

Computational construction of W-graphs  
 associated with Hecke algebras

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1. 序論

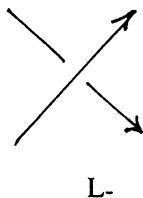
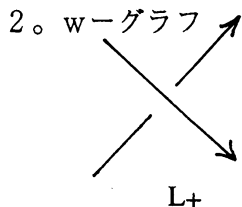
筆者は、1987年以来結び目理論研究支援用ソフトウェア、KnotTheorybyComputer、を研究開発し、国内外で研究発表し、2年位前から国内外にKnotTheorybyComputerを配付してきた。講演をする毎にこのソフトウェアは、機能を拡張し成長してきた。しかしながら、現在のバージョンでも、次に述べるような不満がある。

- (1) 2次元グラフィクスによる、結び目の正則射影（結び目理論を研究するにはこの方法を用いるの現在の処が最適である）を、3次元グラフィクスによるより立体的な表示機能を与えること。
- (2) より交点数の多い結び目（例えば、200交点位まで、または16ブレイド位まで）の多項式不変量を高速に計算すること。
- (3) マウスによる結び目のイソトピー変形を完璧に行なうこと。

(1)の機能に関して、Geometry Center (Minnesota University)が開発したgeomviewは極めて優れたソフトウェアである、しかしgeomviewはvisualizationのみに特化したソフトウェアで結び目のイソトピー変形とか、多項式不変量を計算する機能を全く持たない。

今年3月に、筆者は同センタで開催されたKnot Workshopに参加し、Geometry Centerに1箇月滞在し、同センタが開発したvisualizationに関するソフトウェア技術を修得することに努めた。

Knot Workshopでは、筆者は、"Computer aided Knot theory with OSF/Motif"という題目で、ワークステーション上でのKnotTheorybyComputerについてのデモ講演をすると同時に、(2)に関して最近開発したコンピュータプログラムの紹介をした。ここでは、その方法について解説し、巻末に7、8ブレイドに対応したW-グラフを掲載する。



- (1)  $P_T(t,x) = 1$ 、Tは自明な結び目
- (2)  $t^{-1}P_{L+}(t,x) - tP_{L-}(t,x) = xP_{L_0}(t,x)$

図-1 Conway 関係式による 2 変数Jones polynomial

現在知られている結び目の多項式不変量は、全て結び目の1つの交点における3状態（プラス交点 $L_+$ 、マイナス交点 $L_-$ 、平滑化 $L_0$ （または、逆平滑化を含めた4状態））から派生する二分木（4状態の場合には、3分木）を用いて計算可能である。すなわち、Conway 関係式（図-1）から計算出来る。しかし、この関係式を用いた場合、計算回数は結び目の交点数の冪乗のオーダーで増加する。従って、極めて多くの交点を持つ結び目の多項式不変量を求めるにはもっと別の巧妙な方法が必要である。

1つの方法として、結び目のブレイド表示から行列への表現が有効である。

$n$ 次ブレイド群  $B_n$  とは、次のような群表示を持つ群である。

$$B_n = \langle \sigma_1, \sigma_2, \dots, \sigma_{n-1} \mid \begin{array}{l} \sigma_i \sigma_j = \sigma_j \sigma_i \quad (|i-j| > 1), \\ \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \quad (i=1, 2, \dots, n-2) \end{array} \rangle$$

$$X_{L(q, \lambda)} = (-1 - \lambda q) / (\sqrt{\lambda} (1-q))^{n-1} \sqrt{\lambda} e^{\text{trace}(\pi(b))}$$

$e$  は  $b$  の符号和で、 $\pi$  は  $B_n$  から Hecke algebra  $H(q, n)$  への表現である。

このとき、任意の結び目  $L$  は、ある  $n$ 次ブレイド  $B_n$  の元  $b$  の閉包  $\hat{b}$  としてかけ、 $L$  の2変数 Jones 多項式  $P(L; t, x)$  は、 $B_n$  の行列表現のトレースの1次結合  $X_{L(q, \lambda)}$  としてかける。このとき、 $t = \sqrt{\lambda} \sqrt{q}$ 、 $x = (\sqrt{q} - 1/\sqrt{q})$  である[4]。

問題は、表現行列を具体的にどう求めるかである。

ここでは、Lascoux-Schutzenberger により考案された Young 図形から  $W$ -グラフを構成し、 $W$ -グラフから表現行列を作成する方法を採用する。この方法は、Lascoux から京都大学の行者氏への私信で伝えられたもので[1, 2]、厳密な数学的証明はまだ与えられていないようである。しかし、作成した表現行列の妥当性は、 $B_n$  の関係式を満たすかどうか調べることで確認できる。

Lascoux-Schutzenberger によれば、自然数  $n$  に対して  $H(q, n)$ （または、 $B_n$ ）の行列表現は次のように求められる： $n=6$  の場合について説明する。

$\Lambda(n)$  を  $n$  の分割数、すなわち  $n$  をいくつかの非負の整数の和で表す表し方の数とする。この場合、 $\Lambda(6)=11$  であり、具体的にかくと  $\{\{6\}, \{5, 1\}, \{4, 2\}, \{4, 1, 1\}, \{3, 3\}, \{3, 2, 1\}, \{3, 1, 1, 1\}, \{2, 2, 2\}, \{2, 2, 1, 1\}, \{2, 1, 1, 1, 1\}, \{1, 1, 1, 1, 1, 1\}\}$  となる。

このとき、各分割に対して台を用意する、例えば、 $\{4, 1, 1\}$  に対しては  
を、 $\{3, 3\}$  に対して を用意する。

次に、例えば  $\{3, 2, 1\}$ （この場合、深さ3）に対して用意した台に、数字を左から右に、上から下に数字が増加するように箱に割り振る。例えば、次のようになる。

123	124	134	125	135	123	124	134	125	135	145	126	136
45	35	25	34	24	46	36	26	36	26	26	34	24
6	6	6	6	6	5	5	5	4	4	3	5	5

1 2 6 1 3 6 1 4 6  
 3 5 2 5 2 5  
 4 4 3

これらの16個の台の割当を Young 図形の標準盤と言う。次に、各標準盤に割り当てられた数字を、  
 の順序で数字を並べると、

$x_1=325146, x_2=425136, x_3=435126, x_4=524136, x_5=534126, x_6=326145, x_7=426135,$   
 $x_8=436125, x_9=526134, x_{10}=536124, x_{11}=546123, x_{12}=624135, x_{13}=634125,$   
 $x_{14}=625134, x_{15}=635124, x_{16}=645123$   
 という word が得られる。

これらの word の1つ1つをW-グラフGの頂点 $x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}, x_{12}, x_{13}, x_{14}, x_{15}, x_{16}$ として考える。これらの内、word  $w, w'$ につき、 $w$ の内の2文字 $i, j (i < j)$ を入れ替えると、 $w'$ になり、 $i$ と $j$ の間には、 $i < m < j$ を満たす $m$ が存在しなければ、 $w$ と $w'$ との間には辺があるとす。例えば、325146と425136と、325146と326145との間には辺がある。また、425136と625134とでは、4と6とが入れ替わっているが、5が間に挟まっているのでこの段階では辺で結ばない。

次に、3つの続いた数字、例えば2,3,4に着目する。この時、word  $w$ に対して $wP$ を次のように定義する。

$w = \dots 2 \dots 3 \dots 4 \dots \rightarrow wP$  は定義されない  
 $w = \dots 2 \dots 4 \dots 3 \dots \rightarrow wP = \dots 3 \dots 4 \dots 2 \dots$   
 $w = \dots 3 \dots 2 \dots 4 \dots \rightarrow wP = \dots 4 \dots 2 \dots 3 \dots$   
 $w = \dots 3 \dots 4 \dots 2 \dots \rightarrow wP = \dots 2 \dots 4 \dots 3 \dots$   
 $w = \dots 4 \dots 2 \dots 3 \dots \rightarrow wP = \dots 3 \dots 2 \dots 4 \dots$   
 $w = \dots 4 \dots 3 \dots 2 \dots \rightarrow wP$  は定義されない

つまり、真中に、234のうちの真中の数3がくると、 $wP$ は定義されない。そうでなければ、両端を入れ替える。ここで、 $w_1$ と $w_2$ が既に辺で結ばれていたら、 $w_1P$ と $w_2P$ との間に辺を設ける。例えば、 $w_1 = 325146, w_2 = 534126$ とすれば、これらは辺で結ばれているので、 $w_1P = 425136$ と $w_2P = 524136$ との間に辺を設ける。

2,3,4に限らず、1,2,3にも3,4,5にも4,5,6にも同じことを繰り返して可能な限り辺を付け加える。

更に、このようにして得られた $w_1P$ と $w_2P$ に関しても $(w_1P)P$ と $(w_2P)P$ を作ることにより辺を付け加える。この操作に関しても可能な限り行う。

上のW-グラフの隣接行列Aを書くと、次のようになる：

0100210020000000  
 1011001000000000  
 0100100100000000  
 0100100010210000  
 2011000001001000  
 1000001000002000  
 0100010110010000  
 0010001001001002  
 2001001001000100  
 0000100110100010  
 0002000001000001  
 0001001000001102  
 0000120100010010  
 0000000010010010  
 0000000001001101  
 0000000200120010

左の隣接行列で1は最初の辺の条件から生成される辺を、1より大きい数はそれ以降の生成条件から出来る辺を表す

### 3. 結び目不変量の行列表現

#### 3. 1 行列表現

次に、w-グラフから $B_n$ の行列表現を得るための方法について考える。

まず、w-グラフ $G$ の各頂点 $x$ に対して、 $I(x)$ を $i \in I(=\{1,2,3,4,5\})$ で $i+1$ が $i$ を含む行より下にあるものの集合として定義する。例えば、 $I(x_1) = \{3, 5\}$ である。また、 $IC(x)$ を $I$ における $I(x)$ の補集合とする。すなわち、 $IC(x_1) = \{1, 2, 4\}$ である。この時、 $\sigma_j$ に対応する表現行列 $T_j$ は、16次の正方行列で、対角成分 $(i, i)$ は、 $j \in I(x_i)$ の時、-1で、 $j \notin I(x_i)$ の時 $t$ である。それ以外の成分 $(l, m)$  ( $l \neq m$ )については、 $j \in I(x_l)$ かつ $j \notin I(x_m)$ かつ $A(l, m)$ が0でない時にのみ $\sqrt{q}$ で、それ以外は0である。

#### 3. 2 $B_7$ 、 $B_8$ のw-グラフ

w-グラフの頂点 $x$ は、 $I(x)$ で完全に識別できるので、 $x$ をその $I(x)$ でラベル付け、 $x$ を $I(x)$ で表わす。例えば、 $x_1=325146$ ,  $x_2=425136$ , ...,  $x_{15}=635124$ ,  $x_{16}=645123$ を、 $x_1=[3\ 5]$ ,  $x_2=[2\ 4\ 5]$ , ...,  $x_{15}=[1\ 3]$ ,  $x_{16}=[1\ 2\ 4]$ の様に表わす。

Matrix size = 14, reduced Matrix size = 1

0 1 1 0 2 1 0 4 2 1 3 1 2 3 1 2 5 1 3 4 1 3 6 1 3 8 2 4 7 1 5 6 1 5 9 1  
 6 7 1 6 10 1 7 8 1 7 11 1 7 13 2 8 12 1 9 10 1 10 11 1 11 12 1 12 13 1

上のデータは、w-グラフの隣接行列を表わす。3つ組 $(a, b, c)$ 、例えば0 1 1は $x_1$ と $x_2$ とが辺で結ばれていることを示す。3番目の1は最初の隣接関係で生成された辺であることを示す。0 4 2の場合には、 $x_1$ と $x_5$ が2番目の隣接関係で生成された辺であることを示す。

B7のw-グラフ

B8のw-グラフ

図-2、

図-3

B6のw-グラフは、[6]に言及されているように、Gyoja-Naruseにより既に作られている（ただし、Young 図形の並べ方が[4]によるものと相対になっているので  $I(x)$  が  $IC(x)$  となっている）。ここでは、B7、B8のw-グラフのみを掲載する（図-2、3）。

B9, B10, B11, B12のw-グラフのデータは、次のようにあまりに膨大なので割愛する。

B9            105452 バイト、        B10            594020 バイト、

B11          2854938 バイト、        B12          11318332 バイト

巻末に掲載したプログラムによれば、B16までのw-グラフを作成することが出来る[7]。それ以上のものについては記憶領域を使用する上での特別な配慮が必要とされる。

実は、Lascoux-Schutzenbergerの方法を用いて2年前に大学院の学生が、紙とエンピツで、深さ2までのYoung 図形（これは、1変数Jones多項式）に対応するw-グラフを12ブレイドまで、すべての深さのYoung 図形（これは、2変数Jones多項式）に対応するw-グラフを8ブレイドまで計算した[11]。彼女の計算は、1変数Jones多項式の11ブレイドまでは正しいが、残念ながら12ブレイドの計算に誤りがあった。今年の初めに彼女の計算結果を、KnotTheorybyComputerに移植する際に、始めてその誤りに気づいたときには、筆者も含めて誰もその計算を復元することは不可能であった。

そこで、Lascoux-Schutzenbergerの方法を直接プログラムすることにした。最初は比較的簡単な1変数Jones多項式についてのプログラムの制作から始め、それを2週間位で完成出来た。次に2変数Jones多項式についても2箇月位で完成出来た。実際に使用可能な形のプログラムは、アメリカでの講演が済んで日本に帰国してからである。

また、12ブレイドあたりの2変数Jones多項式（実際には、4ブレイドの3-parallel link に関係する1変数Jones多項式）を求めるには、1変数多項式をエントリに持つ約8000次の正方行列の行列計算が必要となるが、これは大型疎行列を1次元配列を効率よく用いて計算するプログラムを開発することにより可能となった。

ところで、mutantな2つの結び目を一般的に判別する多項式不変量は、現在でもまだ知られていない。筆者は大阪大学の村上順氏の協力を得て、12ブレイドのw-グラフの部分グラフから、表現行列の部分表現行列を作り、これからmutantな2つの結び目を判別する多項式不変量を作った。この多項式不変量は、寺坂一樹下結び目とConway結び目を判別出来る[8]。

