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On the Discrete Spectrum of Schrödinger Operators with Perturbed Magnetic Fields

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1 Introduction

In this note we study a Schrödinger operator with a magnetic field :

$$H = (-i\nabla - b(x))^2 + V(x) \tag{1.1}$$

defined on $C_0^\infty(\mathbf{R}^3)$, where $V \in L^2_{loc}(\mathbf{R}^3)$ is a scalar potential and $b \in C^1(\mathbf{R}^3)^3$ is a vector potential, both of which are real-valued, and $\overrightarrow{B}(x) = \nabla \times b$ is called the magnetic field. Letting $T = -i\nabla - b(x)$, we define the quadratic form q_H by

$$q_H[\phi, \psi] = \int_{\mathbf{R}^3} \left(T\phi \cdot \overline{T\psi} + V\phi\overline{\psi}\right) dx,$$

$$q_H[\phi] = q_H[\phi, \phi]$$

for $\phi, \psi \in C_0^\infty(\mathbf{R}^3)$. We assume that

$$V(x) \to 0 \quad as \quad |x| \to \infty. \tag{V1}$$

Then H admits a unique self-adjoint realization in $L^2(\mathbf{R}^3)$ (denoted by the same notation H) with the domain

$$D(H) = \left\{u \in L^2(\mathbf{R}^3);\ |V|^{\frac{1}{2}}u,\ Tu,\ Hu \in L^2(\mathbf{R}^3)\right\},$$

which is associated with the closure of q_H (denoted by the same notation q_H) with the form domain

$$Q(H) = \left\{u \in L^2(\mathbf{R}^3);\ |V|^{\frac{1}{2}}u,\ Tu \in L^2(\mathbf{R}^3)\right\}.$$

This fact can be proved in the same way as in the cases of the constant magnetic fields ([1] and [6]).

It is well known that, if $\overrightarrow{B}(x) \equiv 0$, then the finiteness or the infiniteness of the discrete spectrum of H depends on the decay order of the potential V, of which the border is $|x|^{-2}$ ([5]). On the other hand, if $\overrightarrow{B}(x) \equiv (0,0,B)$, B being a positive constant, then the number of the discrete spectrum of H is infinite under a suitable negativity assumption of the potential which is independent of the decay order of V. More precisely, the following result was proved by Avron-Herbst-Simon [2].

Theorem 0. ([2]) *Let $\overrightarrow{B}(x) = \nabla \times b = (0,0,B)$, B being a positive constant. Suppose that $V \in L^2 + L^\infty_\varepsilon$ and that V is non-positive, not identically zero and azimuthally symmetric. Then the number of the discrete spectrum of H is infinite.*

Here a function $f(x)$ on $\mathbf{R}^3$ is called azimuthally symmetric (in z-axis) if $f(x)$ depends only on ρ and z. Now a question arizes : What occurs for the discrete spectrum when we perturb slightly the constant magnetic field ? One may well imagine that the infiniteness or the finiteness of the discrete spectrum depends on both of the magnetic vector potential $b(x)$ and the scalar potential $V(x)$. This is certainly true and the aim of this paper is to clearify the relation between $b(x)$ and $V(x)$ for H to have an infinite or a finite discrete spectrum.

To state the main theorem we make some preparations. Let $x = (x_1, x_2, z) \in \mathbf{R}^3, \vec{\rho} = (x_1, x_2), r = |x|, \rho = |\vec{\rho}|$, and $\nabla_2 = (\partial/\partial x_1, \partial/\partial x_2)$. We assume that

$$(V2) \qquad \begin{cases} V \textit{ is azimuthally symmetric, bounded above and there exists} \\ R_0 > 0 \textit{ such that } V \in C^0(|x| \geq R_0), \quad V < 0 \textit{ for } |x| \geq R_0. \end{cases}$$

Let B be a positive constant and

$$b_c(x) = B/2(-x_2, x_1, 0),$$

which satisfies $\nabla \times b_c = (0,0,B)$. For given $b \in C^1(\mathbf{R}^3)^3$, we put

$$b_p(x) = b(x) - b_c(x) = (a_1(x), a_2(x), a_3(x)).$$

By introducing the polar coordinate (ρ, θ) in $\mathbf{R}^2$, we define the set X by

$$X = \Big\{a \in C^1(\mathbf{R}^3); \text{ there exists } N(a) \in \mathbf{N} \text{ such that}$$

$$\int_0^{2\pi} a(x) e^{ik\theta} d\theta = 0 \text{ for } |k| \geq N(a), \ k \in \mathbf{Z}\Big\}.$$

We denote by $\sigma(H)$ the spectrum of H, by $\sigma_d(H)$ the discrete spectrum of H, by $\sigma_e(H)$ the essential spectrum of H and by $\sharp Y$ the cardinal number of a set Y. For two vector potentials $b_1, b_2 \in C^1(\mathbf{R}^3)^3$, we denote $b_1 \sim b_2$ when b_1 is equivalent to b_2 under a gauge transformation, namely, $b_1 - b_2 = \nabla\lambda$ for some $\lambda \in C^2(\mathbf{R}^3)$. Then our main result is the following theorem.

