数理解析研究所講究録
第 873 巻 1994 年 140-155

On the Elliptic-hyperbolic Davey-Stewartson system

HIRATA Hitoshi (平田 均)
Department of Mathematical Science
University of Tokyo
Komaba, Meguro-ku, Tokyo 153, Japan

§1. Introduction and theorem. In this paper, we consider Davey-Stewartson system:

(1) $$i\partial_t u + \partial_x^2 u + \delta\partial_y^2 u = a|u|^2 u + bu\partial_x\psi,$$

(2) $$\partial_x^2\psi + m\partial_y^2\psi = \partial_x(|u|^2).$$

Here $u = u(t,x,y)$ is a complex valued function of $t \in \mathbf{R}$ and $(x,y) \in \mathbf{R}^2$, and $\psi = \psi(t,x,y)$ is a real valued function. This system is introduced by Davey and Stewartson [DS] in 1974, as the equation of 2-dimensional long waves over finite depth liquid. They considered 1-dimensional progressive wave $i\varepsilon\omega\exp[ikx]$ + complex conjugate, and u is related to the distortion from this progressive wave. ψ is related to the velocity potential. And real parameters δ, m, a, b are determined by the wave number k, frequency ω of the first progressive wave, and the depth h.

In their work, $\delta < 0$ and $m > 0$, but considering the effect of the surface tension, (δ, m) may become $(-,+)$, $(+,+)$ and $(+,-)$ physically. (See the work of Ghidaglia and Saut [GS].) We call each of above cases hyperbolic-elliptic case, elliptic-elliptic case, and elliptic-hyperbolic case respectively.

Elliptic-elliptic case is usual 2-dimensional nonlinear Schrödinger equation with non-local interaction, and we can construct a unique solution by usual methods. Since its nonlinear term has non-symmetry, the corresponding stationary problem is quite difficult, and Ohta [Oh1], [Oh2] discussed this problem by using concentration compactness methods. Hyperbolic-elliptic case is so-called *hyperbolic* Schrödinger equation. We can construct the time-local solution of it by similar way for usual nonlinear Schrödinger equation, but we can not obtain a-priori estimate for this solution, so it is unknown that above time-local solution becomes time-global or not.

In this paper, we consider the elliptic-hyperbolic case $(\delta, m) = (+, -)$, and for the simplification, we take $\delta = 1$ and $m = -1$. In this case, since (2) is hyperbolic equation for ψ, ψ is not determined uniquely if we impose no condition for the behavior of ψ as $|x|, |y| \to \infty$. So, we demand following radiation condition:

$$\psi, \nabla\psi \to 0 \quad \text{as } y \pm x \to \infty. \tag{3}$$

Under this condition, we can rewrite (2) as following:

$$\psi = \partial_x K(|u|^2), \quad \text{where} \quad K(f) := \mathcal{F}^{-1}(\eta^2 - \xi^2 + i0)^{-1}\mathcal{F}. \tag{4}$$

Here, $\mathcal{F}$ is Fourier transform from $\mathbf{R}^2_{x,y}$ to $\mathbf{R}^2_{\xi,\eta}$. By using (4), we rewrite (1) and (2) under the condition (3) as following:

$$i\partial_t u = -\Delta_{x,y} u + a|u|^2 u + bu\partial_x^2 K(|u|^2). \tag{5}$$

This equation is a nonlinear Schrödinger equation on $\mathbf{R}^2_{x,y}$, but since the second nonlinear term $bu\partial_x^2 K(|u|^2)$ has so-called *derivative loss*, then we can not directly apply the contraction mapping method. Although, by using the compactness method, there exist some results for the Cauchy problem for (5) with initial data $u(0, x, y) = u_0 \in X$, where X is some Banach space. (See the works of Ghidaglia and Saut [GS], Tsutsumi M. [TM].) They obtain a global weak solution of (5), but one can not obtain uniqueness of the solution. To obtain the uniqueness, we want to solve (5) using contraction mapping method. So we regard the local smoothing property of free Schrödinger equation.

