ON THE ESSENTIAL SELF-ADJOINTNESS OF THE RELATIVISTIC HAMILTONIAN OF A SPINLESS PARTICLE

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ABSTRACT. The relativistic quantum hamiltonian H describing a spinless particle in an electromagnetic field is considered. H is associated with the classical hamiltonian $c\sqrt{m^2c^2 + |p - A(x)|^2} + V(x)$ via Weyl's correspondence. In the preceding papers [12] and [13] the author proved that if $V(x)$ is bounded from below by a polynomial in x, H with domain $C_0^\infty(R^d)$ is essentially self-adjoint. Here we will show that H is essentially self-adjoint if $V(x)$ is bounded from below by $-C \exp a|x|$ for positive constants C and a. These results are quite different from those on the non-relativistic operator, i.e. the Schrödinger operator, but much close to those on the Dirac operator.

1. INTRODUCTION

The result in this note was obtained together with Prof. T. Ichinose at Kanazawa University. Consider a charged particle with charge one and rest mass m in an electromagnetic field. Then its relativistic classical hamiltonian is given by

$$h_A^m(x, p) + V(x) \equiv c\sqrt{m^2c^2 + |p - A(x)|^2} + V(x) \quad (x \in R^d), \quad (1.1)$$

where $A(x) = (A_1(x), \cdots, A_d(x))$ and $V(x)$ imply the vector potential and the scalar one, respectively. c is the velocity of light. The quantum hamiltonian $H_A^m f(x) + V(x)f(x)$ via Weyl's correspondence is formally defined by

$$(2\pi)^{-d} \iint_{R^{2d}} e^{i(x-x')\cdot p} h_A^m(\frac{x+x'}{2}, p) f(x')dx'dp + V(x)f(x). \qquad (1.2)$$

For example, if $A_j(x)$ $(j = 1, 2, \cdots, d)$ are sufficiently smooth and have the bounded derivatives of any positive order on R^d, this quantum hamiltonian defines a linear operator in the space $L^2(R^d)$ of all square integrable functions (e.g. [17]). This quantum hamiltonian can be considered as the hamiltonian describing a relativistic spinless particle (e.g. [18], [7], [3], and Appendix 2 to XIII. 12 in [16]).

When $A(x) = 0$ and $V(x)$ is the Coulomb potential, a Yukawa-type potential, and their sum, the essential self-adjointness and spectral properties of $H_0^m + V(x)$ have been studied in [18], [7]and [3]. As for the general $H_A^m + V(x)$, T. Ichinose proposed the extension of the quantum hamiltonian defined by (1.2) to that for non-smooth $A_j(x)$ in [8] and [11] and proved its essential self-adjointness with domain $C_0^\infty(R^d)$ in [11] under the assumption that $V(x) \in L_{loc}^2(R^d)$ is bounded from below and $A_j(x) \in L_{loc}^{2+\delta}(R^d)$ $(j = 1, 2, \cdots, d)$ for a $\delta > 0$. This extension will be introduced in section 2. $C_0^\infty(R^d)$ denotes the space of all infinitely differentiable functions with compact support and $L_{loc}^2(R^d)$ the space of all locally square integrable functions. Recently the au-

thor proved in [12] and [13] that if $V(x) \in L^2_{loc}(R^d)$ is bounded from below by a polynomial in x, $H^m_A + V(x)$ with domain $C^\infty_0(R^d)$ is essentially self-adjoint under a suitable assumption on sufficiently smooth $A_j(x)$.

Our aim in the present paper is to show that if $V(x) \in L^2_{loc}(R^d)$ satisfies

$$V(x) \geq -Ce^{a|x|} \tag{1.3}$$

for positive constants C and a, then $H^m_A + V(x)$ with domain $C^\infty_0(R^d)$ is essentially self-adjoint under a suitable assumption on non-smooth $A_j(x)$, where $H^m_A + V(x)$ is the extension stated above of (1.2). For example, we can obtain the result below. Let $d \geq 3$ and $V(x) \in L^2_{loc}(R^d)$ be a real valued function such that (1.3) holds for positive constants C and a. Let Z be a non-negative constant less than $(d-2)c/2$ and $A_j(x)$ $(j = 1, 2, \cdots, d)$ a bounded, locally Hölder continuous function. Then $H^m_A - Z/|x| + V(x)$ with domain $C^\infty_0(R^d)$ is essentially self-adjoint (Example in the section 2 of the present paper).

