_i) + \frac{1}{2}(size\ of\ b_{n-1}) = \frac{1}{2}(size\ of\ b_1) + \sum_{i=2}^{n-1}(size\ of\ b_i)\ .$$

The *distance transform*, T(B), is defined as the minimum path length from a point x at the center of the BLACK block B to some point z on the border of the nearest WHITE block. That is,

$$F(B, W) = \min_z L(P_{xz}),\ T(B) = \min_W F(B, W)\ .$$

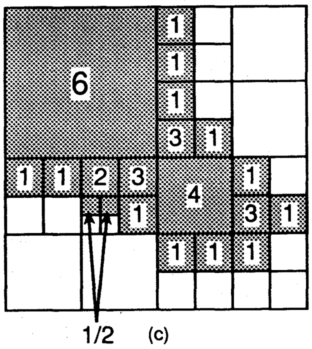


Fig. 2. Distance transform T of the image in Fig.1a.

For example, Fig. 1a is a block decomposition of a given image and its corresponding graph representation is given in Fig. 1b. The distance transform of each BLACK block is given in Fig. 2. Note that the distance transforms of all WHITE blocks are equal to zero.

Almost same arguments as in [11], the following results hold. They are quite useful to define the graph rewriting rules.

Lemma 3.1. The minimum-distance path from a node to the border of the nearest WHITE node does not pass through nodes lager than the starting node.

Lemma 3.2. When calculating the distance of any given node, the node itself cannot participate to the minimum-distance paths of any adjacent nodes.

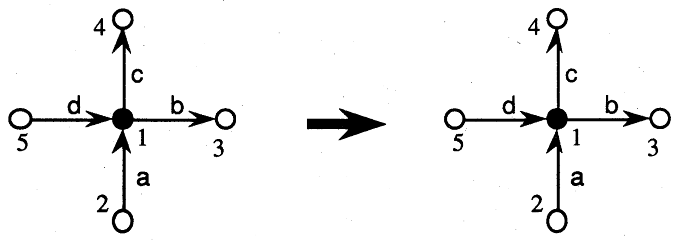
The calculation of distance transform can be done by using a kind of graph rewriting rules. Actually, the rules defined below don't rewrite graphs to other graphs. They manipulate distance transform only for each BLACK block. To define our graph expansion rules, path-controlled embedding rules must be extended to the cases of context-sensitive rewriting rules.

Definition 3.1. A graph production rules is a construction (α, β, ψ), where α and β are connected OS-graphs, ψ is a mapping from $(v\times\Delta_E^+)$ into $(v\times\Delta_E)$ provided that for any $\pi\in\Delta_E^+$, ψ maps π into c, the first label of π, i.e., there exists a $\sigma\in\Delta_E^*$ such that π=cσ

Note here that the extended embedding rules still do not depend on node labels.The definition of graph rewriting rules for image reconstruction are as follows:

Definition 3.2. *Graph rewriting rules for calculating distance transform* T have the following three schemes:

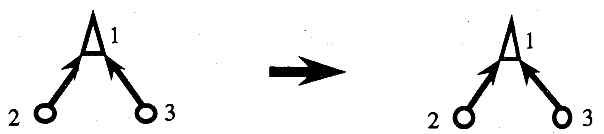
(1)



T(1) = undefined $\quad T(1) = \min_{2 \le i \le 5} T(i) + \frac{1}{2}(\text{size of node } 1)$

where the label of node 1 must be BLACK, and the labels of all edges are from {h, v}. The distance transforms of all nodes other than node 1 must be defined.

(2)



T(1) = undefined $\quad$ T(1) = min (T(2), T(3))

where the labels of nodes 2 and 3 are arbitrary including Δ and ∇. The distance transforms of nodes 2 and 3 must be defined.

(3)

T(1) = undefined $\quad$ T(1) = 0

where the label of node 1 is WHITE.

For each of those rule schemes, ψ is defined such that all nodes in each scheme are connected to previously connected nodes. Since such definitions are quite simple, the definition of ψ is omitted.

Theorem 3.1. For any given graph expression of binary input image, the distance transform T for each BLACK node of the graph expression is determined by the applications of graph rewriting rules in Definition 3.2.