The main result of this paper is following.

THEOREM 1. *Let a be a real constant and b a real function of (x, y) such that $b \in W^{3,\infty}(\mathbf{R}^2_{x,y})$ and $\langle r\rangle^{1/2+\varepsilon} b \in L^\infty(\mathbf{R}^2_{x,y})$ for some $\varepsilon > 0$, where $W^{k,p} := \{f \in L^p : \partial^\alpha f \in L^p \text{ for } |\alpha| \le k\}$ and $\langle r\rangle = (1 + |x|^2 + |y|^2)^{1/2}$. Then, if $u_0 \in \Sigma(5)$ is sufficiently* *small for the norm of this space, where* $H^k = W^{k,2}$, *there exists a unique time-local solution* $u \in X(T)$ *of*

$$(6)\qquad \begin{cases} i\partial_t u = -\Delta_{x,y} u + a|u|^2 u + bu\partial_x^2 K(|u|^2), \\ u(0,x,y) = u_0(x,y). \end{cases}$$

Here,

$$\begin{aligned} \Sigma(k) :=& \{f \in L^2(\mathbf{R}^2_{x,y}) : x^{\beta_1} y^{\beta_2} \partial_x^{\alpha_1} \partial_y^{\alpha_2} f \in L^2(\mathbf{R}^2) \text{ for } \alpha_1 + \alpha_2 + \beta_1 + \beta_2 \le k\} \\ =& H^k(\mathbf{R}^2) \cap L^2_k(\mathbf{R}^2), \\ \|f; \Sigma(k)\| :=& \{ \sum_{\alpha_1+\alpha_2+\beta_1+\beta_2 \le k} \|x^{\beta_1} y^{\beta_2} \partial_x^{\alpha_1} \partial_y^{\alpha_2} f; L^2(\mathbf{R}^2)\|^2 \}^{1/2}, \end{aligned}$$

and

$$\begin{aligned} & X(T) = X_0(T) \cap X_a(T), \\ & X_0(T) := C([0,T]; \Sigma(3)), \\ & X_a(T) := \{ v : \partial_x^\alpha \partial_y^\beta v \in W^{1,\infty}(\mathbf{R}_x; L^2([0,T] \times \mathbf{R}_y)) \cap W^{1,\infty}(\mathbf{R}_y; L^2([0,T] \times \mathbf{R}_x)) \\ & \qquad\qquad \text{for } \alpha + \beta \le 3 \}. \end{aligned}$$

Recently, Linares and Ponce[LP] proved the existence and uniqueness of time-local solution of (6) in the case that b is constant not only a. They showed if $u_0 \in H^{12}(\mathbf{R}^2_{x,y}) \cap H^6(\mathbf{R}^2_{x,y}; \langle r \rangle^6 dxdy)$ is sufficiently small, then there exists unique time-local solution of (6) in suitable space. But the spaces which they used are very complicated and their initial space is too narrow. Here,we show if b decays, one can construct the solution in more wide space.

Originally, Davey-Stewartson system is introduced for a kind of 2-dimensionalization of 1-dimensional cubic nonlinear Schrödinger equation:

$$(7)\qquad i\partial_t u = -\Delta u + \lambda |u|^2 u,$$

and in case that parameters δ, m, a and b satisfy some relations, it becomes a soliton equation as like as (7). So, there exist several works which one deals it as a soliton equation, (e.g. [AbFo], [AnFr], [BC], [FS]), but in this paper, we consider more general situation, and we use functional analysis methods.