Theorem 1. *Assume $(V1), (V2)$ and that $a_j(x) \in X$ $(j = 1,2,3)$. Suppose* *that there exist* $R_1 > 0$ *and positive constants* $c_j (j = 1, 2, 3)$ *such that*

$$(1.2) \qquad \begin{cases} |a_j(x)| \le c_1 \min\left\{|V(x)|^{1/2}, |V(x)|\rho\right\} \ (j = 1, 2), \\ |\nabla_2 a_j(x)| \le c_2 |V(x)| \ (j = 1, 2), \\ |a_3(x)| \le c_3 |V(x)|^{1/2} \end{cases}$$

for $|x| \ge R_1$,

$$(1.3) \qquad 2(c_1^2 + c_2) + c_3^2 + \sqrt{2} c_1 < 1,$$

and also suppose that

$$(1.4) \qquad \partial a_3 / \partial z \to 0 \ \text{as} \ |x| \to \infty.$$

Then $\sigma_e(H) = [B, \infty)$ *and*

$$(1.5) \qquad \sharp \sigma_d(H) = +\infty.$$

Remark 1.1. Let V be as in Theorem 1. If $W \in L^2_{loc}(\mathbf{R}^3)$ satisfies $(V1)$ and $W \le V$, then $\sharp \sigma_d(T^2 + W) = +\infty$ by the min-max principle. Thus we can apply the above theorem to potentials which are not azimuthally symmetric or not continuous on $|x| \ge R_0$.

Remark 1.2. The above theorem of course holds if we replace the vector potential by an equivalent one.

As an example we consider the perturbation of the constant magnetic field on a compact set.

Proposition 1.3. *If there exists* $R_2 > 0$ *such that*

$$\overrightarrow{B}(x) = (0, 0, B) \ \text{for} \ |x| \ge R_2,$$

then one can replace the magnetic vector potential $b(x)$ *by an equivalent one satisfying (1.2),(1.3) and (1.4).*

Proof of Proposition 1.3. It is easy to see that

$$\nabla \times (b - b_c) = 0 \quad (|x| \ge R_2).$$

Hence, there exist $\lambda \in C^2(\mathbf{R}^3)$ such that

$$b - b_c = \nabla \lambda \quad (|x| \ge R_2).$$

We put

$$\tilde{b} = b - \nabla \lambda \ \text{on} \ \mathbf{R}^3.$$

Then $\tilde{b} \sim b$ and $\tilde{b} - b_c = 0$ for $|x| \geq R_2$. For this $\tilde{b}$, (1.2),(1.3) and (1.4) are always satisfied. □

In §2 we explain some examples showing that the above condition in Theorem 1 is almost optimal to guarantee the infiniteness of the discrete spectrum of H. These examples also show that some non-constant magnetic fields decrease the number of bound states in spite of the fact that the number of the discrete spectrum of H with $V = O(r^{-\alpha})$ ($0 < \alpha < 2$) is infinite if $\overrightarrow{B}(x) \equiv 0$ ([5]).

2 Examples

In this section we illustrate some examples showing that the conditions in Theorem 1 are almost optimal. We first prepare the following proposition without a proof.

Proposition 2.1. *If* $|b_p(x)| \to 0, |div b_p(x)| \to 0$ *as* $|x| \to \infty$, *then* $\sigma_e(H) = [B, \infty)$.

For the sake of convenience, we strengthen slightly the conditions in Theorem 1 as follows.

Theorem 1*. *Assume* $(V1), (V2)$ *and that* $a_j(x) \in X$ $(j = 1,2,3)$. *Suppose that*

$$(2.1) \qquad \begin{cases} a_j(x) = o\Big(\min\big\{|V(x)|^{1/2}, |V(x)|\rho\big\}\Big) \ (j = 1,2), \\ \nabla_2 a_j(x) = o\big(|V(x)|\big) \ (j = 1,2), \\ a_3(x) = o\big(|V(x)|^{1/2}\big) \\ \partial a_3/\partial z = o(1) \end{cases}$$

as $|x| \to \infty$. *Then* $\sigma_e(H) = [B, \infty)$ *and*

$$\sharp\sigma_d(H) = +\infty.$$

We give the above mentioned examples in the following form.