巻末に制作したプログラムのリストを掲載したので興味ある読者は利用してください。

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## W-graph for B7

partition: 7 0 0 0 0 0 0

x1 = [1 2 3 4 5 6]

partition: 6 1 0 0 0 0 0

x1 = [2 3 4 5 6] x2 = [1 3 4 5 6] x3 = [1 2 4 5 6] x4 = [1 2 3 5 6] x5 = [1 2 3 4 6] x6 = [1 2 3 4 5]

partition: 5 2 0 0 0 0 0

x1 = [2 4 5 6] x2 = [1 3 4 5 6] x3 = [2 3 5 6] x4 = [1 3 5 6] x5 = [1 2 4 5 6]

x6 = [2 3 4 6] x7 = [1 3 4 6] x8 = [1 2 4 6] x9 = [1 2 3 5 6] x10 = [2 3 4 5]

x11 = [1 3 4 5] x12 = [1 2 4 5] x13 = [1 2 3 5] x14 = [1 2 3 4 6]

partition: 5 1 1 0 0 0 0

x1 = [3 4 5 6] x2 = [2 4 5 6] x3 = [1 4 5 6] x4 = [2 3 5 6] x5 = [1 3 5 6]

x6 = [1 2 5 6] x7 = [2 3 4 6] x8 = [1 3 4 6] x9 = [1 2 4 6] x10 = [1 2 3 6]

x11 = [2 3 4 5] x12 = [1 3 4 5] x13 = [1 2 4 5] x14 = [1 2 3 5] x15 = [1 2 3 4]

partition: 4 3 0 0 0 0 0

x1 = [2 4 6] x2 = [1 3 4 6] x3 = [2 3 5 6] x4 = [1 3 5 6] x5 = [1 2 4 5 6]

x6 = [2 4 5] x7 = [1 3 4 5] x8 = [2 3 5] x9 = [1 3 5] x10 = [1 2 4 5]

x11 = [2 3 4 6] x12 = [1 3 4 6] x13 = [1 2 4 6] x14 = [1 2 3 5 6]

partition: 4 2 1 0 0 0 0

x1 = [3 5 6] x2 = [2 4 5 6] x3 = [1 4 5 6] x4 = [2 5 6] x5 = [1 3 5 6]

x6 = [3 4 6] x7 = [2 4 6] x8 = [1 4 6] x9 = [2 3 5 6] x10 = [1 3 5 6]

x11 = [1 2 5 6] x12 = [2 4 6] x13 = [1 3 4 6] x14 = [2 3 6] x15 = [1 3 6]

x16 = [1 2 4 6] x17 = [3 4 5] x18 = [2 4 5] x19 = [1 4 5] x20 = [2 3 5]

x21 = [1 3 5] x22 = [1 2 5] x23 = [2 3 4 6] x24 = [1 3 4 6] x25 = [1 2 4 6]

x26 = [1 2 3 6] x27 = [2 4 5] x28 = [1 3 4 5] x29 = [2 3 5] x30 = [1 3 5]

x31 = [1 2 4 5] x32 = [2 3 4] x33 = [1 3 4] x34 = [1 2 4] x35 = [1 2 3 5]

partition: 4 1 1 1 0 0 0

x1 = [4 5 6] x2 = [3 5 6] x3 = [2 5 6] x4 = [1 5 6] x5 = [3 4 6]

x6 = [2 4 6] x7 = [1 4 6] x8 = [2 3 6] x9 = [1 3 6] x10 = [1 2 6]

x11 = [3 4 5] x12 = [2 4 5] x13 = [1 4 5] x14 = [2 3 5] x15 = [1 3 5]

x16 = [1 2 5] x17 = [2 3 4] x18 = [1 3 4] x19 = [1 2 4] x20 = [1 2 3]

partition: 3 3 1 0 0 0 0

x1 = [3 5] x2 = [2 4 5] x3 = [1 4 5] x4 = [2 5] x5 = [1 3 5]

x6 = [3 4 6] x7 = [2 4 6] x8 = [1 4 6] x9 = [2 3 5 6] x10 = [1 3 5 6]

x11 = [1 2 5 6] x12 = [2 4 6] x13 = [1 3 4 6] x14 = [2 3 6] x15 = [1 3 6]

x16 = [1 2 4 6] x17 = [2 4] x18 = [1 3 4] x19 = [2 3 5] x20 = [1 3 5] x21 = [1 2 4 5]

partition: 3 2 2 0 0 0 0

x1 = [3 6] x2 = [2 4 6] x3 = [1 4 6] x4 = [2 5 6] x5 = [1 3 5 6]

x6 = [3 5] x7 = [2 4 5] x8 = [1 4 5] x9 = [2 5] x10 = [1 3 5]

x11 = [3 4] x12 = [2 4] x13 = [1 4] x14 = [2 3 5] x15 = [1 3 5]

x16 = [1 2 5] x17 = [2 4 6] x18 = [1 3 4 6] x19 = [2 3 6] x20 = [1 3 6] x21 = [1 2 4 6]

partition: 3 2 1 1 0 0 0

x1 = [4 6] x2 = [3 5 6] x3 = [2 5 6] x4 = [1 5 6] x5 = [3 6]

x6 = [2 4 6] x7 = [1 4 6] x8 = [2 6] x9 = [1 3 6] x10 = [4 5]

x11 = [3 5] x12 = [2 5] x13 = [1 5] x14 = [3 4 6] x15 = [2 4 6]

x16 = [1 4 6] x17 = [2 3 6] x18 = [1 3 6] x19 = [1 2 6] x20 = [3 5]

x21 = [2 4 5] x22 = [1 4 5] x23 = [2 5] x24 = [1 3 5] x25 = [3 4]

$x_{26} = [2\ 4]$   $x_{27} = [1\ 4]$   $x_{28} = [2\ 3\ 5]$   $x_{29} = [1\ 3\ 5]$   $x_{30} = [1\ 2\ 5]$

$x_{31} = [2\ 4]$   $x_{32} = [1\ 3\ 4]$   $x_{33} = [2\ 3]$   $x_{34} = [1\ 3]$   $x_{35} = [1\ 2\ 4]$

partition: 3 1 1 1 1 0 0 0

$x_1 = [5\ 6]$   $x_2 = [4\ 6]$   $x_3 = [3\ 6]$   $x_4 = [2\ 6]$   $x_5 = [1\ 6]$

$x_6 = [4\ 5]$   $x_7 = [3\ 5]$   $x_8 = [2\ 5]$   $x_9 = [1\ 5]$   $x_{10} = [3\ 4]$

$x_{11} = [2\ 4]$   $x_{12} = [1\ 4]$   $x_{13} = [2\ 3]$   $x_{14} = [1\ 3]$   $x_{15} = [1\ 2]$

partition: 2 2 2 1 0 0 0 0

$x_1 = [4]$   $x_2 = [3\ 5]$   $x_3 = [2\ 5]$   $x_4 = [1\ 5]$   $x_5 = [3\ 6]$

$x_6 = [2\ 4\ 6]$   $x_7 = [1\ 4\ 6]$   $x_8 = [2\ 6]$   $x_9 = [1\ 3\ 6]$   $x_{10} = [3]$

$x_{11} = [2\ 4]$   $x_{12} = [1\ 4]$   $x_{13} = [2\ 5]$   $x_{14} = [1\ 3\ 5]$

partition: 2 2 1 1 1 0 0 0

$x_1 = [5]$   $x_2 = [4\ 6]$   $x_3 = [3\ 6]$   $x_4 = [2\ 6]$   $x_5 = [1\ 6]$

$x_6 = [4]$   $x_7 = [3\ 5]$   $x_8 = [2\ 5]$   $x_9 = [1\ 5]$   $x_{10} = [3]$

$x_{11} = [2\ 4]$   $x_{12} = [1\ 4]$   $x_{13} = [2]$   $x_{14} = [1\ 3]$

partition: 2 1 1 1 1 1 0 0

$x_1 = [6]$   $x_2 = [5]$   $x_3 = [4]$   $x_4 = [3]$   $x_5 = [2]$   $x_6 = [1]$

partition: 1 1 1 1 1 1 1 0

Matrix size = 6, reduced Matrix size = 0

0 1 1 1 2 1 2 3 1 3 4 1 4 5 1

Matrix size = 14, reduced Matrix size = 1

0 1 1 0 2 1 0 4 2 1 3 1 2 3 1 2 5 1 3 4 1 3 6 1 3 8 2 4 7 1 5 6 1 5 9 1

6 7 1 6 10 1 7 8 1 7 11 1 7 13 2 8 12 1 9 10 1 10 11 1 11 12 1 12 13 1

Matrix size = 15, reduced Matrix size = 1

0 1 1 1 2 1 1 3 1 2 4 1 3 4 1 3 6 1 4 5 1 4 7 1 5 8 1 6 7 1 6 10 1

7 8 1 7 11 1 8 9 1 8 12 1 9 13 1 10 11 1 11 12 1 12 13 1 13 14 1

Matrix size = 14, reduced Matrix size = 2

0 1 1 0 2 1 0 4 2 0 5 1 0 10 2 0 12 3 1 3 1 1 6 1 1 11 2 2 3 1 2 7 1 2 13 4

3 4 1 3 8 1 4 9 1 5 6 1 5 7 1 5 9 2 6 8 1 7 8 1 7 10 1

8 9 1 8 11 1 8 13 2 9 12 1 10 11 1 11 12 1 12 13 1

Matrix size = 35, reduced Matrix size = 4

0 1 1 0 4 2 0 5 1 0 8 2 1 2 1 1 3 1 1 6 1 2 4 1 2 7 1 3 4 1 3 8 1 3 10 2 3 11 1

4 9 1 4 12 1 5 6 1 5 12 2 5 16 1 6 7 1 6 8 1 6 11 1 6 17 1 6 22 2

7 9 1 7 12 1 7 15 2 7 18 1 7 23 2 8 9 1 8 13 1 8 19 1 9 10 1 9 14 1 9 20 1

10 15 1 10 21 1 11 12 1 11 13 1 11 15 2 11 26 1 12 14 1 12 27 1 13 14 1 13 22 1 13 28 1

14 15 1 14 23 1 14 25 2 14 29 1 15 24 1 15 30 1 16 17 1 16 27 2 17 18 1 17 19 1 17 26 1

18 20 1 18 27 1 18 30 2 19 20 1 19 22 1 19 28 1 20 21 1 20 23 1 20 29 1

21 24 1 21 30 1 21 34 2 22 23 1 22 31 1 23 24 1 23 32 1 24 25 1 24 33 1 25 34 1

26 27 1 26 28 1 26 30 2 27 29 1 28 29 1 28 31 1 29 30 1 29 32 1 29 34 2

30 33 1 31 32 1 32 33 1 33 34 1

Matrix size = 20, reduced Matrix size = 2

0 1 1 1 2 1 1 4 1 2 3 1 2 5 1 3 6 1 4 5 1 4 10 1 5 6 1 5 7 1 5 11 1

6 8 1 6 12 1 7 8 1 7 13 1 8 9 1 8 14 1 9 15 1 10 11 1 11 12 1 11 13 1

12 14 1 13 14 1 13 16 1 14 15 1 14 17 1 15 18 1 16 17 1 17 18 1 18 19 1

Matrix size = 21, reduced Matrix size = 3

0 1 1 0 4 2 0 5 1 0 8 2 1 2 1 1 3 1 1 6 1 2 4 1 2 7 1 2 20 3

3 4 1 3 8 1 3 10 2 3 11 1 3 18 2 3 20 3 4 9 1 4 12 1 4 19 2 5 6 1 5 12 2

6 7 1 6 8 1 6 11 1 7 9 1 7 12 1 7 15 2 8 9 1 8 13 1 9 10 1 9 14 1 10 15 1



11 12 1 11 13 1 11 15 2 11 16 1 12 14 1 12 17 1 13 14 1 13 18 1 14 15 1 14 19 1  
 15 20 1 16 17 1 16 18 1 16 20 2 17 19 1 18 19 1 19 20 1

Matrix size = 21, reduced Matrix size = 3

0 1 1 0 4 2 0 5 1 0 17 3 0 18 3 1 2 1 1 3 1 1 6 1 1 16 2 2 4 1 2 7 1 2 17 2  
 3 4 1 3 8 1 4 9 1 5 6 1 5 9 2 5 10 1 5 13 2 6 7 1 6 8 1 6 11 1 7 9 1 7 12 1  
 8 9 1 8 13 1 8 15 2 8 16 1 9 14 1 9 17 1 10 11 1 10 17 2 11 12 1 11 13 1 11 16 1  
 12 14 1 12 17 1 12 20 2 13 14 1 13 18 1 14 15 1 14 19 1 15 20 1 16 17 1 16 18 1 16 20 2  
 17 19 1 18 19 1 19 20 1

Matrix size = 35, reduced Matrix size = 4

0 1 1 0 5 2 0 9 1 0 13 2 1 2 1 1 4 1 1 10 1 2 3 1 2 5 1 2 11 1 3 6 1 3 12 1  
 4 5 1 4 8 2 4 13 1 4 16 2 4 19 1 5 6 1 5 7 1 5 14 1 5 20 1 6 8 1 6 15 1 6 21 1  
 7 8 1 7 16 1 7 18 2 7 22 1 8 17 1 8 23 1 9 10 1 9 20 2 10 11 1 10 13 1 10 19 1  
 11 12 1 11 14 1 11 20 1 11 27 2 12 15 1 12 21 1 12 28 2 13 14 1 13 24 1  
 14 15 1 14 16 1 14 25 1 15 17 1 15 26 1 16 17 1 16 27 1 17 18 1 17 28 1 18 29 1  
 19 20 1 19 23 2 19 24 1 19 27 2 20 21 1 20 22 1 20 25 1 21 23 1 21 26 1  
 22 23 1 22 27 1 22 29 2 22 30 1 23 28 1 23 31 1 24 25 1 24 31 2 25 26 1 25 27 1 25 30 1  
 26 28 1 26 31 1 26 34 2 27 28 1 27 32 1 28 29 1 28 33 1 29 34 1 30 31 1 30 32 1 30 34 2  
 31 33 1 32 33 1 33 34 1

Matrix size = 15, reduced Matrix size = 1

0 1 1 1 2 1 1 5 1 2 3 1 2 6 1 3 4 1 3 7 1 4 8 1 5 6 1 6 7 1 6 9 1  
 7 8 1 7 10 1 8 11 1 9 10 1 10 11 1 10 12 1 11 13 1 12 13 1 13 14 1

Matrix size = 14, reduced Matrix size = 2

0 1 1 0 5 2 0 11 4 1 2 1 1 4 1 1 13 3 2 3 1 2 5 1 2 12 2 3 6 1 3 13 2  
 4 5 1 4 8 2 4 9 1 5 6 1 5 7 1 5 10 1 6 8 1 6 11 1 7 8 1 7 12 1 8 13 1  
 9 10 1 9 13 2 10 11 1 10 12 1 11 13 1 12 13 1

Matrix size = 14, reduced Matrix size = 1

0 1 1 0 6 2 1 2 1 1 5 1 2 3 1 2 6 1 3 4 1 3 7 1 4 8 1 5 6 1 5 10 2  
 6 7 1 6 9 1 7 8 1 7 10 1 8 11 1 9 10 1 9 13 2 10 11 1 10 12 1 11 13 1 12 13 1

