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LONG-RANGE SCATTERING FOR THREE-BODY STARK HAMILTONIANS

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1. INTRODUCTION

Suppose that a system of three particles interacting on each other via the pair potentials $V_{jk}(r_j - r_k)$, $1 \le j < k \le 3$, is put in a constant electric field $\mathcal{E} \in \mathbf{R}^d$, $\mathcal{E} \neq 0$. If the mass and charge of j-th particle, $j = 1, 2, 3$, are m_j and e_j and we denote its position vector by $r_j \in \mathbf{R}^d$, the motion of the system is governed by the Hamiltonian

$$\tilde{H} = -\sum_{j=1}^{3}(\frac{1}{2m_j}\Delta_{r_j} + e_j\langle \mathcal{E}, r_j\rangle) + \sum_{1\le j<k\le 3} V_{jk}(r_j - r_k), \tag{1.1}$$

where $\langle x, y\rangle = \sum_{j=1}^{d} x_j y_j$ for $x, y \in \mathbf{R}^d$.

The purpose of this article is to study the scattering theory for the system in the case when V_{jk} is long-range: $|V_{jk}(y)| = O(|y|^{-\rho})$ as $|y| \to \infty$ for some $0 < \rho \le \frac{1}{2}$. To state our result, we need introduce some notations. First we remove the center of mass motion from $\tilde{H}$. This is achieved as follows: We introduce the metric $r \cdot \tilde{r} = \sum_{j=1}^{3} m_j\langle r_j, \tilde{r}_j\rangle$ for $r = (r_1, r_2, r_3)$ and $\tilde{r} = (\tilde{r}_1, \tilde{r}_2, \tilde{r}_3) \in \mathbf{R}^d$ and define $X = \{r \in \mathbf{R}^{3d} | \sum_{j=1}^{3} m_j r_j = 0\}$ and $X^\perp = \mathbf{R}^{3d} \ominus X = \{r \in \mathbf{R}^{3d} | r_1 = r_2 = r_3\}$. Then $L^2(\mathbf{R}^{3d})$ is decomposed into $L^2(X) \otimes L^2(X^\perp)$ and, accordingly, $\tilde{H}$ can be written in the form

$$\tilde{H} = H \otimes Id + Id \otimes T^\perp.$$

Here

$$H = -\frac{1}{2}\Delta - E \cdot x + \sum_{1\le j<k\le 3} V_{jk}(r_j - r_k), \quad T^\perp = -\frac{1}{2}\Delta^\perp - E^\perp \cdot x^\perp, \tag{1.2}$$

Δ (resp. $\Delta^\perp$) is the Laplace-Beltrami operator on X (resp. $X^\perp$), x and E (resp. $x^\perp$ and $E^\perp$) are the projections of r and $(\frac{e_1}{m_1}\mathcal{E}, \frac{e_2}{m_2}\mathcal{E}, \frac{e_3}{m_3}\mathcal{E}) \in \mathbf{R}^{3d}$ to X (resp. $X^\perp$), respectively. We write $V = \sum V_{jk}(r_j - r_k)$. If $E \neq 0$, H is called a three-body Stark Hamiltonian and $H_0 = -\frac{1}{2}\Delta - E \cdot x$ the free Stark Hamiltonian. We are concerned with H and H_0 only.

For each pair $\alpha = (j,k)$, we denote by X^α the configuration space for the relative motion of j-th and k-th particles: $X^\alpha = \{r \in X | m_j r_j + m_k r_k = 0\}$ and $X_\alpha = X \ominus X^\alpha = \{r \in X | r_j = r_k\}$ is the configuration space for the motion of the third particle relative to the center of mass of the pair α. We denote by x^α (resp. x_α) the projection of $x \in X$ to X^α (resp. X_α) and write $V_\alpha(x^\alpha) = V_{jk}(r_j - r_k)$.

We say that $V = \sum_\alpha V_\alpha$ satisfies the condition $(V)_{\rho,\mu}$ if, for each pair α, V_α is a real-valued C^∞ function and satisfies the following conditions:

$$(V.1)\ |\partial_{x^\alpha}^m V_\alpha(x^\alpha)| \le C_m \langle x^\alpha \rangle^{-\rho-\mu|m|} \text{ for some } \rho > 0 \text{ and } \mu > 0,$$

$$(V.2)\ \sum_\alpha \frac{1+|\omega_\alpha|}{|\omega^\alpha|^2} \sup_{x^\alpha \in X^\alpha} |\omega^\alpha \cdot (\nabla^\alpha V_\alpha)(x^\alpha)| < |E|,$$

where m is any multi-index, $\omega = \frac{E}{|E|}$, $\langle x \rangle = (1+x^2)^{\frac{1}{2}}$ and ∇^α denotes the gradient on X^α. When V satisfies $(V)_{\rho,\mu}$, both H_0 and H are essentially self-adjoint on $C_0^\infty(X)$, and we denote their closures by the same notations.

The main result of this article is the following theorem.

Theorem 1.1 (Asymptotic Completeness). *Suppose that $E^\alpha \neq 0$ for all pair α and V satisfies the condition $(V)_{\rho,\mu}$ with $0 < \rho \le \frac{1}{2}$ and $\rho + \mu > 1$. Then the modified wave operators*

$$W_0^{G,\pm} = s - \lim_{t \to \pm\infty} e^{itH} e^{-\{itH_0 + i\int_0^t V(\frac{E}{2}\tau^2)d\tau\}} \tag{1.3}$$

exist and are unitary operators on $L^2(X)$. Moreover, $W_0^{G,\pm}$ have the intertwining property

$$e^{itH} W_0^{G,\pm} = W_0^{G,\pm} e^{itH_0}, \quad t \in \mathbf{R}. \tag{1.4}$$

Remark 1.2. For long-range potentials, it is known that in general, the usual wave operators

$$W_0^\pm = s - \lim_{t \to \pm\infty} e^{itH} e^{-itH_0}$$

does not exist even for two-body case (see Ozawa [O]).