$$(2.2) \qquad b = f(r)(-x_2, x_1, 0)$$

where $f \in C^1([0,\infty))$, $f'(0) = 0$ and f is real-valued. In this case $a_1(x) = -(f(r) - B/2)x_2, a_2(x) = (f(r) - B/2)x_1, a_3(x) = 0$, so the assumption that $a_j \in X$ (j=1,2,3) is satisfied. We assume that $V(x)$ is a function of $r = |x|$. Then (2.1) is equivalent to the following

$$
(2.3) \qquad \begin{cases} |f(r) - B/2| = o\Big(\min\big\{|V(x)|^{1/2}r^{-1}, |V(x)|\big\}\Big), \\ \\ |f'(r)| = o\big(|V(x)|r^{-1}\big). \end{cases}
$$

Now we put $V = -r^{-\alpha}$ $(\alpha > 0)$ for $|x| \geq 2$, then (2.3) is equivalent to

$$
(2.4) \qquad \begin{cases} |f(r) - B/2| = o\big(r^{\min\{-1-\alpha/2, -\alpha\}}\big), \\ \\ |f'(r)| = o(r^{-1-\alpha}). \end{cases}
$$

Before showing the examples, we prepare the following proposition.

Proposition 2.2. *For $\phi \in C_0^\infty(\mathbf{R}^3)$, we have the following inequality.*

$$
(2.5) \qquad \int |T\phi|^2 dx \geq \int \big(\partial b_2/\partial x_1 - \partial b_1/\partial x_2\big)|\phi|^2 dx,
$$

where $b = (b_1(x), b_2(x), b_3(x))$.

Cororally. *In the case of (2.2) we have*

$$
\int |T\phi|^2 dx \geq \int (f'(r)\rho^2 r^{-1} + 2f(r))|\phi|^2 dx \quad \text{for} \quad \phi \in C_0^\infty(\mathbf{R}^3).
$$

In particular, if $f'(r) \leq 0$, then

$$
(2.6) \qquad \int |T\phi|^2 dx \geq \int F_f(r)|\phi|^2 dx \quad \text{for} \quad \phi \in C_0^\infty(\mathbf{R}^3),
$$

where $F_f(r) = rf'(r) + 2f(r)$.

Proof of Proposition 2.2. We put

$$
A_1 = \partial/\partial x_1 + b_2, A_2 = \partial/\partial x_2 - b_1, A = A_1 + iA_2 \quad \text{and} \quad P = \partial/\partial z - ib_3.
$$

Then by a straightforward calculation,

$$
\begin{aligned}
A^*A = \;& -\partial^2/\partial x_1^2 - \partial^2/\partial x_2^2 + 2i(b_1\partial/\partial x_1 + b_2\partial/\partial x_2) + i(\partial b_1/\partial x_1 + \partial b_2/\partial x_2) \\
& + |b_1|^2 + |b_2|^2 - \partial b_2/\partial x_1 + \partial b_1/\partial x_2, \\
P^*P = \;& -\partial^2/\partial z^2 + 2ib_3\partial z + i\partial b_3/\partial z + |b_3|^2.
\end{aligned}
$$

Therefore we have

$$
P^*P + A^*A = T^2 - \big(\partial b_2/\partial x_1 - \partial b_1/\partial x_2\big).
$$

Hence, for $\phi \in C_0^\infty(\mathbf{R}^3)$,

$$
\int |T\phi|^2 dx = \big((P^*P + A^*A)\phi, \phi\big)_{L^2} + \int (\partial b_2/\partial x_1 - \partial b_1/\partial x_2)|\phi|^2 dx
$$

$$\geq \int (\partial b_2/\partial x_1 - \partial b_1/\partial x_2)|\phi|^2 dx. \square$$

Example 1. We first take $\alpha = 2$, namely, let

$$V(x) = \begin{cases} -r^{-2} \ (r \geq e^{1/2}), \\ 0 \ (r < e^{1/2}). \end{cases}$$

If $f(r) - B/2 = r^{-\beta}$ for $r \geq e^{1/2}$ $(\beta > 2)$, the condition (2.4) is fulfilled, hence $\sharp\sigma_d(H) = +\infty$. We next see what occurs when this condition is violated. We define $f(r)$ by

$$f(r) = \begin{cases} B/2 + r^{-2} \log r \ (r \geq e^{1/2}), \\ B/2 + 1/(2e) \ (r < e^{1/2}). \end{cases}$$

Then $f \in C^1([0,\infty)), f'(0) = 0, f'(r) \leq 0$, and

$$F_f(r) = \begin{cases} B + r^{-2} \ (r \geq e^{1/2}), \\ B + e^{-1} \ (r < e^{1/2}). \end{cases}$$

Hence, by using (2.6),

$$(2.7) \qquad (H\phi, \phi)_{L^2} \geq \int (F(r) + V)|\phi|^2 dx \geq B\|\phi\|_{L^2} \ \text{for} \ \phi \in C_0^\infty(\mathbf{R}^3).$$

By Proposition 2.1, it is easy to see that $\sigma_e(H) = [B, \infty)$. Hence, by (2.7), we have

$$\sigma_d(H) = \emptyset.$$

Example 2. To consider the case of $0 < \alpha < 2$ we use the almost same but slightly complicated method.