Matrix size = 6, reduced Matrix size = 0

0 1 1 1 2 1 2 3 1 3 4 1 4 5 1

## W—graph for B8

partition: 8 0 0 0 0 0 0 0

x1 = [1 2 3 4 5 6 7]

partition: 7 1 0 0 0 0 0 0

x1 = [2 3 4 5 6 7] x2 = [1 3 4 5 6 7] x3 = [1 2 4 5 6 7] x4 = [1 2 3 5 6 7] x5 = [1 2 3 4 6 7]

x6 = [1 2 3 4 5 7] x7 = [1 2 3 4 5 6]

partition: 6 2 0 0 0 0 0 0

x1 = [2 4 5 6 7] x2 = [1 3 4 5 6 7] x3 = [2 3 5 6 7] x4 = [1 3 5 6 7] x5 = [1 2 4 5 6 7]

x6 = [2 3 4 6 7] x7 = [1 3 4 6 7] x8 = [1 2 4 6 7] x9 = [1 2 3 5 6 7] x10 = [2 3 4 5 7]

x11 = [1 3 4 5 7] x12 = [1 2 4 5 7] x13 = [1 2 3 5 7] x14 = [1 2 3 4 6 7] x15 = [2 3 4 5 6]

x16 = [1 3 4 5 6] x17 = [1 2 4 5 6] x18 = [1 2 3 5 6] x19 = [1 2 3 4 6] x20 = [1 2 3 4 5 7]

partition: 6 1 1 0 0 0 0 0

x1 = [3 4 5 6 7] x2 = [2 4 5 6 7] x3 = [1 4 5 6 7] x4 = [2 3 5 6 7] x5 = [1 3 5 6 7]

x6 = [1 2 5 6 7] x7 = [2 3 4 6 7] x8 = [1 3 4 6 7] x9 = [1 2 4 6 7] x10 = [1 2 3 6 7]

x11 = [2 3 4 5 7] x12 = [1 3 4 5 7] x13 = [1 2 4 5 7] x14 = [1 2 3 5 7] x15 = [1 2 3 4 7]

x16 = [2 3 4 5 6] x17 = [1 3 4 5 6] x18 = [1 2 4 5 6] x19 = [1 2 3 5 6] x20 = [1 2 3 4 6] x21 = [1 2 3 4 5]

partition: 5 3 0 0 0 0 0 0

x1 = [2 4 6 7] x2 = [1 3 4 6 7] x3 = [2 3 5 6 7] x4 = [1 3 5 6 7] x5 = [1 2 4 5 6 7]

x6 = [2 4 5 7] x7 = [1 3 4 5 7] x8 = [2 3 5 7] x9 = [1 3 5 7] x10 = [1 2 4 5 7]

x11 = [2 3 4 6 7] x12 = [1 3 4 6 7] x13 = [1 2 4 6 7] x14 = [1 2 3 5 6 7] x15 = [2 4 5 6]

x16 = [1 3 4 5 6] x17 = [2 3 5 6] x18 = [1 3 5 6] x19 = [1 2 4 5 6] x20 = [2 3 4 6]

x21 = [1 3 4 6] x22 = [1 2 4 6] x23 = [1 2 3 5 6] x24 = [2 3 4 5 7] x25 = [1 3 4 5 7]

x26 = [1 2 4 5 7] x27 = [1 2 3 5 7] x28 = [1 2 3 4 6 7]

partition: 5 2 1 0 0 0 0 0

x1 = [3 5 6 7] x2 = [2 4 5 6 7] x3 = [1 4 5 6 7] x4 = [2 5 6 7] x5 = [1 3 5 6 7]

x6 = [3 4 6 7] x7 = [2 4 6 7] x8 = [1 4 6 7] x9 = [2 3 5 6 7] x10 = [1 3 5 6 7]

x11 = [1 2 5 6 7] x12 = [2 4 6 7] x13 = [1 3 4 6 7] x14 = [2 3 6 7] x15 = [1 3 6 7]

x16 = [1 2 4 6 7] x17 = [3 4 5 7] x18 = [2 4 5 7] x19 = [1 4 5 7] x20 = [2 3 5 7]

x21 = [1 3 5 7] x22 = [1 2 5 7] x23 = [2 3 4 6 7] x24 = [1 3 4 6 7] x25 = [1 2 4 6 7]

x26 = [1 2 3 6 7] x27 = [2 4 5 7] x28 = [1 3 4 5 7] x29 = [2 3 5 7] x30 = [1 3 5 7]

x31 = [1 2 4 5 7] x32 = [2 3 4 7] x33 = [1 3 4 7] x34 = [1 2 4 7] x35 = [1 2 3 5 7]

x36 = [3 4 5 6] x37 = [2 4 5 6] x38 = [1 4 5 6] x39 = [2 3 5 6] x40 = [1 3 5 6]

x41 = [1 2 5 6] x42 = [2 3 4 6] x43 = [1 3 4 6] x44 = [1 2 4 6] x45 = [1 2 3 6]

x46 = [2 3 4 5 7] x47 = [1 3 4 5 7] x48 = [1 2 4 5 7] x49 = [1 2 3 5 7] x50 = [1 2 3 4 7]

x51 = [2 4 5 6] x52 = [1 3 4 5 6] x53 = [2 3 5 6] x54 = [1 3 5 6] x55 = [1 2 4 5 6]

x56 = [2 3 4 6] x57 = [1 3 4 6] x58 = [1 2 4 6] x59 = [1 2 3 5 6] x60 = [2 3 4 5]

x61 = [1 3 4 5] x62 = [1 2 4 5] x63 = [1 2 3 5] x64 = [1 2 3 4 6]

partition: 5 1 1 1 0 0 0 0

x1 = [4 5 6 7] x2 = [3 5 6 7] x3 = [2 5 6 7] x4 = [1 5 6 7] x5 = [3 4 6 7]

x6 = [2 4 6 7] x7 = [1 4 6 7] x8 = [2 3 6 7] x9 = [1 3 6 7] x10 = [1 2 6 7]

x11 = [3 4 5 7] x12 = [2 4 5 7] x13 = [1 4 5 7] x14 = [2 3 5 7] x15 = [1 3 5 7]

x16 = [1 2 5 7] x17 = [2 3 4 7] x18 = [1 3 4 7] x19 = [1 2 4 7] x20 = [1 2 3 7]

x21 = [3 4 5 6] x22 = [2 4 5 6] x23 = [1 4 5 6] x24 = [2 3 5 6] x25 = [1 3 5 6]

x26 = [1 2 5 6] x27 = [2 3 4 6] x28 = [1 3 4 6] x29 = [1 2 4 6] x30 = [1 2 3 6]

x31 = [2 3 4 5] x32 = [1 3 4 5] x33 = [1 2 4 5] x34 = [1 2 3 5] x35 = [1 2 3 4]

partition: 4 4 0 0 0 0 0 0

x1 = [2 4 6] x2 = [1 3 4 6] x3 = [2 3 5 6] x4 = [1 3 5 6] x5 = [1 2 4 5 6]  
 x6 = [2 4 5 7] x7 = [1 3 4 5 7] x8 = [2 3 5 7] x9 = [1 3 5 7] x10 = [1 2 4 5 7]  
 x11 = [2 3 4 6 7] x12 = [1 3 4 6 7] x13 = [1 2 4 6 7] x14 = [1 2 3 5 6 7]

partition: 4 3 1 0 0 0 0 0

x1 = [3 5 7] x2 = [2 4 5 7] x3 = [1 4 5 7] x4 = [2 5 7] x5 = [1 3 5 7]  
 x6 = [3 4 6 7] x7 = [2 4 6 7] x8 = [1 4 6 7] x9 = [2 3 5 6 7] x10 = [1 3 5 6 7]  
 x11 = [1 2 5 6 7] x12 = [2 4 6 7] x13 = [1 3 4 6 7] x14 = [2 3 6 7] x15 = [1 3 6 7]  
 x16 = [1 2 4 6 7] x17 = [2 4 7] x18 = [1 3 4 7] x19 = [2 3 5 7] x20 = [1 3 5 7]  
 x21 = [1 2 4 5 7] x22 = [3 5 6] x23 = [2 4 5 6] x24 = [1 4 5 6] x25 = [2 5 6]  
 x26 = [1 3 5 6] x27 = [3 4 6] x28 = [2 4 6] x29 = [1 4 6] x30 = [2 3 5 6]  
 x31 = [1 3 5 6] x32 = [1 2 5 6] x33 = [2 4 6] x34 = [1 3 4 6] x35 = [2 3 6]  
 x36 = [1 3 6] x37 = [1 2 4 6] x38 = [3 4 5 7] x39 = [2 4 5 7] x40 = [1 4 5 7]  
 x41 = [2 3 5 7] x42 = [1 3 5 7] x43 = [1 2 5 7] x44 = [2 3 4 6 7] x45 = [1 3 4 6 7]  
 x46 = [1 2 4 6 7] x47 = [1 2 3 6 7] x48 = [2 4 5 7] x49 = [1 3 4 5 7] x50 = [2 3 5 7]  
 x51 = [1 3 5 7] x52 = [1 2 4 5 7] x53 = [2 3 4 7] x54 = [1 3 4 7] x55 = [1 2 4 7]  
 x56 = [1 2 3 5 7] x57 = [2 4 6] x58 = [1 3 4 6] x59 = [2 3 5 6] x60 = [1 3 5 6]  
 x61 = [1 2 4 5 6] x62 = [2 4 5] x63 = [1 3 4 5] x64 = [2 3 5] x65 = [1 3 5]  
 x66 = [1 2 4 5] x67 = [2 3 4 6] x68 = [1 3 4 6] x69 = [1 2 4 6] x70 = [1 2 3 5 6]

partition: 4 2 2 0 0 0 0 0

x1 = [3 6 7] x2 = [2 4 6 7] x3 = [1 4 6 7] x4 = [2 5 6 7] x5 = [1 3 5 6 7]  
 x6 = [3 5 7] x7 = [2 4 5 7] x8 = [1 4 5 7] x9 = [2 5 7] x10 = [1 3 5 7]  
 x11 = [3 4 7] x12 = [2 4 7] x13 = [1 4 7] x14 = [2 3 5 7] x15 = [1 3 5 7]  
 x16 = [1 2 5 7] x17 = [2 4 6 7] x18 = [1 3 4 6 7] x19 = [2 3 6 7] x20 = [1 3 6 7]  
 x21 = [1 2 4 6 7] x22 = [3 5 6] x23 = [2 4 5 6] x24 = [1 4 5 6] x25 = [2 5 6]  
 x26 = [1 3 5 6] x27 = [3 4 6] x28 = [2 4 6] x29 = [1 4 6] x30 = [2 3 5 6]  
 x31 = [1 3 5 6] x32 = [1 2 5 6] x33 = [2 4 6] x34 = [1 3 4 6] x35 = [2 3 6]  
 x36 = [1 3 6] x37 = [1 2 4 6] x38 = [3 4 5] x39 = [2 4 5] x40 = [1 4 5]  
 x41 = [2 3 5] x42 = [1 3 5] x43 = [1 2 5] x44 = [2 3 4 6] x45 = [1 3 4 6]  
 x46 = [1 2 4 6] x47 = [1 2 3 6] x48 = [2 4 5 7] x49 = [1 3 4 5 7] x50 = [2 3 5 7]  
 x51 = [1 3 5 7] x52 = [1 2 4 5 7] x53 = [2 3 4 7] x54 = [1 3 4 7] x55 = [1 2 4 7] x56 = [1 2 3 5 7]

partition: 4 2 1 1 0 0 0 0

x1 = [4 6 7] x2 = [3 5 6 7] x3 = [2 5 6 7] x4 = [1 5 6 7] x5 = [3 6 7]  
 x6 = [2 4 6 7] x7 = [1 4 6 7] x8 = [2 6 7] x9 = [1 3 6 7] x10 = [4 5 7]  
 x11 = [3 5 7] x12 = [2 5 7] x13 = [1 5 7] x14 = [3 4 6 7] x15 = [2 4 6 7]  
 x16 = [1 4 6 7] x17 = [2 3 6 7] x18 = [1 3 6 7] x19 = [1 2 6 7] x20 = [3 5 7]  
 x21 = [2 4 5 7] x22 = [1 4 5 7] x23 = [2 5 7] x24 = [1 3 5 7] x25 = [3 4 7]  
 x26 = [2 4 7] x27 = [1 4 7] x28 = [2 3 5 7] x29 = [1 3 5 7] x30 = [1 2 5 7]  
 x31 = [2 4 7] x32 = [1 3 4 7] x33 = [2 3 7] x34 = [1 3 7] x35 = [1 2 4 7]  
 x36 = [4 5 6] x37 = [3 5 6] x38 = [2 5 6] x39 = [1 5 6] x40 = [3 4 6]  
 x41 = [2 4 6] x42 = [1 4 6] x43 = [2 3 6] x44 = [1 3 6] x45 = [1 2 6]  
 x46 = [3 4 5 7] x47 = [2 4 5 7] x48 = [1 4 5 7] x49 = [2 3 5 7] x50 = [1 3 5 7]  
 x51 = [1 2 5 7] x52 = [2 3 4 7] x53 = [1 3 4 7] x54 = [1 2 4 7] x55 = [1 2 3 7]  
 x56 = [3 5 6] x57 = [2 4 5 6] x58 = [1 4 5 6] x59 = [2 5 6] x60 = [1 3 5 6]  
 x61 = [3 4 6] x62 = [2 4 6] x63 = [1 4 6] x64 = [2 3 5 6] x65 = [1 3 5 6]  
 x66 = [1 2 5 6] x67 = [2 4 6] x68 = [1 3 4 6] x69 = [2 3 6] x70 = [1 3 6]  
 x71 = [1 2 4 6] x72 = [3 4 5] x73 = [2 4 5] x74 = [1 4 5] x75 = [2 3 5]  
 x76 = [1 3 5] x77 = [1 2 5] x78 = [2 3 4 6] x79 = [1 3 4 6] x80 = [1 2 4 6]