4. Graph medial axis transform

The graph medial axis transform (GMAT) is a graph expression for digitized images. It divides the input image into a set of nondisjoint circle. Unlike from Semet's quadtree medial axis transform (QMAT) [8], GMAT adopts path-length transform T discussed previous section. So GMAT's circles do not become squares.

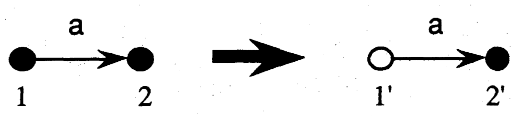
To define the graph medial axis transform, we must define the graph expression skeleton. Let the set of all BLACK blocks in the input image be B. For each BLACK block $b_i \in B$, let $SQ(b_i)$ be the part of the input image whose center is x_i (center of b_i) and whose side length is equal to $2*T(b_i)$.

Definition 4.1. The *graph expression skeleton* is the set, SK, of BLACK blocks satisfying the following properties:
(1) B = UNION(SQ(b_i)),
(2) for any $s_i \in$ SK, there isn't b_k in B which is not equal to s_i and SQ(s_i)$\subseteq$SQ(b_k),
(3) for any $b_i \in$ B, there exists t_i in SK such that SQ(b_i)$\subseteq$SQ(t_i).

Definition 4.2. The *graph medial axis transform* is the graph expression whose BLACK nodes came from graph expression skeleton and each of them has an path-length transform associated to it.

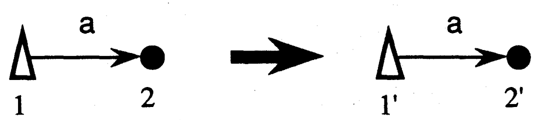
For example, the GMAT for the digitized image of Fig. 1a is represented in Fig. 3. Each BLACK node of it has an value of distance transform given in Fig. 2.

Definition 4.3. *Graph rewriting rules for GMAT* have the following three schemes:
(1)



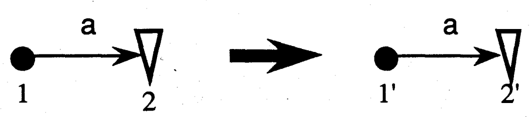
if $\text{size}(1) + \frac{1}{2}\text{size}(2) + T(1) \leq T(2)$

where the label for node 1 is BLACK, for node 2 and 2' are BLACK, Δ, or ∇, for node 1' is WHITE. The edge label a is from {h, v, h', v'}.
(2)



T(1') = T(2) and T(2') = T(2)

where the labels for nodes 2 and 2' are BLACK, or Δ. The edge label a is from {h, v, h_U, h_D, v_L, v_R}.
(3)



T(1') = T(1) and T(2') = T(1)

where the labels for nodes 1 and 1' are BLACK, or Δ. The edge label a is from {h, v, h_U, h_D, v_L, v_R}.

For each of those rule schemes, ψ is defined such that all nodes in each scheme are connected to previously connected nodes. Again they are omitted.

To compress the graph expression around newly introduced white nodes, we need graph compression rules in [7].

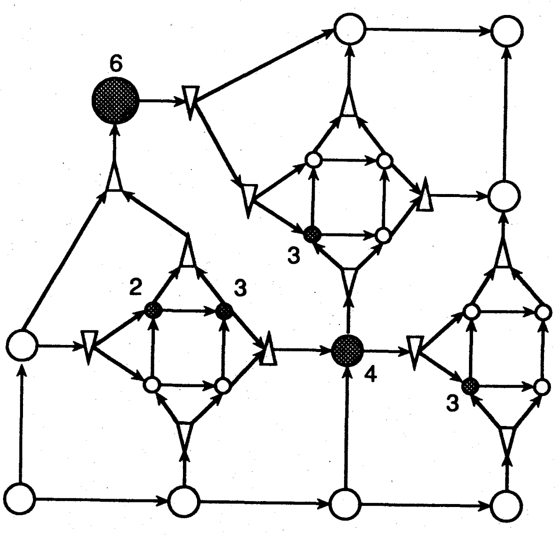


Fig. 3. GMAT representation for the image in Fig. 1a.

Theorem 4.1. For any given graph expression of binary input image, its GMAT is obtained by the applications of graph rewriting rules in Definition 4.3 providing that the distance transform T for each BLACK node of the graph expression is given

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