Let

$$V(x) = \begin{cases} -r^{-\alpha} \ (r \geq 2), \ 0 < \alpha < 2, \\ 0 \ (r < 2). \end{cases}$$

If $f(r) - B/2 = (\text{constant}) \cdot r^{-\beta}$ for $r \geq 2$ $(\beta > 1 + \alpha/2)$, the condition (2.4) is fulfilled, hence $\sharp\sigma_d(H) = +\infty$. When $\beta = \alpha$ $(< 1 + \alpha/2)$, H does not always have infinitely many bound states, although the difference $(1 + \alpha/2) - \alpha \to 0$ as $\alpha \to 2$. In fact, We define $f(r)$ by

$$f(r) = \begin{cases} B/2 + r^{-\alpha}/(2-\alpha) \ (r \geq 2), \\ B/2 + \{2^{-\alpha} + 2^{-\alpha-2}\alpha r(2-r)\}/(2-\alpha) \ (1 < r < 2), \\ B/2 + 2^{-\alpha-2}(4+\alpha)/(2-\alpha) \ (r \leq 1). \end{cases}$$

Then $f \in C^1([0,\infty)), f'(0) = 0, f'(r) \le 0$, and

$$F_f(r) = \begin{cases} B + r^{-\alpha} \ (r \ge 2), \\ B + 2^{-\alpha-1}\{-2\alpha r^2 + 3\alpha r + 4\}/(2-\alpha) \ \ (1 < r < 2), \\ B + 2^{-\alpha-1}(4+\alpha)/(2-\alpha) \ \ (r \le 1), \end{cases}$$

so

$$F_f(r) + V(x) \ge B \ \ (0 \le r < \infty).$$

Hence, by using (2.6), we have

$$(H\phi, \phi)_{L^2} \ge B\|\phi\|_{L^2}^2 \ \text{ for } \ \phi \in C_0^\infty(\mathbf{R}^3).$$

So, in the case of $1 < \alpha < 2$, by the same reasoning as before, we have $\sigma(H) = \sigma_e(H) = [B, \infty)$, hence

$$\sigma_d(H) = \emptyset.$$

In the case of $0 < \alpha \le 1$, we need another proof that $\sigma_e(H) = [B,\infty)$, which is due to [4] (p117).

We next show that the negativity assumption $(V2)$ is necessary for the infiniteness of the discrete spectrum under the situation that V is bounded above.

Example 3. Let

$$f(r) = \begin{cases} B/2 \ \ (r \ge 2), \\ B/2 + \exp(1/(r-2)) \ (3/2 \le r < 2), \\ B/2 + 2e^{-2} - \exp(-1/(r-1)) \ \ (1 \le r < 3/2), \\ B/2 + 2e^{-2} \ \ (0 \le r < 1). \end{cases}$$

Then we have $f \in C^1([0,\infty)), f'(0) = 0, f'(r) \le 0$, and

$$F_f(r) = \begin{cases} B \ \ (r \ge 2), \\ B + 4e^{-2} \ \ (0 \le r \le 1). \end{cases}$$

Now we define $V(x)$ by

$$V(x) = \begin{cases} 0 \ \ (r \ge 2), \\ \max(0, B - F_f(r)) \ (1 < r < 2), \\ v(r) \ \ (0 \le r \le 1), \end{cases}$$

where $|v(r)| \le 4e^{-2}$. We remark that, in this case, (2.3) is satisfied but $V(x)$ does not satisfy $(V2)$. We also have

$$(H\phi, \phi)_{L^2} \ge \int (F_f(r) + V)|\phi|^2 dx \ge B\|\phi\|_{L^2}^2,$$

so

$$\sigma_e(H) = [B, \infty), \sigma_d(H) = \emptyset.$$

Finally we show an example of the magnetic bottle (see [2]) which means a magnetic Schrödinger operator without the static potential term having a non-empty discrete spectrum.

Example 4. Let

$$\beta = \inf \left\{ (-\Delta\phi, \phi)_{L^2} \, ; \phi \in C_0^\infty(|x| \le 1), \ \|\phi\|_{L^2} = 1 \right\}. \tag{2.8}$$

We pick up $f \in C^1([0, \infty))$ such that

$$f(r) = \begin{cases} 0 \quad (0 \le r \le 1), \\ (\beta + 1)/2 \quad (r \ge 2). \end{cases}$$

Then, by Proposition 2.1, $\sigma_e(T^2) = [\beta + 1, \infty)$, so, by (2.8), it is easy to see that

$$\inf \sigma(T^2) \le \beta < \inf \sigma_e(T^2),$$

so we have $\sigma_d(T^2) \ne \emptyset$.

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