$x_{81} = [1\ 2\ 3\ 6]$   $x_{82} = [2\ 4\ 5]$   $x_{83} = [1\ 3\ 4\ 5]$   $x_{84} = [2\ 3\ 5]$   $x_{85} = [1\ 3\ 5]$   
 $x_{86} = [1\ 2\ 4\ 5]$   $x_{87} = [2\ 3\ 4]$   $x_{88} = [1\ 3\ 4]$   $x_{89} = [1\ 2\ 4]$   $x_{90} = [1\ 2\ 3\ 5]$   
 partition: 4 1 1 1 1 0 0 0 0  
 $x_1 = [5\ 6\ 7]$   $x_2 = [4\ 6\ 7]$   $x_3 = [3\ 6\ 7]$   $x_4 = [2\ 6\ 7]$   $x_5 = [1\ 6\ 7]$   
 $x_6 = [4\ 5\ 7]$   $x_7 = [3\ 5\ 7]$   $x_8 = [2\ 5\ 7]$   $x_9 = [1\ 5\ 7]$   $x_{10} = [3\ 4\ 7]$   
 $x_{11} = [2\ 4\ 7]$   $x_{12} = [1\ 4\ 7]$   $x_{13} = [2\ 3\ 7]$   $x_{14} = [1\ 3\ 7]$   $x_{15} = [1\ 2\ 7]$   
 $x_{16} = [4\ 5\ 6]$   $x_{17} = [3\ 5\ 6]$   $x_{18} = [2\ 5\ 6]$   $x_{19} = [1\ 5\ 6]$   $x_{20} = [3\ 4\ 6]$   
 $x_{21} = [2\ 4\ 6]$   $x_{22} = [1\ 4\ 6]$   $x_{23} = [2\ 3\ 6]$   $x_{24} = [1\ 3\ 6]$   $x_{25} = [1\ 2\ 6]$   
 $x_{26} = [3\ 4\ 5]$   $x_{27} = [2\ 4\ 5]$   $x_{28} = [1\ 4\ 5]$   $x_{29} = [2\ 3\ 5]$   $x_{30} = [1\ 3\ 5]$   
 $x_{31} = [1\ 2\ 5]$   $x_{32} = [2\ 3\ 4]$   $x_{33} = [1\ 3\ 4]$   $x_{34} = [1\ 2\ 4]$   $x_{35} = [1\ 2\ 3]$   
 partition: 3 3 2 0 0 0 0 0 0  
 $x_1 = [3\ 6]$   $x_2 = [2\ 4\ 6]$   $x_3 = [1\ 4\ 6]$   $x_4 = [2\ 5\ 6]$   $x_5 = [1\ 3\ 5\ 6]$   
 $x_6 = [3\ 5\ 7]$   $x_7 = [2\ 4\ 5\ 7]$   $x_8 = [1\ 4\ 5\ 7]$   $x_9 = [2\ 5\ 7]$   $x_{10} = [1\ 3\ 5\ 7]$   
 $x_{11} = [3\ 4\ 7]$   $x_{12} = [2\ 4\ 7]$   $x_{13} = [1\ 4\ 7]$   $x_{14} = [2\ 3\ 5\ 7]$   $x_{15} = [1\ 3\ 5\ 7]$   
 $x_{16} = [1\ 2\ 5\ 7]$   $x_{17} = [2\ 4\ 6\ 7]$   $x_{18} = [1\ 3\ 4\ 6\ 7]$   $x_{19} = [2\ 3\ 6\ 7]$   $x_{20} = [1\ 3\ 6\ 7]$   
 $x_{21} = [1\ 2\ 4\ 6\ 7]$   $x_{22} = [3\ 5]$   $x_{23} = [2\ 4\ 5]$   $x_{24} = [1\ 4\ 5]$   $x_{25} = [2\ 5]$   
 $x_{26} = [1\ 3\ 5]$   $x_{27} = [3\ 4\ 6]$   $x_{28} = [2\ 4\ 6]$   $x_{29} = [1\ 4\ 6]$   $x_{30} = [2\ 3\ 5\ 6]$   
 $x_{31} = [1\ 3\ 5\ 6]$   $x_{32} = [1\ 2\ 5\ 6]$   $x_{33} = [2\ 4\ 6]$   $x_{34} = [1\ 3\ 4\ 6]$   $x_{35} = [2\ 3\ 6]$   $x_{36} = [1\ 3\ 6]$   
 $x_{37} = [1\ 2\ 4\ 6]$   $x_{38} = [2\ 4\ 7]$   $x_{39} = [1\ 3\ 4\ 7]$   $x_{40} = [2\ 3\ 5\ 7]$   $x_{41} = [1\ 3\ 5\ 7]$   $x_{42} = [1\ 2\ 4\ 5\ 7]$   
 partition: 3 3 1 1 0 0 0 0 0  
 $x_1 = [4\ 6]$   $x_2 = [3\ 5\ 6]$   $x_3 = [2\ 5\ 6]$   $x_4 = [1\ 5\ 6]$   $x_5 = [3\ 6]$   
 $x_6 = [2\ 4\ 6]$   $x_7 = [1\ 4\ 6]$   $x_8 = [2\ 6]$   $x_9 = [1\ 3\ 6]$   $x_{10} = [4\ 5\ 7]$   
 $x_{11} = [3\ 5\ 7]$   $x_{12} = [2\ 5\ 7]$   $x_{13} = [1\ 5\ 7]$   $x_{14} = [3\ 4\ 6\ 7]$   $x_{15} = [2\ 4\ 6\ 7]$   
 $x_{16} = [1\ 4\ 6\ 7]$   $x_{17} = [2\ 3\ 6\ 7]$   $x_{18} = [1\ 3\ 6\ 7]$   $x_{19} = [1\ 2\ 6\ 7]$   $x_{20} = [3\ 5\ 7]$   
 $x_{21} = [2\ 4\ 5\ 7]$   $x_{22} = [1\ 4\ 5\ 7]$   $x_{23} = [2\ 5\ 7]$   $x_{24} = [1\ 3\ 5\ 7]$   $x_{25} = [3\ 4\ 7]$   
 $x_{26} = [2\ 4\ 7]$   $x_{27} = [1\ 4\ 7]$   $x_{28} = [2\ 3\ 5\ 7]$   $x_{29} = [1\ 3\ 5\ 7]$   $x_{30} = [1\ 2\ 5\ 7]$   
 $x_{31} = [2\ 4\ 7]$   $x_{32} = [1\ 3\ 4\ 7]$   $x_{33} = [2\ 3\ 7]$   $x_{34} = [1\ 3\ 7]$   $x_{35} = [1\ 2\ 4\ 7]$   
 $x_{36} = [3\ 5]$   $x_{37} = [2\ 4\ 5]$   $x_{38} = [1\ 4\ 5]$   $x_{39} = [2\ 5]$   $x_{40} = [1\ 3\ 5]$   
 $x_{41} = [3\ 4\ 6]$   $x_{42} = [2\ 4\ 6]$   $x_{43} = [1\ 4\ 6]$   $x_{44} = [2\ 3\ 5\ 6]$   $x_{45} = [1\ 3\ 5\ 6]$   
 $x_{46} = [1\ 2\ 5\ 6]$   $x_{47} = [2\ 4\ 6]$   $x_{48} = [1\ 3\ 4\ 6]$   $x_{49} = [2\ 3\ 6]$   $x_{50} = [1\ 3\ 6]$   
 $x_{51} = [1\ 2\ 4\ 6]$   $x_{52} = [2\ 4]$   $x_{53} = [1\ 3\ 4]$   $x_{54} = [2\ 3\ 5]$   $x_{55} = [1\ 3\ 5]$   $x_{56} = [1\ 2\ 4\ 5]$   
 partition: 3 2 2 1 0 0 0 0 0  
 $x_1 = [4\ 7]$   $x_2 = [3\ 5\ 7]$   $x_3 = [2\ 5\ 7]$   $x_4 = [1\ 5\ 7]$   $x_5 = [3\ 6\ 7]$   
 $x_6 = [2\ 4\ 6\ 7]$   $x_7 = [1\ 4\ 6\ 7]$   $x_8 = [2\ 6\ 7]$   $x_9 = [1\ 3\ 6\ 7]$   $x_{10} = [3\ 7]$   
 $x_{11} = [2\ 4\ 7]$   $x_{12} = [1\ 4\ 7]$   $x_{13} = [2\ 5\ 7]$   $x_{14} = [1\ 3\ 5\ 7]$   $x_{15} = [4\ 6]$   
 $x_{16} = [3\ 5\ 6]$   $x_{17} = [2\ 5\ 6]$   $x_{18} = [1\ 5\ 6]$   $x_{19} = [3\ 6]$   $x_{20} = [2\ 4\ 6]$   
 $x_{21} = [1\ 4\ 6]$   $x_{22} = [2\ 6]$   $x_{23} = [1\ 3\ 6]$   $x_{24} = [4\ 5]$   $x_{25} = [3\ 5]$   
 $x_{26} = [2\ 5]$   $x_{27} = [1\ 5]$   $x_{28} = [3\ 4\ 6]$   $x_{29} = [2\ 4\ 6]$   $x_{30} = [1\ 4\ 6]$   
 $x_{31} = [2\ 3\ 6]$   $x_{32} = [1\ 3\ 6]$   $x_{33} = [1\ 2\ 6]$   $x_{34} = [3\ 5\ 7]$   $x_{35} = [2\ 4\ 5\ 7]$   
 $x_{36} = [1\ 4\ 5\ 7]$   $x_{37} = [2\ 5\ 7]$   $x_{38} = [1\ 3\ 5\ 7]$   $x_{39} = [3\ 4\ 7]$   $x_{40} = [2\ 4\ 7]$   
 $x_{41} = [1\ 4\ 7]$   $x_{42} = [2\ 3\ 5\ 7]$   $x_{43} = [1\ 3\ 5\ 7]$   $x_{44} = [1\ 2\ 5\ 7]$   $x_{45} = [2\ 4\ 7]$   
 $x_{46} = [1\ 3\ 4\ 7]$   $x_{47} = [2\ 3\ 7]$   $x_{48} = [1\ 3\ 7]$   $x_{49} = [1\ 2\ 4\ 7]$   $x_{50} = [3\ 6]$   
 $x_{51} = [2\ 4\ 6]$   $x_{52} = [1\ 4\ 6]$   $x_{53} = [2\ 5\ 6]$   $x_{54} = [1\ 3\ 5\ 6]$   $x_{55} = [3\ 5]$   
 $x_{56} = [2\ 4\ 5]$   $x_{57} = [1\ 4\ 5]$   $x_{58} = [2\ 5]$   $x_{59} = [1\ 3\ 5]$   $x_{60} = [3\ 4]$   
 $x_{61} = [2\ 4]$   $x_{62} = [1\ 4]$   $x_{63} = [2\ 3\ 5]$   $x_{64} = [1\ 3\ 5]$   $x_{65} = [1\ 2\ 5]$   
 $x_{66} = [2\ 4\ 6]$   $x_{67} = [1\ 3\ 4\ 6]$   $x_{68} = [2\ 3\ 6]$   $x_{69} = [1\ 3\ 6]$   $x_{70} = [1\ 2\ 4\ 6]$   
 partition: 3 2 1 1 1 0 0 0 0

$x_1 = [5\ 7]$   $x_2 = [4\ 6\ 7]$   $x_3 = [3\ 6\ 7]$   $x_4 = [2\ 6\ 7]$   $x_5 = [1\ 6\ 7]$   
 $x_6 = [4\ 7]$   $x_7 = [3\ 5\ 7]$   $x_8 = [2\ 5\ 7]$   $x_9 = [1\ 5\ 7]$   $x_{10} = [3\ 7]$   
 $x_{11} = [2\ 4\ 7]$   $x_{12} = [1\ 4\ 7]$   $x_{13} = [2\ 7]$   $x_{14} = [1\ 3\ 7]$   $x_{15} = [5\ 6]$   
 $x_{16} = [4\ 6]$   $x_{17} = [3\ 6]$   $x_{18} = [2\ 6]$   $x_{19} = [1\ 6]$   $x_{20} = [4\ 5\ 7]$   
 $x_{21} = [3\ 5\ 7]$   $x_{22} = [2\ 5\ 7]$   $x_{23} = [1\ 5\ 7]$   $x_{24} = [3\ 4\ 7]$   $x_{25} = [2\ 4\ 7]$   
 $x_{26} = [1\ 4\ 7]$   $x_{27} = [2\ 3\ 7]$   $x_{28} = [1\ 3\ 7]$   $x_{29} = [1\ 2\ 7]$   $x_{30} = [4\ 6]$   
 $x_{31} = [3\ 5\ 6]$   $x_{32} = [2\ 5\ 6]$   $x_{33} = [1\ 5\ 6]$   $x_{34} = [3\ 6]$   $x_{35} = [2\ 4\ 6]$   
 $x_{36} = [1\ 4\ 6]$   $x_{37} = [2\ 6]$   $x_{38} = [1\ 3\ 6]$   $x_{39} = [4\ 5]$   $x_{40} = [3\ 5]$   
 $x_{41} = [2\ 5]$   $x_{42} = [1\ 5]$   $x_{43} = [3\ 4\ 6]$   $x_{44} = [2\ 4\ 6]$   $x_{45} = [1\ 4\ 6]$   
 $x_{46} = [2\ 3\ 6]$   $x_{47} = [1\ 3\ 6]$   $x_{48} = [1\ 2\ 6]$   $x_{49} = [3\ 5]$   $x_{50} = [2\ 4\ 5]$   
 $x_{51} = [1\ 4\ 5]$   $x_{52} = [2\ 5]$   $x_{53} = [1\ 3\ 5]$   $x_{54} = [3\ 4]$   $x_{55} = [2\ 4]$   
 $x_{56} = [1\ 4]$   $x_{57} = [2\ 3\ 5]$   $x_{58} = [1\ 3\ 5]$   $x_{59} = [1\ 2\ 5]$   $x_{60} = [2\ 4]$   
 $x_{61} = [1\ 3\ 4]$   $x_{62} = [2\ 3]$   $x_{63} = [1\ 3]$   $x_{64} = [1\ 2\ 4]$

partition: 3 1 1 1 1 1 0 0 0

$x_1 = [6\ 7]$   $x_2 = [5\ 7]$   $x_3 = [4\ 7]$   $x_4 = [3\ 7]$   $x_5 = [2\ 7]$   
 $x_6 = [1\ 7]$   $x_7 = [5\ 6]$   $x_8 = [4\ 6]$   $x_9 = [3\ 6]$   $x_{10} = [2\ 6]$   
 $x_{11} = [1\ 6]$   $x_{12} = [4\ 5]$   $x_{13} = [3\ 5]$   $x_{14} = [2\ 5]$   $x_{15} = [1\ 5]$   
 $x_{16} = [3\ 4]$   $x_{17} = [2\ 4]$   $x_{18} = [1\ 4]$   $x_{19} = [2\ 3]$   $x_{20} = [1\ 3]$   $x_{21} = [1\ 2]$

partition: 2 2 2 2 0 0 0 0 0

$x_1 = [4]$   $x_2 = [3\ 5]$   $x_3 = [2\ 5]$   $x_4 = [1\ 5]$   $x_5 = [3\ 6]$   
 $x_6 = [2\ 4\ 6]$   $x_7 = [1\ 4\ 6]$   $x_8 = [2\ 6]$   $x_9 = [1\ 3\ 6]$   $x_{10} = [3\ 7]$   
 $x_{11} = [2\ 4\ 7]$   $x_{12} = [1\ 4\ 7]$   $x_{13} = [2\ 5\ 7]$   $x_{14} = [1\ 3\ 5\ 7]$

partition: 2 2 2 1 1 0 0 0 0

$x_1 = [5]$   $x_2 = [4\ 6]$   $x_3 = [3\ 6]$   $x_4 = [2\ 6]$   $x_5 = [1\ 6]$   
 $x_6 = [4\ 7]$   $x_7 = [3\ 5\ 7]$   $x_8 = [2\ 5\ 7]$   $x_9 = [1\ 5\ 7]$   $x_{10} = [3\ 7]$   
 $x_{11} = [2\ 4\ 7]$   $x_{12} = [1\ 4\ 7]$   $x_{13} = [2\ 7]$   $x_{14} = [1\ 3\ 7]$   $x_{15} = [4]$   
 $x_{16} = [3\ 5]$   $x_{17} = [2\ 5]$   $x_{18} = [1\ 5]$   $x_{19} = [3\ 6]$   $x_{20} = [2\ 4\ 6]$   
 $x_{21} = [1\ 4\ 6]$   $x_{22} = [2\ 6]$   $x_{23} = [1\ 3\ 6]$   $x_{24} = [3]$   $x_{25} = [2\ 4]$   
 $x_{26} = [1\ 4]$   $x_{27} = [2\ 5]$   $x_{28} = [1\ 3\ 5]$

partition: 2 2 1 1 1 1 0 0 0

$x_1 = [6]$   $x_2 = [5\ 7]$   $x_3 = [4\ 7]$   $x_4 = [3\ 7]$   $x_5 = [2\ 7]$   
 $x_6 = [1\ 7]$   $x_7 = [5]$   $x_8 = [4\ 6]$   $x_9 = [3\ 6]$   $x_{10} = [2\ 6]$   
 $x_{11} = [1\ 6]$   $x_{12} = [4]$   $x_{13} = [3\ 5]$   $x_{14} = [2\ 5]$   $x_{15} = [1\ 5]$   
 $x_{16} = [3]$   $x_{17} = [2\ 4]$   $x_{18} = [1\ 4]$   $x_{19} = [2]$   $x_{20} = [1\ 3]$

partition: 2 1 1 1 1 1 1 0 0

$x_1 = [7]$   $x_2 = [6]$   $x_3 = [5]$   $x_4 = [4]$   $x_5 = [3]$   $x_6 = [2]$   $x_7 = [1]$

partition: 1 1 1 1 1 1 1 1 0

Matrix size = 7, reduced Matrix size = 0

0 1 1 1 2 1 2 3 1 3 4 1 4 5 1 5 6 1

Matrix size = 20, reduced Matrix size = 0

0 1 1 0 2 1 0 4 2 1 3 1 2 3 1 2 5 1 3 4 1 3 6 1 3 8 2 4 7 1  
5 6 1 5 9 1 6 7 1 6 10 1 7 8 1 7 11 1 7 13 2 8 12 1 9 10 1 9 14 1  
10 11 1 10 15 1 11 12 1 11 16 1 12 13 1 12 17 1 12 19 2 13 18 1  
14 15 1 15 16 1 16 17 1 17 18 1 18 19 1