§2. Proof of theorem. First, we change the variables x, y for $x' = \frac{1}{\sqrt{2}}(y+x)$ and $y' = \frac{1}{\sqrt{2}}(y-x)$, and rewrite the equation (6). The rewritten equation is

(8)
$$\begin{cases} i\partial_t u = -\Delta_{x,y}u + a|u|^2u + bu\left(\int_x^\infty \partial_y|u(\xi,y)|^2d\xi + \int_y^\infty \partial_x|u(x,\eta)|^2d\eta\right), \\ u(0,x,y) = u_0(x,y). \end{cases}$$

Here, to simplify the notations, we denote again by x, y in place of x', y'. The corresponding integral equation for (8) is

$$\text{(9)} \qquad \begin{cases} u(t) = U(t)u_0 - i\int_0^t U(t-s)N(u(s))ds, \\ N(u) := a|u|^2u + bu\left(\int_x^\infty \partial_y|u(\xi,y)|^2d\xi + \int_y^\infty \partial_x|u(x,\eta)|^2d\eta\right), \end{cases}$$

where $U(t) := \exp[it\Delta_{x,y}]$ is free Schrödinger propagator. Since (8) and (9) are equivalent if $u \in X(T)$, it suffices to prove that (9) has a unique solution $u \in X(T)$. Now, remark that the free Schrödinger propagator U satisfies following well-known estimates.

LEMMA 2. *(1) $U : \phi \mapsto U\phi$ is a bounded operator from $\Sigma(k)$ to $C([0,T];\Sigma(k))$ for any $k \in \overline{\mathbf{N}}$, and*

$$\|U\phi; C([0,T];\Sigma(k))\| \le C(1+T^k)\|\phi;\Sigma(k)\|.$$

(2) $S : f \mapsto Sf := \int_0^t U(t-s)f(s)ds$ is a bounded operator from $L^1([0,T];\Sigma(k))$ to $C([0,T];\Sigma(k))$ for any $k \in \overline{\mathbf{N}}$, and

$$\|Sf; C([0,T];\Sigma(k))\| \le C(1+T^k)\|f; L^1([0,T];\Sigma(k))\|.$$

Moreover, S satisfies following key estimate.

LEMMA 3. *S is a bounded operator from $L^1\left(\mathbf{R}_x; L^2([0,T]\times\mathbf{R}_y)\right)$ to $W^{1,\infty}\left(\mathbf{R}_x; L^2([0,T]\times\mathbf{R}_y)\right)$, and*

$$\|Sf; W^{1,\infty}\left(\mathbf{R}_x; L^2([0,T]\times\mathbf{R}_y)\right)\| \le C(1+T)\|f; L^1\left(\mathbf{R}_x; L^2([0,T]\times\mathbf{R}_y)\right)\|.$$

This estimate means if f decays for x-direction, then Sf becomes more regular with respect to x-variable than f at least locally. That is, this estimate means some kind of local smoothing effect of S.

Then, we define following auxiliary space:

$$
\begin{aligned}
&Y(T) = Y_0(T) \cap Y_a(T),\\
&Y_0(T) := L^1\big([0,T];\Sigma(3)\big),\\
&Y_a(T) := \big\{v : \partial_x^\alpha \partial_y^\beta v \in L^1\big(\mathbf{R}_x; L^2([0,T]\times \mathbf{R}_y)\big) \cap L^1\big(\mathbf{R}_y; L^2([0,T]\times \mathbf{R}_x)\big)\\
&\qquad\qquad \text{for } \alpha+\beta \le 3\big\}.
\end{aligned}
$$

Lemma 2 and Lemma 3 show the operator S maps $Y(T)$ to $X(T)$.

Next, we estimate the nonlinear term $N(u)$. Let $N_1(u) := bu\int_x^\infty \partial_y |u(\xi,y)|^2 d\xi$. Remark following facts:

(i) By Sobolev's inequality, we have

$$\|f; L^p(\mathbf{R}^2)\| \le C\|f; H^s(\mathbf{R}^2)\| \quad \text{for } 1/p = 1/2 - s/2, \quad 0 \le s < 1,$$

and

$$\|f; L^\infty(\mathbf{R}^2)\| \le C\|f; H^{1+\varepsilon}(\mathbf{R}^2)\|.$$

From here, ε means a certain small positive number.