Remark 1.3. The condition that $E^\alpha \neq 0$ for all pair α means that the electric field $\mathcal{E}$ is effective on each pair separately, which leads to the unitarity of $W_0^{G,\pm}$. *N*-body systems with different $\frac{e_j}{m_j}$, $j = 1, \cdots, N$, can be treated by our method and the modified wave operators $W_0^{G,\pm}$ for such systems exist and are unitary under the additional assumption that V satisfies the condition corresponding to $(V)_{\rho,\mu}$ with $0 < \rho \le \frac{1}{2}$ and $\rho + \mu > 1$. However, we shall restrict ourselves here to three-body systems for simplicity.

When $E^\alpha = 0$ for some pair α (note that only one E^α can vanish if $E \neq 0$), the Hamiltonian for the pair α

$$H^\alpha = -\frac{1}{2}\Delta^\alpha + V_\alpha(x^\alpha),$$

where Δ^α is the Laplace-Beltrami operator on X^α, can have bound states and the scattering for H may be multi-channel scattering. Such systems are now under investigation.

Remark 1.4. The condition $(V.2)$ that the pair interactions are small compared with $|E|$ is necessary for us to prove the Mourre estimate and one of propagation estimates. We want to eliminate this condition eventually.

Remark 1.5. The superscript G of the modified wave operators $W_0^{G,\pm}$ indicates that the modification is of Graf-type (see Graf [Gr2] and Jensen-Ozawa [JO2]). In [Gr2], Graf proved the existence and the asymptotic completeness of the modified wave operators $W_0^{G,\pm}$ for two-body systems under the condition that $V \in C^1(\mathbf{R}^d)$ and $\nabla V(x) = O(|x|^{-1-\epsilon})$ as $|x| \to \infty$ for some $\epsilon > 0$. Our assumption $\rho + \mu > 1$ corresponds to this condition.

Remark 1.6. Under the assumption as in Theorem 1.1, we can prove that the Dollard-type modified wave operators

$$W_0^{D,\pm} = s - \lim_{t\to\pm\infty} e^{itH}U_0^D(t) \tag{1.5}$$

exist and are unitary operators on $L^2(X)$, where $U_0^D(t)$ is the propagator generated by the time-dependent Hamiltonian

$$H_0^D(t) = H_0 + V(pt - \frac{E}{2}t^2), \tag{1.6}$$

i.e. $\{U_0^D(t)\}_{t\in\mathbf{R}}$ is a family of unitary operators such that for $\psi \in \mathcal{D}(H_0)$, $\psi_t := U_0^D(t)\psi$ is a strong solution of $i\frac{\partial \psi_t}{\partial t} = H_0^D(t)\psi_t$, $\psi_0 = \psi$. For two-body case, Jensen and Yajima [JY] proved this when $d = 1$ and $\mu = \frac{1}{2}$. But we need the assumption $\rho + \mu > 1$.

Remark 1.7. For short-range case, when each pair potential V_α satisfies $|V_\alpha(x^\alpha)| = O(|x^\alpha|^{-\rho})$ as $|x^\alpha| \to \infty$ for some $\rho > \frac{1}{2}$, the multi-channel scattering mentioned in Remark 1.3 has been studied by Korotyaev [Ko] and Tamura [T2] (see Tamura [T3-4] for general N-body case).

Remark 1.8. When $E = 0$, the problem of the existence and the asymptotic completeness of the multi-channel wave operators has been recently solved for short-range potentials as well as for a class of long-range potentials (cf. Sigal-Soffer [SS1], Graf [Gr1], Kitada [Ki1-2], Tamura [T1] and Yafaev [Y] for short-range case, and Enss [E], Dereziński [D1-2], Gérard [G] and Wang [Wa] for long-range case).

2. An outline of the proof of Theorem 1.1

In this section, we give an outline of the proof of Theorem 1.1. For brevity, we omit the proof of the existence of the modified wave operators $W_0^{G,\pm}$. When we admit the existence of $W_0^{G,\pm}$, the intertwining property (1.4) can be proved by the standard way.

First we need the following proposition.

Proposition 2.1. *Assume that V satisfies $(V)_{\rho,\mu}$ with $\rho+\mu>1$. Let $J\subset\mathbf{R}$ be any bounded interval. Then for $\psi\in RanE_J(H)$, there exist $\psi_0^\pm\in L^2(X)$ such that*

$$e^{-itH}\psi = U_0^G(t)\psi_0^\pm + o(1) \text{ as } t\to\pm\infty. \tag{2.1}$$

We should note that this proposition is a key step for the proof of the asymptotic completeness of $W_0^{G,\pm}$. The property stated in this proposition is called asymptotic clustering by some authors ([SS2], [DG] and [Ki2]). The proof is completed by proving some propagation estimates, but we omit it.

Now we shall show the asymptotic completeness of $W_0^{G,+}$ only. For $W_0^{G,-}$, the proof is similar. In virtue of Proposition 2.1 and the existence of the modified wave operator $W_0^{G,+}$, we see that for $\psi\in RanE_J(H)$, there exists $\psi_0^+\in L^2(X)$ such that $\psi=W_0^{G,+}\psi_0^+$, which implies $RanE_J(H)\subset RanW_0^{G,+}$. If we note $W_0^{G,+}$ is an isometry and $J\subset\mathbf{R}$ is any bounded interval, this implies that $W_0^{G,+}$ is a unitary operator on $L^2(X)$. This completes the proof of Theorem 1.1.

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