Matrix size = 21, reduced Matrix size = 0

0 1 1 1 2 1 1 3 1 2 4 1 3 4 1 3 6 1 4 5 1 4 7 1 5 8 1  
6 7 1 6 10 1 7 8 1 7 11 1 8 9 1 8 12 1 9 13 1 10 11 1 10 15 1

11 12 1 11 16 1 12 13 1 12 17 1 13 14 1 13 18 1 14 19 1 15 16 1

16 17 1 17 18 1 18 19 1 19 20 1

Matrix size = 28, reduced Matrix size = 1

0 1 1 0 2 1 0 4 2 0 5 1 0 10 2 0 12 3 1 3 1 1 6 1 1 11 2 2 3 1 2 7 1 2 13 4  
 3 4 1 3 8 1 4 9 1 5 6 1 5 7 1 5 9 2 5 14 1 6 8 1 6 15 1 7 8 1 7 10 1 7 16 1 7 23 2  
 8 9 1 8 11 1 8 13 2 8 17 1 8 24 2 8 26 3 9 12 1 9 18 1 9 25 2 10 11 1 10 19 1  
 11 12 1 11 20 1 11 27 4 12 13 1 12 21 1 13 22 1 14 15 1 14 16 1 14 18 2 15 17 1  
 16 17 1 16 19 1 17 18 1 17 20 1 17 22 2 18 21 1 19 20 1 19 23 1 20 21 1 20 24 1  
 21 22 1 21 25 1 21 27 2 22 26 1 23 24 1 24 25 1 25 26 1 26 27 1

Matrix size = 64, reduced Matrix size = 2

0 1 1 0 4 2 0 5 1 0 8 2 1 2 1 1 3 1 1 6 1 2 4 1 2 7 1 3 4 1 3 8 1 3 10 2 3 11 1  
 4 9 1 4 12 1 5 6 1 5 12 2 5 16 1 6 7 1 6 8 1 6 11 1 6 17 1 6 22 2  
 7 9 1 7 12 1 7 15 2 7 18 1 7 23 2 8 9 1 8 13 1 8 19 1 9 10 1 9 14 1 9 20 1  
 10 15 1 10 21 1 11 12 1 11 13 1 11 15 2 11 26 1 12 14 1 12 27 1 13 14 1 13 22 1 13 28 1  
 14 15 1 14 23 1 14 25 2 14 29 1 15 24 1 15 30 1 16 17 1 16 27 2 16 35 1  
 17 18 1 17 19 1 17 26 1 17 36 1 18 20 1 18 27 1 18 30 2 18 37 1  
 19 20 1 19 22 1 19 28 1 19 38 1 19 45 2 20 21 1 20 23 1 20 29 1 20 39 1 20 46 2  
 21 24 1 21 30 1 21 34 2 21 40 1 21 47 2 22 23 1 22 31 1 22 41 1  
 23 24 1 23 32 1 23 42 1 24 25 1 24 33 1 24 43 1 25 34 1 25 44 1  
 26 27 1 26 28 1 26 30 2 26 50 1 27 29 1 27 51 1 28 29 1 28 31 1 28 52 1  
 29 30 1 29 32 1 29 34 2 29 53 1 30 33 1 30 54 1 31 32 1 31 45 1 31 55 1  
 32 33 1 32 46 1 32 56 1 33 34 1 33 47 1 33 49 2 33 57 1 34 48 1 34 58 1  
 35 36 1 35 51 2 36 37 1 36 38 1 36 50 1 37 39 1 37 51 1 37 54 2 38 39 1 38 41 1 38 52 1  
 39 40 1 39 42 1 39 53 1 40 43 1 40 54 1 40 58 2 41 42 1 41 45 1 41 55 1  
 42 43 1 42 46 1 42 56 1 43 44 1 43 47 1 43 57 1 44 48 1 44 58 1 44 63 2  
 45 46 1 45 59 1 46 47 1 46 60 1 47 48 1 47 61 1 48 49 1 48 62 1 49 63 1  
 50 51 1 50 52 1 50 54 2 51 53 1 52 53 1 52 55 1 53 54 1 53 56 1 53 58 2  
 54 57 1 55 56 1 55 59 1 56 57 1 56 60 1 57 58 1 57 61 1 57 63 2 58 62 1  
 59 60 1 60 61 1 61 62 1 62 63 1

Matrix size = 35, reduced Matrix size = 1

0 1 1 1 2 1 1 4 1 2 3 1 2 5 1 3 6 1 4 5 1 4 10 1 5 6 1 5 7 1 5 11 1  
 6 8 1 6 12 1 7 8 1 7 13 1 8 9 1 8 14 1 9 15 1 10 11 1 10 20 1  
 11 12 1 11 13 1 11 21 1 12 14 1 12 22 1 13 14 1 13 16 1 13 23 1 14 15 1 14 17 1 14 24 1  
 15 18 1 15 25 1 16 17 1 16 26 1 17 18 1 17 27 1 18 19 1 18 28 1 19 29 1 20 21 1  
 21 22 1 21 23 1 22 24 1 23 24 1 23 26 1 24 25 1 24 27 1 25 28 1 26 27 1 26 30 1  
 27 28 1 27 31 1 28 29 1 28 32 1 29 33 1 30 31 1 31 32 1 32 33 1 33 34 1

Matrix size = 14, reduced Matrix size = 1

0 1 1 0 2 1 0 4 2 0 5 1 0 10 2 0 12 3 1 3 1 1 6 1 1 11 2 2 3 1 2 7 1 2 13 4  
 3 4 1 3 8 1 4 9 1 5 6 1 5 7 1 5 9 2 6 8 1 7 8 1 7 10 1 8 9 1 8 11 1 8 13 2  
 9 12 1 10 11 1 11 12 1 12 13 1

Matrix size = 70, reduced Matrix size = 4

0 1 1 0 4 2 0 5 1 0 8 2 0 21 1 0 37 2 0 40 3 1 2 1 1 3 1 1 6 1 1 22 1 1 38 2  
 2 4 1 2 7 1 2 20 3 2 23 1 2 39 2 3 4 1 3 8 1 3 10 2 3 11 1 3 18 2 3 20 3 3 24 1  
 3 40 2 3 42 3 3 47 2 4 9 1 4 12 1 4 19 2 4 25 1 4 41 2 4 48 2  
 5 6 1 5 12 2 5 26 1 5 43 4 6 7 1 6 8 1 6 11 1 6 27 1 7 9 1 7 12 1 7 15 2 7 28 1  
 8 9 1 8 13 1 8 29 1 9 10 1 9 14 1 9 30 1 10 15 1 10 31 1  
 11 12 1 11 13 1 11 15 2 11 16 1 11 32 1 11 43 3 11 45 4 12 14 1 12 17 1 12 33 1 12 44 3

13 14 1 13 18 1 13 34 1 13 46 5 14 15 1 14 19 1 14 35 1 15 20 1 15 36 1  
 16 17 1 16 18 1 16 20 2 16 47 1 16 52 2 16 54 3 16 56 1 17 19 1 17 48 1 17 53 2 17 57 1  
 18 19 1 18 49 1 18 55 4 18 58 1 19 20 1 19 50 1 19 59 1 20 51 1 20 60 1  
 21 22 1 21 25 2 21 26 1 21 29 2 22 23 1 22 24 1 22 27 1 23 25 1 23 28 1 23 60 3  
 24 25 1 24 29 1 24 31 2 24 32 1 24 58 2 24 60 3 25 30 1 25 33 1 25 59 2  
 26 27 1 26 33 2 26 37 1 27 28 1 27 29 1 27 32 1 27 38 1 27 43 2  
 28 30 1 28 33 1 28 36 2 28 39 1 28 44 2 29 30 1 29 34 1 29 40 1  
 30 31 1 30 35 1 30 41 1 31 36 1 31 42 1 31 69 3 32 33 1 32 34 1 32 36 2 32 47 1 32 56 1  
 33 35 1 33 48 1 33 57 1 34 35 1 34 43 1 34 49 1 34 58 1 34 66 2  
 35 36 1 35 44 1 35 46 2 35 50 1 35 59 1 35 67 2 35 69 3 36 45 1 36 51 1 36 60 1 36 68 2  
 37 38 1 37 48 2 38 39 1 38 40 1 38 47 1 39 41 1 39 48 1 39 51 2  
 40 41 1 40 43 1 40 49 1 41 42 1 41 44 1 41 50 1 42 45 1 42 51 1 42 55 2  
 43 44 1 43 52 1 44 45 1 44 53 1 45 46 1 45 54 1 46 55 1 47 48 1 47 49 1 47 51 2 47 61 1  
 48 50 1 48 62 1 49 50 1 49 52 1 49 63 1 50 51 1 50 53 1 50 55 2 50 64 1  
 51 54 1 51 65 1 52 53 1 52 66 1 53 54 1 53 67 1 54 55 1 54 68 1 55 69 1  
 56 57 1 56 58 1 56 60 2 56 61 1 56 66 2 56 68 3 57 59 1 57 62 1 57 67 2  
 58 59 1 58 63 1 58 69 4 59 60 1 59 64 1 60 65 1 61 62 1 61 63 1 61 65 2  
 62 64 1 63 64 1 63 66 1 64 65 1 64 67 1 64 69 2 65 68 1 66 67 1 67 68 1 68 69 1

Matrix size = 56, reduced Matrix size = 3

0 1 1 0 4 2 0 5 1 0 17 3 0 18 3 1 2 1 1 3 1 1 6 1 1 16 2 2 4 1 2 7 1 2 17 2  
 3 4 1 3 8 1 4 9 1 5 6 1 5 9 2 5 10 1 5 13 2 5 21 1 6 7 1 6 8 1 6 11 1 6 22 1  
 7 9 1 7 12 1 7 23 1 8 9 1 8 13 1 8 15 2 8 16 1 8 24 1 8 47 2  
 9 14 1 9 17 1 9 25 1 9 48 2 10 11 1 10 17 2 10 26 1 10 48 3  
 11 12 1 11 13 1 11 16 1 11 27 1 11 47 2 11 52 3  
 12 14 1 12 17 1 12 20 2 12 28 1 12 48 2 12 51 3 12 53 3  
 13 14 1 13 18 1 13 29 1 13 49 2 14 15 1 14 19 1 14 30 1 14 50 2  
 15 20 1 15 31 1 15 51 2 16 17 1 16 18 1 16 20 2 16 32 1  
 17 19 1 17 33 1 18 19 1 18 34 1 19 20 1 19 35 1 20 36 1  
 21 22 1 21 25 2 21 26 1 21 29 2 22 23 1 22 24 1 22 27 1 23 25 1 23 28 1  
 24 25 1 24 29 1 24 31 2 24 32 1 25 30 1 25 33 1 26 27 1 26 33 2 26 37 1  
 27 28 1 27 29 1 27 32 1 27 38 1 27 43 2 28 30 1 28 33 1 28 36 2 28 39 1 28 44 2  
 29 30 1 29 34 1 29 40 1 30 31 1 30 35 1 30 41 1 31 36 1 31 42 1  
 32 33 1 32 34 1 32 36 2 32 47 1 33 35 1 33 48 1 34 35 1 34 43 1 34 49 1  
 35 36 1 35 44 1 35 46 2 35 50 1 36 45 1 36 51 1 37 38 1 37 48 2  
 38 39 1 38 40 1 38 47 1 39 41 1 39 48 1 39 51 2 40 41 1 40 43 1 40 49 1  
 41 42 1 41 44 1 41 50 1 42 45 1 42 51 1 42 55 2 43 44 1 43 52 1  
 44 45 1 44 53 1 45 46 1 45 54 1 46 55 1 47 48 1 47 49 1 47 51 2  
 48 50 1 49 50 1 49 52 1 50 51 1 50 53 1 50 55 2 51 54 1 52 53 1 53 54 1 54 55 1

Matrix size = 90, reduced Matrix size = 5

0 1 1 0 5 2 0 9 1 0 13 2 1 2 1 1 4 1 1 10 1 2 3 1 2 5 1 2 11 1  
 3 6 1 3 12 1 4 5 1 4 8 2 4 13 1 4 16 2 4 19 1 5 6 1 5 7 1 5 14 1 5 20 1  
 6 8 1 6 15 1 6 21 1 7 8 1 7 16 1 7 18 2 7 22 1 8 17 1 8 23 1  
 9 10 1 9 20 2 9 35 1 10 11 1 10 13 1 10 19 1 10 36 1 10 45 2  
 11 12 1 11 14 1 11 20 1 11 27 2 11 37 1 11 46 2 12 15 1 12 21 1 12 28 2 12 38 1 12 47 2  
 13 14 1 13 24 1 13 39 1 14 15 1 14 16 1 14 25 1 14 40 1 15 17 1 15 26 1 15 41 1  
 16 17 1 16 27 1 16 42 1 17 18 1 17 28 1 17 43 1 18 29 1 18 44 1  
 19 20 1 19 23 2 19 24 1 19 27 2 19 55 1 20 21 1 20 22 1 20 25 1 20 56 1

21 23 1 21 26 1 21 57 1 22 23 1 22 27 1 22 29 2 22 30 1 22 58 1 23 28 1 23 31 1 23 59 1  
 24 25 1 24 31 2 24 45 1 24 60 1 25 26 1 25 27 1 25 30 1 25 46 1 25 51 2 25 61 1  
 26 28 1 26 31 1 26 34 2 26 47 1 26 52 2 26 62 1 27 28 1 27 32 1 27 48 1 27 63 1  
 28 29 1 28 33 1 28 49 1 28 64 1 29 34 1 29 50 1 29 65 1 30 31 1 30 32 1 30 34 2 30 66 1  
 31 33 1 31 67 1 32 33 1 32 51 1 32 68 1 33 34 1 33 52 1 33 54 2 33 69 1  
 34 53 1 34 70 1 35 36 1 35 56 2 36 37 1 36 39 1 36 55 1 37 38 1 37 40 1 37 56 1 37 63 2  
 38 41 1 38 57 1 38 64 2 39 40 1 39 45 1 39 60 1 40 41 1 40 42 1 40 46 1 40 61 1  
 41 43 1 41 47 1 41 62 1 42 43 1 42 48 1 42 63 1 42 77 2 43 44 1 43 49 1 43 64 1 43 78 2  
 44 50 1 44 65 1 44 79 2 45 46 1 45 71 1 46 47 1 46 48 1 46 72 1 47 49 1 47 73 1  
 48 49 1 48 51 1 48 74 1 49 50 1 49 52 1 49 75 1 50 53 1 50 76 1 51 52 1 51 77 1  
 52 53 1 52 78 1 53 54 1 53 79 1 54 80 1 55 56 1 55 59 2 55 60 1 55 63 2  
 56 57 1 56 58 1 56 61 1 57 59 1 57 62 1 58 59 1 58 63 1 58 65 2 58 66 1  
 59 64 1 59 67 1 60 61 1 60 67 2 60 71 1 61 62 1 61 63 1 61 66 1 61 72 1 61 77 2  
 62 64 1 62 67 1 62 70 2 62 73 1 62 78 2 63 64 1 63 68 1 63 74 1 64 65 1 64 69 1 64 75 1  
 65 70 1 65 76 1 66 67 1 66 68 1 66 70 2 66 81 1 67 69 1 67 82 1 68 69 1 68 77 1 68 83 1  
 69 70 1 69 78 1 69 80 2 69 84 1 70 79 1 70 85 1 71 72 1 71 82 2 72 73 1 72 74 1 72 81 1  
 73 75 1 73 82 1 73 85 2 74 75 1 74 77 1 74 83 1 75 76 1 75 78 1 75 84 1  
 76 79 1 76 85 1 76 89 2 77 78 1 77 86 1 78 79 1 78 87 1 79 80 1 79 88 1  
 80 89 1 81 82 1 81 83 1 81 85 2 82 84 1 83 84 1 83 86 1 84 85 1 84 87 1 84 89 2  
 85 88 1 86 87 1 87 88 1 88 89 1