(ii) By Hölder's inequality, we have

$$\|f; L^p(\mathbf{R}_z^n)\| \le C\|\langle z\rangle^{s+\varepsilon} f; L^q(\mathbf{R}_z^n)\| \quad \text{for } s/n = 1/p - 1/q.$$

(iii) For any $p \in [1,\infty]$,

$$
\begin{aligned}
&\|f; L^p\big(\mathbf{R}_x; L^p([0,T]\times\mathbf{R}_y)\big)\| = \|f; L^p\big(\mathbf{R}_y; L^p([0,T]\times\mathbf{R}_x)\big)\|\\
&= \|f; L^p([0,T]; L^p(\mathbf{R}^2))\| = \|f; L^p([0,T]\times\mathbf{R}^2)\|.
\end{aligned}
$$

(iv) If $\alpha+\beta \le k$, then

$$\|\langle r\rangle^\alpha \partial_{x,y}^\beta f; L^2(\mathbf{R}^2)\| \le C\|f; \Sigma(k)\|.$$

We have to estimate $\|N_1(u); Y_0(T)\|$ and $\|N_1(u); Y_a(T)\|$ by $\|u; X\|$.

Let $u \in X(T)$. Then we have $\|u; C([0,T];\Sigma(3))\| < \infty$ and $\|\partial_x^4 u; L^\infty\big(\mathbf{R}_x; L^2([0,T]\times \mathbf{R}_y)\big)\| < \infty$.

Remark if we differentiate $N_1(u)$ with respect to x, the term which this differentiation operates integral factor is low order. So, we only consider the derivation with respect to y. By Leibniz's rule, we have

$$\begin{aligned}&\partial_y^3 N_1(u)\\ &= \sum_{\alpha_1+\alpha_2+\alpha_3+\alpha_4=3} \frac{12}{\alpha_1!\alpha_2!\alpha_3!\alpha_4!}\partial_y^{\alpha_1} b\, \partial_y^{\alpha_2} u \int_x^\infty \mathrm{Re}\{\partial_y^{\alpha_3+1}u(\xi,y)\partial_y^{\alpha_4}\overline{u}(\xi,y)\}d\xi.\end{aligned}$$

Taking care of the derivative order, we use following term. We denote $\partial_y^{\alpha_1} b\, \partial_y^{\alpha_2} u \int_x^\infty \partial_y^{\alpha_3+1}u(\xi,y)\partial_y^{\alpha_4}\overline{u}(\xi,y)d\xi$ by $(\alpha_1,\alpha_2,\alpha_3+1,\alpha_4)$-term. Then, appearing terms in $\partial_y^3 N_1(u)$ are following: (0,0,4,0), (0,0,3,1), (0,0,2,2), (0,0,1,3), (0,1,3,0), (0,1,2,1), (0,1,1,2), (0,2,2,0), (0,2,1,1), (0,3,1,0), (1,0,3,0), (1,0,2,1), (1,0,1,2), (1,1,2,0), (1,1,1,1), (1,2,1,0), (2,0,2,0), (2,0,1,1), (2,1,1,0) and (3,0,1,0). But since third and fourth factors have even role, we can quit (0,0,1,3), (0,1,1,2) and (1,0,1,2)-terms.

