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Wagner's theorem and combinatorial enumeration of 3-polytopes

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Abstract

Wagner's Theorem states the reducibility of any planar triangulations to the canonical triangulation using a finite number of elementary operations. In this note, we study this theorem dually in terms of 3-polytopes, and we obtain a modified proof of the theorem which can be used for listing all combinatorial types of 3-polytopes.

1 Introduction

Wagner's theorem [6] states that any planar triangulations can be reduced to the canonical triangulation using a finite number of elementary flip operations. By duality, and using the well-known Steinitz's theorem [4, 5], Wagner's theorem can be stated in terms of simple polytopes.

Theorem 1.1 **[6]** *Using a finite number of elementary operations, every* 3*-dimensional simple polytope can be reduced to a shell i.e. a simple* 3*-polytope* S *such that all vertices of* S *belong to exactly two adjacent facets.*

Our objective is to use this theorem for listing all different combinatorial types of a simple 3-polytope with a given number n of vertices. Although the proof of Wagner's theorem can be considered as a finite algorithm, we present in this note some modifications which lead to an algorithm with the following advantages:

a) the number of steps is reduced,
b) the algorithm is history-free and a reverse search type method can be efficiently implemented,
c) the high cost of eliminating isomorphic duplicates is reduced.

In section 2 and 3 a modified proof of Wagner's theorem and our algorithm for listing all combinatorial types of 3-polytopes are presented.

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2 Wagner's reduction of simple polytope to a shell

The elementary operation used in the reduction of a simple polytope to a shell is called *edge twist* operation and is represented in Figure 2.1. It is the dual operation of the well-known flip operation for triangulation.

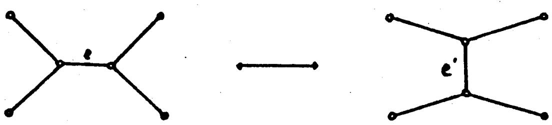

Figure 2.1: edge twist

With G denoting the edge-graph of a simple 3-polytope P and G' denoting the graph obtained after twisting one edge e of G, one can easily check that G' is also polytopal if and only if the two non-adjacent facets incident to e do not intersect. In this case e is said to be *twistable*.

Then starting with any simple 3-polytope with n vertices, the reduction to a shell is done by first choosing two adjacent facets and a direction (right or left) and then by somehow zipping in this direction the polytope up the shell with n vertices using the fact that two incident edges can not be both non-twistable. Since there are $O(n)$ paths joining the two prescribed facets and since any of those path contains at most $O(n)$ edges, this proof gives a $O(n^2)$ steps algorithm to reduce any simple 3-polytope to a shell.

We modify the proof in the following way: first for any pair (F, F') of adjacent facets, i.e. for any edge e, we define the vector $(k, -l)$ where k denotes the number of consecutive edges joining the two adjacent facets and l denotes the number of edges on the shortest path next to the k edges joining the two facets as illustrated in Figure 2.2. Then the algorithm is given by:

while $k < \frac{n}{2} - 1$ do,
 determine the edge e such that $(k, -l)$ is maximum,
 then twist one edge on the path containing l edges.

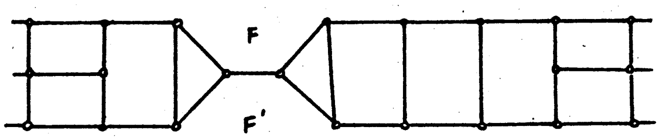

Figure 2.2: case (k,-l)=(3,-2)

At each step $(k, -l)$ is increasing: either l decreases to 1 which means that k increases by 1 or l simply decreases by 1. Therefore the algorithm stops after a finite number of steps with the edge-graph of the shell with n vertices. Obviously our modifications reduce the number of steps although the complexity is still $O(n^2)$. This algorithm is history-free since no facet is prescribed and there is no need to store information about the previous stages reached during the process of the algorithm. This last property leads to an easy implementation of a reverse search type algorithm for the listing of all combinatorial types of a simple polytope presented in the next section.

3 Combinatorial enumeration of 3-polytopes

In this section we introduce a reverse search algorithm for enumerating all combinatorial types of 3-polytopes. The main idea of reverse search was given by Avis and Fukuda in [1]. Here, for our purpose we use a slightly more general form of the algorithm.

Let $G = (V, E)$ be a graph with vertex set V and edge set E. We consider the set V to be the set of objects to be enumerated and thus G is not explicitly given. In our application, we set $V = V(n)$ to be the set of all combinatorial types of 3-polytopes with n vertices, and we call two vertices (polytopes) *adjacent* if and only if one can be obtained from the other by one twist operation. This determines an undirected graph $G = G(n)$.