Matrix size = 35, reduced Matrix size = 2

0 1 1 1 2 1 1 5 1 2 3 1 2 6 1 3 4 1 3 7 1 4 8 1 5 6 1 5 15 1  
 6 7 1 6 9 1 6 16 1 7 8 1 7 10 1 7 17 1 8 11 1 8 18 1 9 10 1 9 19 1  
 10 11 1 10 12 1 10 20 1 11 13 1 11 21 1 12 13 1 12 22 1 13 14 1 13 23 1  
 14 24 1 15 16 1 16 17 1 16 19 1 17 18 1 17 20 1 18 21 1 19 20 1 19 25 1  
 20 21 1 20 22 1 20 26 1 21 23 1 21 27 1 22 23 1 22 28 1 23 24 1 23 29 1  
 24 30 1 25 26 1 26 27 1 26 28 1 27 29 1 28 29 1 28 31 1 29 30 1 29 32 1  
 30 33 1 31 32 1 32 33 1 33 34 1

Matrix size = 42, reduced Matrix size = 3

0 1 1 0 4 2 0 5 1 0 17 3 0 18 3 0 26 2 0 29 3 0 35 5  
 1 2 1 1 3 1 1 6 1 1 16 2 1 27 2 1 36 4 2 4 1 2 7 1 2 17 2 2 28 2  
 3 4 1 3 8 1 3 29 2 3 31 3 4 9 1 4 30 2 5 6 1 5 9 2 5 10 1 5 13 2 5 21 1 5 40 4  
 6 7 1 6 8 1 6 11 1 6 22 1 6 41 5 7 9 1 7 12 1 7 23 1 8 9 1 8 13 1 8 15 2 8 16 1 8 24 1  
 9 14 1 9 17 1 9 25 1 10 11 1 10 17 2 10 26 1 10 38 3 11 12 1 11 13 1 11 16 1 11 27 1 11 37 2  
 12 14 1 12 17 1 12 20 2 12 28 1 12 38 2 12 41 3 13 14 1 13 18 1 13 29 1 13 39 2  
 14 15 1 14 19 1 14 30 1 14 40 2 15 20 1 15 31 1 15 41 2 16 17 1 16 18 1 16 20 2 16 32 1  
 17 19 1 17 33 1 18 19 1 18 34 1 19 20 1 19 35 1 20 36 1 21 22 1 21 25 2 21 26 1 21 29 2  
 22 23 1 22 24 1 22 27 1 23 25 1 23 28 1 23 41 3 24 25 1 24 29 1 24 31 2 24 32 1 24 39 2 24 41 3  
 25 30 1 25 33 1 25 40 2 26 27 1 26 33 2 27 28 1 27 29 1 27 32 1 28 30 1 28 33 1 28 36 2  
 29 30 1 29 34 1 30 31 1 30 35 1 31 36 1 32 33 1 32 34 1 32 36 2 32 37 1  
 33 35 1 33 38 1 34 35 1 34 39 1 35 36 1 35 40 1 36 41 1 37 38 1 37 39 1 37 41 2  
 38 40 1 39 40 1 40 41 1

Matrix size = 56, reduced Matrix size = 4

0 1 1 0 5 2 0 9 1 0 13 2 1 2 1 1 4 1 1 10 1 2 3 1 2 5 1 2 11 1 2 43 3  
 3 6 1 3 12 1 3 44 3 4 5 1 4 8 2 4 13 1 4 16 2 4 19 1 4 40 2 4 43 3  
 5 6 1 5 7 1 5 14 1 5 20 1 5 41 2 6 8 1 6 15 1 6 21 1 6 42 2  
 7 8 1 7 16 1 7 18 2 7 22 1 7 43 2 7 45 3 7 46 2 8 17 1 8 23 1 8 44 2 8 47 2



9 10 1 9 20 2 10 11 1 10 13 1 10 19 1 11 12 1 11 14 1 11 20 1 11 27 2  
 12 15 1 12 21 1 12 28 2 13 14 1 13 24 1 14 15 1 14 16 1 14 25 1 15 17 1 15 26 1  
 16 17 1 16 27 1 17 18 1 17 28 1 18 29 1 19 20 1 19 23 2 19 24 1 19 27 2 19 35 1  
 20 21 1 20 22 1 20 25 1 20 36 1 21 23 1 21 26 1 21 37 1 22 23 1 22 27 1 22 29 2 22 30 1 22 38 1  
 23 28 1 23 31 1 23 39 1 24 25 1 24 31 2 24 40 1 25 26 1 25 27 1 25 30 1 25 41 1  
 26 28 1 26 31 1 26 34 2 26 42 1 27 28 1 27 32 1 27 43 1 28 29 1 28 33 1 28 44 1  
 29 34 1 29 45 1 30 31 1 30 32 1 30 34 2 30 46 1 31 33 1 31 47 1 32 33 1 32 48 1  
 33 34 1 33 49 1 34 50 1 35 36 1 35 39 2 35 40 1 35 43 2 36 37 1 36 38 1 36 41 1  
 37 39 1 37 42 1 37 55 3 38 39 1 38 43 1 38 45 2 38 46 1 38 53 2 38 55 3  
 39 44 1 39 47 1 39 54 2 40 41 1 40 47 2 41 42 1 41 43 1 41 46 1 42 44 1 42 47 1 42 50 2  
 43 44 1 43 48 1 44 45 1 44 49 1 45 50 1 46 47 1 46 48 1 46 50 2 46 51 1  
 47 49 1 47 52 1 48 49 1 48 53 1 49 50 1 49 54 1 50 55 1 51 52 1 51 53 1 51 55 2  
 52 54 1 53 54 1 54 55 1

Matrix size = 70, reduced Matrix size = 5

0 1 1 0 5 2 0 11 4 0 14 1 0 34 3 0 38 3 1 2 1 1 4 1 1 13 3 1 15 1 1 33 2  
 2 3 1 2 5 1 2 12 2 2 16 1 2 34 2 3 6 1 3 13 2 3 17 1 3 35 2  
 4 5 1 4 8 2 4 9 1 4 18 1 5 6 1 5 7 1 5 10 1 5 19 1 6 8 1 6 11 1 6 20 1  
 7 8 1 7 12 1 7 21 1 8 13 1 8 22 1 9 10 1 9 13 2 9 33 1 9 45 3 9 46 3 9 49 1  
 10 11 1 10 12 1 10 34 1 10 44 2 10 50 1 11 13 1 11 35 1 11 45 2 11 51 1  
 12 13 1 12 36 1 12 52 1 13 37 1 13 53 1 14 15 1 14 19 2 14 23 1 14 27 2 14 51 4  
 15 16 1 15 18 1 15 24 1 15 53 3 16 17 1 16 19 1 16 25 1 16 52 2  
 17 20 1 17 26 1 17 53 2 18 19 1 18 22 2 18 27 1 18 30 2 18 33 1 18 49 1  
 19 20 1 19 21 1 19 28 1 19 34 1 19 50 1 20 22 1 20 29 1 20 35 1 20 51 1  
 21 22 1 21 30 1 21 32 2 21 36 1 21 52 1 21 65 2 22 31 1 22 37 1 22 53 1 22 66 2  
 23 24 1 23 34 2 23 56 5 24 25 1 24 27 1 24 33 1 24 58 4  
 25 26 1 25 28 1 25 34 1 25 41 2 25 57 3 26 29 1 26 35 1 26 42 2 26 58 3 26 64 4  
 27 28 1 27 38 1 27 66 3 28 29 1 28 30 1 28 39 1 28 65 2 29 31 1 29 40 1 29 66 2 29 69 3  
 30 31 1 30 41 1 30 67 2 31 32 1 31 42 1 31 68 2 32 43 1 32 69 2  
 33 34 1 33 37 2 33 38 1 33 41 2 33 54 1 34 35 1 34 36 1 34 39 1 34 55 1  
 35 37 1 35 40 1 35 56 1 36 37 1 36 41 1 36 43 2 36 44 1 36 57 1  
 37 42 1 37 45 1 37 58 1 38 39 1 38 45 2 38 59 1 39 40 1 39 41 1 39 44 1 39 60 1  
 40 42 1 40 45 1 40 48 2 40 61 1 41 42 1 41 46 1 41 62 1 42 43 1 42 47 1 42 63 1  
 43 48 1 43 64 1 44 45 1 44 46 1 44 48 2 44 65 1 45 47 1 45 66 1 46 47 1 46 67 1  
 47 48 1 47 68 1 48 69 1 49 50 1 49 53 2 49 54 1 49 66 3 49 67 3  
 50 51 1 50 52 1 50 55 1 50 65 2 51 53 1 51 56 1 51 66 2 52 53 1 52 57 1  
 53 58 1 54 55 1 54 58 2 54 59 1 54 62 2 55 56 1 55 57 1 55 60 1 56 58 1 56 61 1  
 57 58 1 57 62 1 57 64 2 57 65 1 58 63 1 58 66 1 59 60 1 59 66 2 60 61 1 60 62 1 60 65 1  
 61 63 1 61 66 1 61 69 2 62 63 1 62 67 1 63 64 1 63 68 1 64 69 1  
 65 66 1 65 67 1 65 69 2 66 68 1 67 68 1 68 69 1

Matrix size = 64, reduced Matrix size = 4

0 1 1 0 6 2 0 14 1 0 19 2 1 2 1 1 5 1 1 15 1 2 3 1 2 6 1 2 16 1  
 3 4 1 3 7 1 3 17 1 4 8 1 4 18 1 5 6 1 5 10 2 5 19 1 5 23 2 5 29 1  
 6 7 1 6 9 1 6 20 1 6 30 1 7 8 1 7 10 1 7 21 1 7 31 1 8 11 1 8 22 1 8 32 1  
 9 10 1 9 13 2 9 23 1 9 26 2 9 33 1 10 11 1 10 12 1 10 24 1 10 34 1  
 11 13 1 11 25 1 11 35 1 12 13 1 12 26 1 12 28 2 12 36 1 13 27 1 13 37 1  
 14 15 1 14 30 2 15 16 1 15 19 1 15 29 1 16 17 1 16 20 1 16 30 1 16 42 2  
 17 18 1 17 21 1 17 31 1 17 43 2 18 22 1 18 32 1 18 44 2

19 20 1 19 38 1 20 21 1 20 23 1 20 39 1 21 22 1 21 24 1 21 40 1 22 25 1 22 41 1  
 23 24 1 23 42 1 24 25 1 24 26 1 24 43 1 25 27 1 25 44 1 26 27 1 26 45 1  
 27 28 1 27 46 1 28 47 1 29 30 1 29 34 2 29 38 1 29 42 2 30 31 1 30 33 1 30 39 1  
 31 32 1 31 34 1 31 40 1 32 35 1 32 41 1 33 34 1 33 37 2 33 42 1 33 45 2 33 48 1  
 34 35 1 34 36 1 34 43 1 34 49 1 35 37 1 35 44 1 35 50 1 36 37 1 36 45 1 36 47 2 36 51 1  
 37 46 1 37 52 1 38 39 1 38 49 2 39 40 1 39 42 1 39 48 1 40 41 1 40 43 1 40 49 1 40 56 2  
 41 44 1 41 50 1 41 57 2 42 43 1 42 53 1 43 44 1 43 45 1 43 54 1 44 46 1 44 55 1  
 45 46 1 45 56 1 46 47 1 46 57 1 47 58 1 48 49 1 48 52 2 48 53 1 48 56 2  
 49 50 1 49 51 1 49 54 1 50 52 1 50 55 1 51 52 1 51 56 1 51 58 2 51 59 1  
 52 57 1 52 60 1 53 54 1 53 60 2 54 55 1 54 56 1 54 59 1 55 57 1 55 60 1 55 63 2  
 56 57 1 56 61 1 57 58 1 57 62 1 58 63 1 59 60 1 59 61 1 59 63 2 60 62 1 61 62 1 62 63 1

Matrix size = 21, reduced Matrix size = 1

0 1 1 1 2 1 1 6 1 2 3 1 2 7 1 3 4 1 3 8 1 4 5 1 4 9 1 5 10 1 6 7 1  
 7 8 1 7 11 1 8 9 1 8 12 1 9 10 1 9 13 1 10 14 1 11 12 1 12 13 1 12 15 1  
 13 14 1 13 16 1 14 17 1 15 16 1 16 17 1 16 18 1 17 19 1 18 19 1 19 20 1

Matrix size = 14, reduced Matrix size = 1

0 1 1 0 5 2 0 11 4 1 2 1 1 4 1 1 13 3 2 3 1 2 5 1 2 12 2 3 6 1 3 13 2  
 4 5 1 4 8 2 4 9 1 5 6 1 5 7 1 5 10 1 6 8 1 6 11 1 7 8 1 7 12 1 8 13 1  
 9 10 1 9 13 2 10 11 1 10 12 1 11 13 1 12 13 1

Matrix size = 28, reduced Matrix size = 2

0 1 1 0 6 2 0 16 4 1 2 1 1 5 1 1 19 3 2 3 1 2 6 1 2 18 2 3 4 1 3 7 1 3 19 2  
 4 8 1 4 20 2 5 6 1 5 10 2 5 14 1 6 7 1 6 9 1 6 15 1 7 8 1 7 10 1 7 16 1  
 8 11 1 8 17 1 9 10 1 9 13 2 9 18 1 10 11 1 10 12 1 10 19 1 11 13 1 11 20 1  
 12 13 1 12 21 1 13 22 1 14 15 1 14 19 2 14 25 4 15 16 1 15 18 1 15 27 3  
 16 17 1 16 19 1 16 26 2 17 20 1 17 27 2 18 19 1 18 22 2 18 23 1 19 20 1 19 21 1 19 24 1  
 20 22 1 20 25 1 21 22 1 21 26 1 22 27 1 23 24 1 23 27 2 24 25 1 24 26 1 25 27 1 26 27 1

Matrix size = 20, reduced Matrix size = 1

0 1 1 0 7 2 1 2 1 1 6 1 2 3 1 2 7 1 3 4 1 3 8 1 4 5 1 4 9 1 5 10 1  
 6 7 1 6 12 2 7 8 1 7 11 1 8 9 1 8 12 1 9 10 1 9 13 1 10 14 1 11 12 1 11 16 2  
 12 13 1 12 15 1 13 14 1 13 16 1 14 17 1 15 16 1 15 19 2 16 17 1 16 18 1 17 19 1 18 19 1