Firstly, we estimate $\|N_1(u); L^1(0,T;\Sigma(3))\|$. First, (0,0,4,0)-term is estimated as following:

$$\begin{aligned}&\left\|bu\int_x^\infty \partial_y^4 u\overline{u}d\xi; L^1(0,T;L^2(\mathbf{R}^2))\right\|\\
\leq& T^{1/2}\left\|bu\int_x^\infty \partial_y^4 u\overline{u}d\xi; L^2([0,T]\times\mathbf{R}^2)\right\|\\
\leq& T^{1/2}\|b;L^\infty\|\|u;L^2(\mathbf{R}_x;L^\infty([0,T]\times\mathbf{R}_y))\|\\
&\times\left\|\int_x^\infty \partial_y^4 u\overline{u}d\xi; L^\infty(\mathbf{R}_x;L^2([0,T]\times\mathbf{R}_y))\right\|\\
\leq& CT^{1/2}\|b;L^\infty\|\|\langle x\rangle^{1/2+\epsilon}u;L^\infty([0,T]\times\mathbf{R}^2)\|\|\partial_y^4 u\overline{u};L^1(\mathbf{R}_x;L^2([0,T]\times\mathbf{R}_y))\|\\
\leq& CT^{1/2}\|b;L^\infty\|\|\langle x\rangle^{1/2+\epsilon}u;L^\infty([0,T]\times\mathbf{R}^2)\|\|\partial_y^4 u\cdot\langle x\rangle^{1/2+\epsilon}\overline{u};L^2([0,T]\times\mathbf{R}^2)\|\\
\leq& CT^{1/2}\|b;L^\infty\|\|\langle x\rangle^{1/2+\epsilon}u;C([0,T];H^{1+\epsilon}(\mathbf{R}^2))\|\\
&\times\|\partial_y^4 u;L^\infty(\mathbf{R}_y;L^2([0,T]\times\mathbf{R}_x))\|\|\langle x\rangle^{1/2+\epsilon}u;L^2(\mathbf{R}_y;L^\infty([0,T]\times\mathbf{R}_x))\|\\
\leq& CT^{1/2}\|b;L^\infty\|\|\langle x\rangle^{1/2+\epsilon}u;C([0,T];H^{1+\epsilon}(\mathbf{R}^2))\|\\
&\times\|\partial_y^4 u;L^\infty(\mathbf{R}_y;L^2([0,T]\times\mathbf{R}_x))\|\|\langle x\rangle^{1/2+\epsilon}\langle y\rangle^{1/2+\epsilon}u;L^\infty([0,T]\times\mathbf{R}^2)\|\\
\leq& CT^{1/2}\|b;L^\infty\|\|u;C([0,T];\Sigma(2))\|\|u;X_a\|\\
&\times\|\langle x\rangle^{1/2+\epsilon}\langle y\rangle^{1/2+\epsilon}u;C([0,T];H^{1+\epsilon}(\mathbf{R}^2))\|\\
\leq& CT^{1/2}\|b;L^\infty\|\|u;C([0,T];\Sigma(2))\|\|u;C([0,T];\Sigma(3))\|\|u;X_a\|.\end{aligned}$$

By similar way, the setimate of (0,0,3,1)-term is

$$\left\|bu\int_x^\infty \partial_y^3 u\partial_y\overline{u}d\xi; L^1(0,T;L^2(\mathbf{R}^2))\right\|$$
$$\le CT^{1/2}\|b;L^\infty\|\|u;C([0,T];\Sigma(2))\|\|\partial_y^3 u\partial_y\overline{u}; L^1(\mathbf{R}_x;L^2([0,T]\times\mathbf{R}_y))\|$$
$$\le CT^{1/2}\|b;L^\infty\|\|u;C([0,T];\Sigma(2))\|\|\partial_y^3 u\cdot\langle x\rangle^{1/2+\epsilon}\partial_y\overline{u}; L^2([0,T]\times\mathbf{R}^2))\|$$
$$\le CT^{1/2}\|b;L^\infty\|\|u;C([0,T];\Sigma(2))\|\|\partial_y^3 u;C([0,T];L^2(\mathbf{R}^2))\|$$
$$\times\|\langle x\rangle^{1/2+\epsilon}\partial_y u;L^2([0,T];L^\infty(\mathbf{R}^2))\|$$
$$\le CT\|b;L^\infty\|\|u;C([0,T];\Sigma(2))\|\|\partial_y^3 u;C([0,T];L^2(\mathbf{R}^2))\|$$
$$\times\|\langle x\rangle^{1/2+\epsilon}\partial_y u;C([0,T];H^{1+\epsilon}(\mathbf{R}^2))\|$$
$$\le CT\|b;L^\infty\|\|u;C([0,T];\Sigma(2))\|\|u;C([0,T];\Sigma(3))\|^2.$$