As for the Schrödinger operator $-\frac{1}{2m}\Delta + V_S(x)$, we know that we need for its essential self-adjointness the limitation on the decreasing rate at infinity of the negative part of $V_S(x)$ (e.g. Theorem 2 in [4] and page 157 in [1]). On the other hand as for the Dirac operator, we know from Theorem 2.1 in [2] that such a limitation is not necessary at all for its essential self-adjointness. The decreasing rate for the essential self-adjontness obtained by us is quite different from that on the Schrödinger operator, but much close to that on the

Dirac operator.

Our proof is done by using the commutator theorem in [6] as in [13] and using the results obtained in [8] - [11].

2. Theorems

Through the present paper $A_j(x)$ $(j = 1, 2, \cdots, d)$ and $V(x)$ are assumed to be real valued. We first introduce the extension of H_A^m given in (1.2) from [8] and [11]. This extension is given by

$$H_A^m f(x) = mc^2 f(x) - \lim_{r \downarrow 0} \int_{r \le |y|} \{e^{-iy \cdot A(x+y/2)} f(x+y) - f(x)\} n^m(y) dy,$$

$$(2.1)$$

where $n^m(y)$ is defined by

$$n^m(y) = \begin{cases} 2c(2\pi)^{-(d+1)/2}(mc)^{(d+1)/2}|y|^{-(d+1)/2}K_{(d+1)/2}(mc|y|), & m > 0, \\ \\ c\pi^{-(d+1)/2}\Gamma((d+1)/2)|y|^{-(d+1)}, & m = 0 \end{cases}$$

$$(2.2)$$

$K_\nu(z)$ is the modified Bessel function of the third kind of order ν (e.g. pages 5 and 9 in [5]) and $\Gamma(z)$ the gamma function. We note $n^m(y) > 0$ for any $y \ne 0$.

Let $m \ge 0$ and $A_j(x) \in L_{loc}^{2+\delta}(R^d)$ $(j = 1, 2, \cdots, d)$ for a $\delta > 0$. Then H_A^m with domain $C_0^\infty(R^d)$ defined by (2.1) determines a symmetric operator in $L^2(R^d)$ (e.g. Lemma 2.2, its remark, and (2.20) in [10]). In more details this H_A^m with domain $C_0^\infty(R^d)$ is essentially self-adjoint (Theorem 1.1 in [11]). Assuming that $A_j(x)$ $(j = 1, 2, \cdots, d)$ are sufficiently smooth and have the

bounded derivatives of any positive order on R^d, H_A^m defined by (2.1) is equal to that done in (1.2) (Lemma 2.2 in [8]). Hereafter we always consider H_A^m defined by (2.1).

We state the assumption $(A)_m$ for the main theorem.

$(A)_m$: (i) $\Phi(x)$ is a real valued function in $L_{loc}^2(R^d)$. (ii) $H_A^m + \Phi(x)$ with domain $C_0^\infty(R^d)$ is bounded from below as the quadratic form. (iii) $H_A^m + \Phi(x) + W(x)$ with domain $C_0^\infty(R^d)$ is essentially self-adjoint for any $W(x)$ being in $L_{loc}^2(R^d)$ with $W(x) \geq 0$ almost everywhere (a.e.).

(ii) in $(A)_m$ means that

$$(\{H_A^m + \Phi(x)\}f(x),\ f(x)) \geq -C(f(x),\ f(x))$$

is valid for all $f(x) \in C_0^\infty(R^d)$, which we denote by $H_A^m + \Phi(x) \geq -C$ on $C_0^\infty(R^d)$, where C is a constant and $(\cdot,\ \cdot)$ the inner product of $L^2(R^d)$.

Remark 2.1. We know from Theorem 2.3 in [9] that $H_A^m - H_A^{m'}$ makes a bounded operator on $L^2(R^d)$ for arbitrary non-negative constants m and m', when $A_j(x) \in L_{loc}^{2+\delta}(R^d)$ $(j = 1, 2, \cdots, d)$ for a $\delta > 0$. So we can see by Kato-Rellich's theorem (e.g. Theorem X.12 in [15]) that the assumption $(A)_m$ is equivalent to $(A)_0$ for any $m > 0$.

The following theorem is the main theorem.