Matrix size = 7, reduced Matrix size = 0

0 1 1 1 2 1 2 3 1 3 4 1 4 5 1 5 6 1

```

#include <stdio.h>
#define max_rep_size 200
#define max_matrix_size 25000
#define size_tableau 200
#define max_index 100
#define red_matrix_size 500
typedef struct wptr {
    short index;
    short a2;
    struct wptr *next;
} WPTR;
char *tableau[size_tableau];
char *local_alpha, *alpha[max_rep_size];
unsigned int *local_I_set, *I_set[max_rep_size];
WPTR *incident_matrix[max_matrix_size];
int g_l, g_bl, g_num_of_rep, g_pos, g_I_pos, g_num_vertex;
int g_matrix_size[size_tableau];
int real_max_matrix_size;
int g_depth[size_tableau], max_depth, dim;
int g_red_entry[red_matrix_size];
int g_WI = 0;
FILE *g_fp_Y;

int I_search_number(j, i)
int j, i;
{
    register int r, len;
    len = g_depth[j+1];
    for (r = 0; r < len; r++) if (i == (int)(tableau[r][j] - '@')) return(r);
    return(-1);
}

int get_number(i, k)
int i, k;
{
    register int p, q;
    for (p = 0; p <= g_num_of_rep; p++)
        for (q = k; q < g_depth[p+1]; q++) if (i == (int)(tableau[q][p] - '@')) return(1);
    return(0);
}

void I_set_bit(i)
int i;
{
    switch (i) {
    case 1:
        local_I_set[g_I_pos] |= 0x0001; break;
    case 2:
        local_I_set[g_I_pos] |= 0x0002; break;
    }
}

```

```

case 3:
    local_I_set[g_I_pos] |= 0x0004; break;
case 4:
    local_I_set[g_I_pos] |= 0x0008; break;
case 5:
    local_I_set[g_I_pos] |= 0x0010; break;
case 6:
    local_I_set[g_I_pos] |= 0x0020; break;
case 7:
    local_I_set[g_I_pos] |= 0x0040; break;
case 8:
    local_I_set[g_I_pos] |= 0x0080; break;
case 9:
    local_I_set[g_I_pos] |= 0x0100; break;
case 10:
    local_I_set[g_I_pos] |= 0x0200; break;
case 11:
    local_I_set[g_I_pos] |= 0x0400; break;
case 12:
    local_I_set[g_I_pos] |= 0x0800; break;
case 13:
    local_I_set[g_I_pos] |= 0x1000; break;
case 14:
    local_I_set[g_I_pos] |= 0x2000; break;
case 15:
    local_I_set[g_I_pos] |= 0x4000; break;
}
}
int get_class_number(i, p)
int i, p;
{
    switch (i) {
    case 1:
        if (local_I_set[p] & 0x0001) return(1); break;
    case 2:
        if (local_I_set[p] & 0x0002) return(2); break;
    case 3:
        if (local_I_set[p] & 0x0004) return(3); break;
    case 4:
        if (local_I_set[p] & 0x0008) return(4); break;
    case 5:
        if (local_I_set[p] & 0x0010) return(5); break;
    case 6:
        if (local_I_set[p] & 0x0020) return(6); break;
    case 7:
        if (local_I_set[p] & 0x0040) return(7); break;
    case 8:

```

```

    if (local_I_set[p] & 0x0080) return(8); break;
case 9:
    if (local_I_set[p] & 0x0100) return(9); break;
case 10:
    if (local_I_set[p] & 0x0200) return(10); break;
case 11:
    if (local_I_set[p] & 0x0400) return(11); break;
case 12:
    if (local_I_set[p] & 0x0800) return(12); break;
case 13:
    if (local_I_set[p] & 0x1000) return(13); break;
case 14:
    if (local_I_set[p] & 0x2000) return(14); break;
case 15:
    if (local_I_set[p] & 0x4000) return(15); break;
}
return(0);
}
void I_complement(I_class)
char *I_class;
{
    register int i, j;
    for (i = 1; i < g_bl; i++) {
        for (j = 0; I_class[j] != '@'; j++)
            if (i == (int)(I_class[j] - '@')) I_set_bit(i);
    }
}
void I_tableau()
{
    register int i, j, k, pos;
    char I_class[max_rep_size];
    pos = 0;
    if (g_num_of_rep == 0) {
        for (i = 1; i < g_bl; i++) I_class[pos++] = (char)(i + 64);
        I_class[pos] = '@';
    }
    else {
        for (i = 1; i < g_bl; i++) {
            for (j = 0; j <= g_num_of_rep; j++) {
                if ((k = I_search_number(j, i)) >= 0) {
                    if (get_number(i+1, k+1)) {
                        I_class[pos++] = (char)(i + 64);
                        break;
                    }
                }
            }
        }
    }
}
}

```

```

    I_class[pos] = '@';
}
local_I_set[g_I_pos] = 0; /* bit clear */
I_complement(I_class); g_I_pos++;
}
void alpha_tableau()
{
    register int i, j;
    if (g_num_of_rep == 0) {
        for (i = 0; i < g_depth[1]; i++) local_alpha[g_pos++] = tableau[i][0];
        g_matrix_size[0]++;
    } else {
        for (i = g_l; i > 0; i--)
            for (j = 0; j < g_depth[i]; j++) local_alpha[g_pos++] = tableau[j][i-1];
        g_matrix_size[g_num_of_rep]++;
    }
    if (g_pos % g_bl != 0) {
        printf("mismatch in [alpha_tableau].%n");
        exit(0);
    }
}
void young_tableau(bl, depth)
int bl, depth[];
{
    int i, j;
    char temp;
    if (bl == 1) {
        tableau[0][0] = 'A'; alpha_tableau(); I_tableau();
        if (g_WI) {
            printf("x%01d = [", ++g_num_vertex);
            for (i = 1; i < g_bl; i++) {
                j = get_class_number(i, g_I_pos-1);
                if (j) printf("%01d ", j);
            }
            if (g_num_vertex % 5) printf(" ");
            else printf("]n");
        }
    } else {
        for (i = 0; depth[i] > 0; i++) {
            if (depth[i] > depth[i+1]) {
                temp = tableau[depth[i]-1][i];
                tableau[depth[i]-1][i] = (char)(bl + 64);
                depth[i]--;
                young_tableau(bl-1, depth);
                depth[i]++;
                tableau[depth[i]-1][i] = temp;
            }
        }
    }
}

```

```

int check_identity(w1, w2)
char w1[], w2[];
{
    register int i;
    i = 0;
    while (i < g_bl) {
        if (w1[i] != w2[i]) return(0);
        else i++;
    }
    return(1);
}

int phase1_check(w1, w2)
char w1[], w2[];
{
    register int i, j;
    for (i = 0; i < g_bl-2; i++) { /* 93, 4/30 */
        for (j = i+2; j < g_bl; j++) {
            if (pair_between(w1, i, j)) {
                ch_exchange(&w1[i], &w1[j]);
                if (check_identity(w1, w2)) return(1);
                ch_exchange(&w1[i], &w1[j]);
            }
        }
    }
    return(0);
}

int search_number_position(i, w)
int i;
char w[];
{
    register int j;
    for (j = 0; j < g_bl; j++)
        if (i == (int)(w[j]-'@')) return(j);
    printf("An error has happened in [search_number_position].%n");
    exit(0);
}

int search_word_position(cp, w)
int cp;
char w[];
{
    register int i, k, p;
    for (i = 0; i < g_matrix_size[cp]; i++) {
        for (p = 0, k = g_bl*i; k < g_bl*(i+1); k++, p++) {
            if (w[p] != alpha[cp][k]) {
                p = 0;
                break;
            }
        }
    }
}

```

```

    }
    if (p > 0) return(i);
}
printf("An error happened in [search_word_position].\n");
exit(0);
}
void set_W_graph_entry(i, j, mp)
int i, j, mp;
{
    WPTR *temp, *inf;
    if (j < i) exchange(&i, &j);
    temp = incident_matrix[i];
    if (temp == NULL) {
        incident_matrix[i] = (WPTR *)malloc(sizeof(WPTR));
        incident_matrix[i]->index = mp;
        incident_matrix[i]->a2 = j;
        incident_matrix[i]->next = NULL;
    }
    else {
        if (j < temp->a2) {
            temp = incident_matrix[i]->next;
            incident_matrix[i] = (WPTR *)malloc(sizeof(WPTR));
            incident_matrix[i]->index = mp;
            incident_matrix[i]->a2 = j;
            incident_matrix[i]->next = temp;
            return;
        }
        while (temp != NULL) {
            if (temp->a2 == j) {
                temp->index = mp;
                return;
            }
            else if (temp->a2 < j) {
                if (temp->next == NULL) {
                    temp->next = (WPTR *)malloc(sizeof(WPTR));
                    temp->next->index = mp;
                    temp->next->a2 = j;
                    temp->next->next = NULL;
                    return;
                }
            }
            else if (j < temp->next->a2) {
                inf = temp->next;
                temp->next = (WPTR *)malloc(sizeof(WPTR));
                temp->next->index = mp;
                temp->next->a2 = j;
                temp->next->next = inf;
                return;
            }
        }
    }
}

```



```

    }
    }
    temp = temp->next;
}
}
void phase2_check(cp, mp, w1, w2)
int cp, mp;
char w1[], w2[];
{
    register int i;
    int p1, p2, p3, q1, q2, q3, s1, s2;

    for (i = 1; i <= g_bl-2; i++) {
        p1 = search_number_position(i, w1);
        p2 = search_number_position(i+1, w1);
        p3 = search_number_position(i+2, w1);
        if ((p1 < p2 && p2 < p3) || (p3 < p2 && p2 < p1)) ;
        else {
            q1 = search_number_position(i, w2);
            q2 = search_number_position(i+1, w2);
            q3 = search_number_position(i+2, w2);
            if ((q1 < q2 && q2 < q3) || (q3 < q2 && q2 < q1)) ;
            else {
                if ((p1 < p3 && p3 < p2) || (p2 < p3 && p3 < p1))
                    ch_exchange(&w1[p1], &w1[p2]);
                else if ((p2 < p1 && p1 < p3) || (p3 < p1 && p1 < p2))
                    ch_exchange(&w1[p2], &w1[p3]);

                if ((q1 < q3 && q3 < q2) || (q2 < q3 && q3 < q1))
                    ch_exchange(&w2[q1], &w2[q2]);
                else if ((q2 < q1 && q1 < q3) || (q3 < q1 && q1 < q2))
                    ch_exchange(&w2[q2], &w2[q3]);

                s1 = search_word_position(cp, w1);
                s2 = search_word_position(cp, w2);

                if (!check_W_graph_entry(s1, s2, 0))
                    set_W_graph_entry(s1, s2, mp+1);

                if ((p1 < p3 && p3 < p2) || (p2 < p3 && p3 < p1))
                    ch_exchange(&w1[p1], &w1[p2]);
                else if ((p2 < p1 && p1 < p3) || (p3 < p1 && p1 < p2))
                    ch_exchange(&w1[p2], &w1[p3]);

                if ((q1 < q3 && q3 < q2) || (q2 < q3 && q3 < q1))
                    ch_exchange(&w2[q1], &w2[q2]);
            }
        }
    }
}

```

```

        else if ((q2 < q1 && q1 < q3) || (q3 < q1 && q1 < q2))
            ch_exchange(&w2[q2], &w2[q3]);
    }
}
}
}
void make_W_graph1(cp, i, j)
int    cp, i, j;
{
    register int    k, p;
    char            word1[size_tableau], word2[size_tableau];
    for (p = 0, k = g_bl*i; k < g_bl*(i+1); k++, p++) word1[p] = alpha[cp][k];
    for (p = 0, k = g_bl*j; k < g_bl*(j+1); k++, p++) word2[p] = alpha[cp][k];
    if (phase1_check(word1, word2)) set_W_graph_entry(i, j, 1);
}
void make_W_graph2(cp, mp, i, j)
int    cp, mp, i, j;
{
    register int    k, p;
    char            word1[size_tableau], word2[size_tableau];
    for (p = 0, k = g_bl*i; k < g_bl*(i+1); k++, p++) word1[p] = alpha[cp][k];
    for (p = 0, k = g_bl*j; k < g_bl*(j+1); k++, p++) word2[p] = alpha[cp][k];
    phase2_check(cp, mp, word1, word2);
}
int inclusion(k, j, m)
int    k, j, m;
{
    int    res;
    switch (j) {
    case 1:
        res = I_set[k][m] & 0x0001; break;
    case 2:
        res = I_set[k][m] & 0x0002; break;
    case 3:
        res = I_set[k][m] & 0x0004; break;
    case 4:
        res = I_set[k][m] & 0x0008; break;
    case 5:
        res = I_set[k][m] & 0x0010; break;
    case 6:
        res = I_set[k][m] & 0x0020; break;
    case 7:
        res = I_set[k][m] & 0x0040; break;
    case 8:
        res = I_set[k][m] & 0x0080; break;
    case 9:
        res = I_set[k][m] & 0x0100; break;
    }
}

```

```

case 10:
    res = I_set[k][m] & 0x0200; break;
case 11:
    res = I_set[k][m] & 0x0400; break;
case 12:
    res = I_set[k][m] & 0x0800; break;
case 13:
    res = I_set[k][m] & 0x1000; break;
case 14:
    res = I_set[k][m] & 0x2000; break;
case 15:
    res = I_set[k][m] & 0x4000; break;
    }
return(res);
}
int check_W_graph_entry(i, j, k)
int    i, j, k;
{
    WPTR *temp;
    if (j < i) exchange(&i, &j);
    temp = incident_matrix[i];
    if (temp == NULL) return(0);
    else {
        if (j < temp->a2) return(0);
        while (temp != NULL) {
            if (temp->a2 == j) {
                if (k == 0) return(1);
                else if (temp->index == k) return(k);
                else return(0);
            }
            else if (temp->a2 < j) {
                if (temp->next == NULL) return(0);
                else if (j < temp->next->a2) return(0);
            }
            temp = temp->next;
        }
    }
return(0);
}
void
make_representation(k, s)
int    k, s;
{
    int    j, m, n;
    char  buf[50];
    FILE  *fp_w;
    for (j = 0; j < g_bl-1; j++) {