By Wagner's theorem, we know that this graph $G(n)$ is connected. A primitive algorithm to enumerate all vertices of $G(n)$ is to apply depth-first-search to trace this graph starting from any polytope. For this, we must use an appropriate graph isomorphism checking algorithm to decide whether a newly found polytope is combinatorially equivalent to any of the polytopes generated earlier. This seems to be unavoidable because of the combinatorial nature of the problem. However, this algorithm has two other serious drawbacks. The first one is that tracing the whole graph $G(n)$ is costly: there are $O(n)$ neighbours for each polytope and the algorithm generates $O(n)$ duplicates for each polytope. Secondly, the isomorphism checking is absolutely necessary for the depth-first-search to be applied. One cannot divide the algorithm into two apparently independent parts: the listing of all polytopes with repetitions and the duplicates removal.

The new algorithm we propose resolves these two drawbacks. In fact, it traces only a small spanning subgraph of $G(n)$, and it can be performed in two phases; the first phase to list all polytopes with possible repetitions and the second to eliminate duplications. Furthermore, the first phase requires very little memory: $O(n)$. The second part requires, in principle, a large amount of memory, because we must store all polytopes generated by the first phase. Nevertheless, one can partition the polytopes into obviously different subclasses first, say each classes having the same dual degree sequences, and then apply the duplicates removal to each of these classes. (The dual degree sequence of a 3-polytope is the degree sequence of the graph of its dual polytope.)

Let us describe our algorithm. Recall that our modified proof of Wagner's theorem uses an algorithm to select an edge to be twisted. Let us call this algorithm **MoveToShell.** Although the selection of a twisting edge might not be unique in **MoveToShell**, the algorithm is finite because the associated vector $(k, -l)$ increases monotonically. Furthermore the algorithm is purely combinatorial in the sense that the selection of a twisting edge depends only on the combinatorial type of the polytope but nothing else (e.g. depends neither on the numbering of vertices nor the previous selections of twisting). This fact enables us to define the function f by:

$$f : V(n) \setminus \{S(n)\} \rightarrow 2^{V(n)}$$
$$f(P) = \{P' : P' \text{ is obtained with one twist from } P \text{ by the algorithm } \textbf{MoveToShell}\}.$$

where $S(n)$ is the shell with n vertices.

Then we define the *trace* of the function f as the directed subgraph $T(n) = (V(n), E(f))$ of $G(n)$ where

$$E(f) = \{(P, P') : P \in V(n) \setminus \{S(n)\} \text{ and } P' \in f(v)\}.$$

The trace $T(n)$ is simply the digraph induced by those edges of $G(n)$ used by the algorithm **MoveToShell.** The trace $T(n)$ has two nice properties,

(1) it is a spanning connected subgraph of $G(n)$,

(2) it is acyclic and the shell $S(n)$ is the unique sink.

The first property implies that we can apply depth-first-search to $T(n)$ to enumerate all vertices of $G(n)$. Since $T(n)$ is a subgraph of G (and perhaps much smaller than $G(n)$), it is faster and generates fewer duplicates of polytopes than the primitive algorithm to trace $G(n)$ itself. The second property is more important. By this property, it is possible to visit all vertices by depth-first-search without storing any polytopes generated during computation. Only one point requires attention: when tracing in the graph $T(n)$, we start from the shell $S(n)$ and go against the orientation of the graph. We will list all 3-polytopes with each polytope P repeated as many times as its out-degree in $T(n)$. It should be emphasized that we could use the reverse search technique because we modified the original proof of Wagner's theorem which is not combinatorial in the sense above and also history-dependent.

We have mentioned some practical advantages of our algorithm over the primitive method. Unfortunately we are not be able to claim that our method is theoretically better in the sense of time complexity. We hope that our method can be shown to be better in a theoretical sense also. For instance, we believe that the average out-degree of $T(n)$ is small and much less than $O(n)$. The determination of the average degree would be an interesting open problem for its own sake. Another problem is to make a refinement of the algorithm **MoveToShell** so that the out-degree of each polytope P in $T(n)$ is as small as possible. This amounts to restrict the choice of allowable twist operations in the algorithm as much as possible. It is quite likely that such flexibility can be reduced to the number of automorphisms of a polytope.

References

[1] Avis D. and Fukuda K., "A pivoting algorithm for convex hulls and vertex enumeration of arrangements and polyhedra," Discrete Comput. Geometry 8 (1992), 295-313.

[2] Brøndsted A., *An Introduction to Convex Polytopes*, Graduate Texts in Mathematics, 90, New York-Heidelberg-Berlin, Springer-Verlag, 1983.

[3] Fukuda K. and Rosta V., "Edge and face contraction in 3-connected planar graphs," in preparation.

[4] Grünbaum B., *Convex Polytopes*, London-New York-Sydney Wiley, 1967.

[5] Steinitz E. and Rademacher H., "Vorlesungen über die Theorie der Polyeder," Berlin, Springer-Verlag, 1934. Reprint: New York-Heidelberg-Berlin, Springer-Verlag, 1976.

[6] Wagner K., "Bemerkungen zum Vierfarbenproblem," J. der Deut. Math. Ver. 46, Abt. 1 (1936), 26-32.