Theorem 2.1. *Assume $(A)_0$. Moreover we suppose that $A_j(x)$ $(j = 1, 2, \cdots, d)$ and $V(x)$ satisfy (B.1) or (B.2) below. Then $H_A^m + \Phi(x) + V(x)$ with domain $C_0^\infty(R^d)$ is essentially self-adjoint for any $m \geq 0$.*

(B.1): *(i) $|A_j(x)|$ $(j = 1, 2, \cdots, d)$ is bounded by a polynomial in x. (ii) $V(x) \in L_{loc}^2(R^d)$ is bounded from below by $-C \exp(a|x|^{1-b})$, where $0 < b \leq 1, C \geq 0$, and $a \geq 0$ are constants.*

(B.2): *(i) $A_j(x)$ $(j = 1, 2, \cdots, d)$ is a bounded function on R^d. (ii) $V(x) \in L_{loc}^2(R^d)$ is bounded from below by $-C \exp(a|x|)$, where C and a are positive constants.*

Corollary 2.2. *Suppose that $A_j(x)$ $(j = 1, 2, \cdots, d)$ and $V(x)$ satisfy (B.1) or (B.2) in Theorem 2.1. Then $H_A^m + V(x)$ with domain $C_0^\infty(R^d)$ is essentially self-adjoint for any $m \geq 0$.*

Proof. We have only to prove that the assumption $(A)_0$ where $\Phi(x) = 0$ is satisfied. Then Corollary 2.2 follows from Theorem 2.1. We can see from Theorem 1.1 in [11] that $H_A^0 \geq 0$ on $C_0^\infty(R^d)$ holds and that $H_A^0 + W(x)$ with domain $C_0^\infty(R^d)$ is essentially self-adjoint for any $W(x) \in L_{loc}^2(R^d)$ with $W(x) \geq 0$ a.e. Thus the proof is completed. Q.E.D.

Theorem 2.3. *Let $\Phi(x)$ be a real valued function in $L_{loc}^2(R^d)$ and a H_0^0-bounded multiplication operator with relative bound less than one. Suppose*

that $A_j(x)$ $(j = 1, 2, \cdots, d)$ and $V(x)$ satisfy (B.1) or (B.2) in Theorem 2.1.

Moreover we assume

$$(*) \qquad \int_{0 < y \leq 1} |y \cdot \{A(x + y/2) - A(x)\}| |y|^{-(d+1)} dy \in L^2_{loc}(R^d).$$

Then $H^m_A + \Phi(x) + V(x)$ with domain $C^\infty_0(R^d)$ is essentially self-adjoint for

any $m \geq 0$.

Remark 2.2. Corollary 2.2 and Theorem 2.3 in the present paper give the

generalization of Theorem 2.2 and Theorem 2.3 in [13], respectively.

Example. Let $d \geq 3$ and $\Phi(x) = -Z/|x|$, where $0 \leq Z < (d-2)c/2$ is a

constant. We know Hardy's inequality

$$\left(\frac{d-2}{2}\right)^2 \|\frac{\psi(x)}{|x|}\|^2 \leq \sum_{j=1}^{d} \|\frac{\partial \psi}{\partial x_j}(x)\|^2$$

(e.g. page 169 in [15] or (2.9) in [7]), where $\| \cdot \|$ denotes the L^2-norm. We

denote the Fourier transformation $\int e^{-ix \cdot \xi} \psi(x) dx$ of $\psi(x) \in C^\infty_0(R^d)$ by $\hat{\psi}(\xi)$.

Then we have

$$\left(\frac{d-2}{2}\right)^2 \|\frac{Z}{|x|}\psi(x)\|^2 \leq (2\pi)^{-d} Z^2 \int |\xi|^2 |\hat{\psi(\xi)}|^2 d\xi = c^{-2} Z^2 \|H^0_0 \psi(x)\|^2$$

for $\psi(x) \in C^\infty_0(R^d)$ by using $H^0_0 \psi(x) = c(2\pi)^{-d} \int e^{ix \cdot \xi} |\xi| \hat{\psi}(\xi) d\xi$. So it follows

from the assumption on Z that $-Z/|x|$ is H^0_0-bounded with relative bound less

than one. Let $A_j(x)$ $(j = 1, 2, \cdots, d)$ be a locally Hölder continuous function

on R^d and assume that $A_j(x)$ and $V(x)$ satisfy (B.1) or (B.2). Then $(*)$ in

Theorem 2.3 follows from the local Hölder continuity of $A_j(x)$. It is easy to see

from Theorem 2.3 that $H_A^m - Z/|x| + V(x)$ with domain $C_0^\infty(R^d)$ is essentially self-adjoint for any $m \geq 0$.

The proofs of Theorems 2.1 and 2.3 will be published eleswhere.

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