```

```

sprintf(buf, "R%02d%02d%02d", g_bl, k, j+1);
if ((fp_w = fopen(buf, "w")) == NULL) {
    printf("File %s can not open.\n", buf);
    exit(0);
}
for (n = 0; n < s; n++) {
    for (m = 0; m < n; m++) {
        if (inclusion(k, j+1, m) && !inclusion(k, j+1, n) && check_W_graph_entry(m, n, 0))
            fprintf(fp_w, "%1d %1d 1 ", m, n);
    }
    if (inclusion(k, j+1, n)) fprintf(fp_w, "%1d %1d -1 ", n, n);
    else fprintf(fp_w, "%1d %1d 2 ", n, n);
    for (m = n+1; m < s; m++) {
        if (inclusion(k, j+1, m) && !inclusion(k, j+1, n) && check_W_graph_entry(m, n, 0))
            fprintf(fp_w, "%1d %1d 1 ", m, n);
    }
    fprintf(fp_w, "\n");
}
fprintf(fp_w, "%d %d\n", 9999, 9999);
fclose(fp_w);
}
}
int get_partition(n)
int    n;
{
    int    i;
    int    depth[size_tableau];
    if (n > 0) {
        for (i = g_depth[g_l]; i > 0; i--) {
            g_l++;
            g_depth[g_l] = i;
            get_partition(n-i);
            g_l--;
        }
    } else if (n == 0) {
        printf("\npartition: ");
        for (i = 0; i <= g_bl; i++) {
            if (i < g_l) depth[i] = g_depth[i+1];
            else depth[i] = 0;
            printf(" %d", depth[i]);
        }
        if (depth[max_depth] != 0) printf("\n");
        else {
            for (i = 0; i < max_depth; i++) fprintf(g_fp_Y, " %d", depth[i]);
            fprintf(g_fp_Y, "\n");
            printf("\n");
            g_matrix_size[g_num_of_rep] = g_pos = g_l_pos = g_num_vertex = 0;
        }
    }
}

```

```

young_tableau(g_bl, depth);
alpha[g_num_of_rep] = (char *)malloc((long)sizeof(char)*g_matrix_size[g_num_of_rep]*g_bl);
if (alpha[g_num_of_rep] == NULL) {
    printf("Memory exhausted in [alpha[%d]].\n", g_num_of_rep);
    exit(0);
}
I_set[g_num_of_rep] = (unsigned int *)malloc((long)sizeof(int)*g_matrix_size[g_num_of_rep]);
if (I_set[g_num_of_rep] == NULL) {
    printf("Memory exhausted in [I_set[%d]].\n", g_num_of_rep);
    exit(0);
}
for (i = 0; i < g_matrix_size[g_num_of_rep]*g_bl; i++)
    alpha[g_num_of_rep][i] = local_alpha[i];
for (i = 0; i < g_matrix_size[g_num_of_rep]; i++)
    I_set[g_num_of_rep][i] = local_I_set[i];
if (real_max_matrix_size < g_matrix_size[g_num_of_rep])
    real_max_matrix_size = g_matrix_size[g_num_of_rep];
g_num_of_rep++;
}
}
}
int partition(n)
int n;
{
    g_bl = g_depth[0] = n;
    g_l = g_num_of_rep = 0;
    local_alpha = (char *)malloc((long)sizeof(char)*max_rep_size*max_matrix_size);
    local_I_set = (unsigned int *)malloc((long)sizeof(int)*max_rep_size*max_matrix_size);
    get_partition(n); free(local_alpha); free(local_I_set);
}
void write_W_matrix(k)
int k;
{
    int m;
    WPTR *temp;
    printf("Matrix size = %d, reduced Matrix size = %d\n", g_matrix_size[k], dim);
    if (g_WI < 2) return;
    for (m = 0; m < g_matrix_size[k]; m++) {
        temp = incident_matrix[m];
        while (temp != NULL) {
            /* printf("%2d %2d %2d ", m, temp->a2, temp->index); */
            printf("%2d %2d ", m, temp->a2);
            temp = temp->next;
        }
        printf("\n");
    }
    printf("\n");
}

```

```

}
void match_braid_index(k, i)
int k, i;
{
    switch (g_bl) {
        case 6:
            if ((I_set[k][i]&0x0012) == 0x0012 && !(I_set[k][i]&0x0009)) g_red_entry[dim++] = i;
            break;
        case 7:
            if ((I_set[k][i]&0x0012) == 0x0012 && !(I_set[k][i]&0x0009)) g_red_entry[dim++] = i;
            break;
        case 8:
            if ((I_set[k][i]&0x0012) == 0x0012 && !(I_set[k][i]&0x0049)) g_red_entry[dim++] = i;
            break;
        case 9:
            if ((I_set[k][i]&0x0092) == 0x0092 && !(I_set[k][i]&0x0049)) g_red_entry[dim++] = i;
            break;
        case 10:
            if ((I_set[k][i]&0x0092) == 0x0092 && !(I_set[k][i]&0x0049)) g_red_entry[dim++] = i;
            break;
        case 11:
            if ((I_set[k][i]&0x0092) == 0x0092 && !(I_set[k][i]&0x0249)) g_red_entry[dim++] = i;
            break;
        case 12:
            if ((I_set[k][i]&0x0492) == 0x0492 && !(I_set[k][i]&0x0249)) g_red_entry[dim++] = i;
            break;
    }
}
void reduction_space(k, s)
int k, s;
{
    int i;
    for (i = 0; i < s; i++) match_braid_index(k, i);
}
static void release_memory()
{
    int i;
    for (i = 0; i < g_num_of_rep; i++) free(alpha[i]);
    for (i = 0; i < g_num_of_rep; i++) free(I_set[i]);
}
main()
{
    int i, j, k, m, bl, p0, p1;
    char buf[50];
    FILE *fp_w;
    WPTR *temp, *temp1;
    printf("maximum depth ? ");
}

```

```

scanf("%d", &max_depth);
printf("braid index ? ");
scanf("%d", &bl);
printf("no output (0) or Young tableau (1) and W-graph (2) ? ");
scanf("%d", &g_WI);
for (i = 0; i < size_tableau; i++) {
    tableau[i] = (char *)malloc((long)sizeof(char)*size_tableau);
    if (tableau[i] == NULL) {
        for (j = 0; j < i; j++) free(tableau[j]);
        printf("Memory exhausted in [tableau].%n");
        exit(0);
    }
}
sprintf(buf, "Y%03d", bl);
if ((g_fp_Y = fopen(buf, "w")) == NULL) {
    printf("File %s can not open.%n", buf);
    exit(0);
}
real_max_matrix_size = 1;
partition(bl);
for (i = 0; i < size_tableau; i++) free(tableau[i]);
for (i = 0; i < max_matrix_size; i++) incident_matrix[i] = NULL;
sprintf(buf, "RR%03d", bl);
if ((fp_w = fopen(buf, "w")) == NULL) {
    printf("File %s can not open.%n", buf);
    exit(0);
}
fprintf(fp_w, "%d\n", g_num_of_rep);
for (k = 0; k < g_num_of_rep; k++) {
    if (g_matrix_size[k] > 1) {
        for (i = 0; i < g_matrix_size[k]-1; i++)
            for (j = i+1; j < g_matrix_size[k]; j++) make_W_graph1(k, i, j);
        p1 = 1;
        do {
            p0 = p1;
            for (i = 0; i < g_matrix_size[k]-1; i++) {
                for (j = i+1; j < g_matrix_size[k]; j++) {
                    if (check_W_graph_entry(i, j, p0)) {
                        if (p0 == p1) p1++;
                        make_W_graph2(k, p0, i, j);
                    }
                }
            }
        } while (p1 < p0);
        if (p1 >= max_index) {
            printf("Index overflow.%n");
            release_memory();
            exit(0);
        }
    }
}

```





## 結び目理論研究支援ソフトウェア KnotTheorybyComputer

KNOTTHEORYbyCOMPUTER is a computer software for Apple Macintosh computers being developed in order to assist researchers in knot theory. This program

- (1) draws a picture of a knot or a link on the display from a given P-data.
- (2) allows the user to draw a picture of a knot by the mouse for inputting a knot.
- (3) simplifies a given diagram of a link, and determines whether the knot represented by a diagram of about 50 crossings or less is trivial or not.
- (4) computes polynomial invariants of links such as those of Alexander, Conway, Jones, HOMFLY, Q, and Kauffman.
- (5) allows the user to draw a picture of a braid, and compute the Alexander and the Jones polynomial of the closed braid, using Matrix representations of the braid (up to 10 string braids).
- (6) allows the user to apply Reidemeister, Markov, and other moves to a shown diagram by clicking the mouse.
- (7) allows the user to decompose a knot into a tangle, and execute 3 kinds of mutations of the tangle.
- (8) allows the user to deform the link by edge-moves or vertex-moves using tracking mouse in the PL-drawing mode.
- (9) computes 3-parallel polynomial invariants of 3 and 4 braids which can recognize Kinoshita-Terasaka and Conway knots.

The program has many more features (some of which have not been documented).

### Getting started.

Start the program by double-clicking on its icon. A blank window will appear, and the menu bar will show 6 items: File, Edit, DRAW, ACTION, DEFORM, and MODE&UTIL.

The first two of them are used for manipulating text files only. You can use them to prepare P-data files or to edit log files. Note that the name of a P-data file must end with the extension '.prd', as in 'trivial.prd'. The program supports HFS file system, but all the log files are created in the folder which contains the application.

The third menu DRAW is used for inputting a knot using the mouse. Choose 'Draw a Knot'. Click the mouse to start drawing a knot. Move the mouse at a moderate speed while drawing. When the mouse pointer comes back to the starting point, the program converts the picture to a diagram with a small circle over each crossing. You can then change any of the crossings by clicking over the small circle. When you click somewhere else, the program creates the P-data for the knot and outputs it in the current log file. If no log file is opened, the P-data is written in the file named 'WORK.LOG'. The current version of

KNOTTHEORYbyCOMPUTER does not support inputting a link by this method.

You can alternatively draw a picture of a knot or a link by 'PL Draw a link'. By choosing this menu, one enters the PL draw mode. One can then draw a diagram of a knot or a link "piecewise linearly" by clicking vertices. One can cancel drawing the last edge by pressing the "Delete" key. Note that one remains in the PL draw mode until one chooses the "..Exit" from the "DRAW" menu. Once you have constructed a knot and link, you may modify it by clicking any vertex and then by pressing the "Delete" key. The selected vertex becomes the starting vertex of the current PL-data. Furthermore, you can save and load the PL-data by the corresponding menus appearing when you enter into the PL draw mode.

This mode also contains a facility to compute rapidly the one variable Alexander polynomial of a link with a great many crossing points using the Alexander matrix of the link. The routine makes use of the fraction free Gaussian elimination method to compute determinants of Alexander matrixes and so has been implemented multiple precision integers to compute Alexander polynomials of large links to avoid coefficient expansions.

Moreover, you can draw a picture of an n-string braid by 'Draw a braid'. Entering this menu, you input a braid as follows:

abcdefghABCDEFGHb<sup>5</sup>

where letters express generators of Braid groups, capital letters do inverses of them, and the symbol ^ gives a power of generators.

You can also input three commands 'open', 'read', and 'close' to substitute to input a word ( see Moody.data in the disk). Note that in a data file, one data must be terminated at a carriage return. We will improve the user interface of this routine in the next version.

#### The ACTION menu.

The main menu is ACTION. It contains 13 submenus. You choose 'OPEN a data file' submenu to open a P-data file. 'READ a P-data' reads one P-data. Choosing 'COMPUTE invariants' computes polynomial invariants of links using the Conway relation and writes them in the current log file. By default the program computes the Jones polynomial only; you can change it by the 'SET invariants' submenu in the 'MODE&UTIL' menu. Although all the computed invariants are outputted in the log file, only the default invariant is displayed on the window within a certain screen limit size; scrolling the window displaying the invariant is not supported in the current version of the program.

After reading a P-data, you can see a diagram of the knot by choosing 'SHOW a link' submenu. By choosing 'SELECT a domain' in the DEFORM menu, one can specify the outer region for drawing the diagram.

'JUDGE a knot to be trivial' in the ACTION menu determines if the given knot is a trivial knot by searching for up to 2-waves. If no 2-waves are found, the program applies Reidemeister moves of types II and III under a certain rule to find 2-waves. The judgment is correct for almost all diagrams of about 50 crossings or less, although some diagrams of the trivial knot for which the program fails to judge have been found (see Figure 1.0).

**About P-data.**

KNOTTHEORYbyCOMPUTER uses a data structure called P-data to internally represent a knot or a link. A P-data is defined as follows: Given a diagram of a knot or a link, trace a component starting from an arbitrary point and number each crossing one encounters (1, ...,  $m_1$ ) until one returns to the starting point. For a link diagram, trace another component and continue numbering crossings ( $m_1+1, \dots$ ). When the process is done, each crossing should be assigned two numbers, one for the undercrossing arc and the other for the overcrossing arc. The P-data is

$2n \ c$

$m_1 \ m_2 \ \dots \ m_c$

$a_1 \ a_2 \ \dots \ a_n$

where  $n$  is the number of crossings,  $c$  is the number of components,  $m_i$  is the number of crossings on the  $i$ -th component. When one traces the components of the diagram in the same way as he/she numbered crossings, at the  $i$ -th overcrossing point, the number attached to the corresponding undercrossing point is the absolute value of  $a_i$ . The sign of  $a_i$  represents the algebraic crossing number at the crossing.

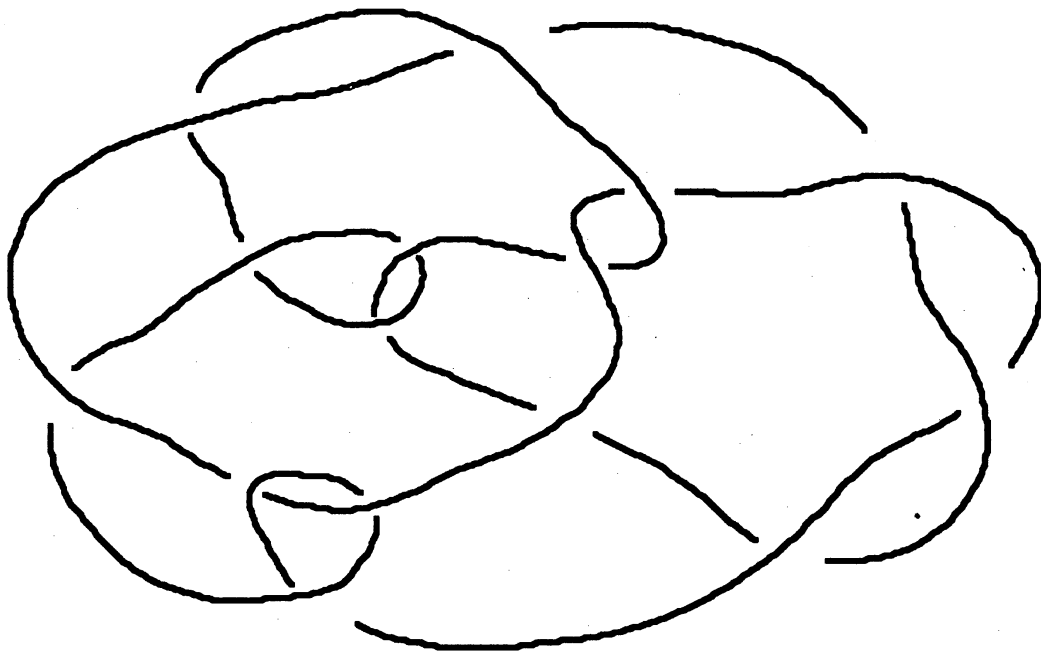


Figure 1.0

For instance, the above diagram of the trivial knot is drawn by the program from the following P-data

```
30 1
30
11 17 -14 -27 25 -20 1 23 3 -10 15 21 -29 -7 13
555
```

(The 555 in the last line indicates the end of the P-data).

Comments and inquiries are welcomed, and send them to  
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P.S. I have planning to improve the user interface of the next version greatly and the improved version will be distributed only by Network, where the network address is [wuarhive.wustl.edu](http://wuarhive.wustl.edu). Please contact professor Earl D. Fife, if you have an any interest. If you are not available for E-mail, please inform me of your hope by a real-mail.

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