Next, (0,0,2,2)-term is estimated as following:

$$\left\|bu\int_x^\infty |\partial_y^2 u|^2 d\xi; L^1(0,T;L^2(\mathbf{R}^2))\right\|$$
$$\le CT^{1/2}\|b;L^\infty\|\|u;C([0,T];\Sigma(2))\|\|\partial_y^2 u; L^2(\mathbf{R}_x;L^4([0,T]\times\mathbf{R}_y))\|^2$$
$$\le CT^{1/2}\|b;L^\infty\|\|u;C([0,T];\Sigma(2))\|\|\langle x\rangle^{1/4+\epsilon}\partial_y^2 u;L^4([0,T]\times\mathbf{R}^2)\|^2$$
$$\le CT\|b;L^\infty\|\|u;C([0,T];\Sigma(2))\|\|\langle x\rangle^{1/4+\epsilon}\partial_y^2 u;C([0,T];L^4(\mathbf{R}^2))\|^2$$
$$\le CT\|b;L^\infty\|\|u;C([0,T];\Sigma(2))\|\|\langle x\rangle^{1/4+\epsilon}\partial_y^2 u;C([0,T];H^{1/2}(\mathbf{R}^2))\|^2$$
$$\le CT\|b;L^\infty\|\|u;C([0,T];\Sigma(2))\|\|u;C([0,T];\Sigma(3))\|^2.$$

Since second factor is estimated by $C([0,T];\Sigma(2))$-norm not $C([0,T];\Sigma(3))$, (0,1,3,0)-term and (0,1,2,1)-term can be dealt by similar way. Besides, the estimate of (0,2,2,0)-term is

$$\left\|b\partial_y^2 u\int_x^\infty \partial_y^2 u\overline{u}d\xi; L^1(0,T;L^2(\mathbf{R}^2))\right\|$$
$$\le T^{1/2}\|b;L^\infty\|\|\partial_y^2 u; L^2(\mathbf{R}_x;L^3([0,T]\times\mathbf{R}_y))\|\|\partial_y^2 u\overline{u}; L^1(\mathbf{R}_x;L^6([0,T]\times\mathbf{R}_y))\|$$
$$\le CT^{1/2}\|b;L^\infty\|\|\langle x\rangle^{1/6+\epsilon}\partial_y^2 u;L^3([0,T]\times\mathbf{R}^2)\|\|\partial_y^2 u;L^6([0,T]\times\mathbf{R}^2)\|$$
$$\times\|u;L^{6/5}(\mathbf{R}_x;L^\infty([0,T]\times\mathbf{R}_y))\|$$
$$\le CT\|b;L^\infty\|\|\langle x\rangle^{1/6+\epsilon}\partial_y^2 u;C([0,T];L^3(\mathbf{R}^2))\|\|\partial_y^2 u;C([0,T];L^6(\mathbf{R}^2))\|$$
$$\times\|\langle x\rangle^{5/6+\epsilon}u;C([0,T];L^\infty(\mathbf{R}^2))\|$$

$$\leq CT\|b;L^\infty\|\|\langle x\rangle^{1/6+\epsilon}\partial_y^2 u;C([0,T];H^{1/3}(\mathbf{R}^2))\|\|\partial_y^2 u;C([0,T];H^{2/3}(\mathbf{R}^2))\|$$
$$\times\|\langle x\rangle^{5/6+\epsilon}u;C([0,T];H^{1+\epsilon}(\mathbf{R}^2))\|$$
$$\leq CT\|b;L^\infty(\mathbf{R}^2)\|\|u;C([0,T];\Sigma(3))\|^2\|u;C([0,T];\Sigma(2))\|.$$

Since fourth factor is estimated by $C([0,T];\Sigma(2))$-norm, (0,2,1,1)-term can be dealt with similar way. Furthermore, (0,3,1,0)-term is

$$\left\|b\partial_y^3 u\int_x^\infty \partial_y u\overline{u}d\xi;L^1(0,T;L^2(\mathbf{R}^2))\right\|$$
$$\leq\|b;L^\infty\|\|\partial_y^3 u;C([0,T];L^2(\mathbf{R}^2))\|\left\|\int_x^\infty \partial_y u\overline{u}d\xi;L^1(0,T;L^\infty(\mathbf{R}^2))\right\|$$
$$\leq T\|b;L^\infty\|\|u;C([0,T];\Sigma(3))\|\|\partial_y u\overline{u};L^1\left(\mathbf{R}_x;L^\infty([0,T]\times\mathbf{R}_y)\right)\|$$
$$\leq CT\|b;L^\infty\|\|u;C([0,T];\Sigma(3))\|\|\partial_y u;L^\infty([0,T]\times\mathbf{R}^2)\|$$
$$\times\|\langle x\rangle^{1+\epsilon}u;L^\infty([0,T]\times\mathbf{R}^2))\|$$
$$\leq CT\|b;L^\infty\|\|u;C([0,T];\Sigma(3))\|\|\partial_y u;C([0,T];H^{1+\epsilon}(\mathbf{R}^2))\|$$
$$\times\|\langle x\rangle^{1+\epsilon}u;C([0,T];H^{1+\epsilon}(\mathbf{R}^2))\|$$
$$\leq CT\|b;L^\infty\|\|u;C([0,T];\Sigma(3))\|^3.$$

The estimates for the terms which have non-zero derivative's order to b with respect to y are easier than non-derivative terms, so we omit them.

Next, we consider the estimation for the term which has weight of the spatial variables. We only need to estimate following term.

$$\left\|\langle r\rangle^3 bu\int_x^\infty \partial_y u\overline{u}d\xi;L^1(0,T;L^2(\mathbf{R}^2))\right\|$$
$$\leq\|\langle r\rangle^3 u;C([0,T];L^2(\mathbf{R}^2))\|\left\|b\int_x^\infty \partial_y u\overline{u}d\xi;L^1(0,T;L^\infty(\mathbf{R}^2))\right\|$$
$$\leq T\|b;L^\infty\|\|u;C([0,T];\Sigma(3))\|\|\partial_y u\overline{u};L^1\left(\mathbf{R}_x;L^\infty([0,T]\times\mathbf{R}_y)\right)\|$$
$$\leq CT\|b;L^\infty\|\|u;C([0,T];\Sigma(3))\|\|\partial_y u\cdot\langle x\rangle^{1+\epsilon}\overline{u};L^\infty([0,T]\times\mathbf{R}^2)\|$$
$$\leq CT\|b;L^\infty\|\|u;C([0,T];\Sigma(3))\|\|\partial_y u;L^\infty([0,T]\times\mathbf{R}^2)\|\|\langle x\rangle^{1+\epsilon}u;L^\infty([0,T]\times\mathbf{R}^2)\|$$
$$\leq CT\|b;L^\infty\|\|u;C([0,T];\Sigma(3))\|\|\partial_y u;C([0,T];H^{1+\epsilon}(\mathbf{R}^2))\|$$
$$\times\|\langle x\rangle^{1+\epsilon}u;C([0,T];H^{1+\epsilon}(\mathbf{R}^2))\|$$
$$\leq CT\|b;L^\infty\|\|u;C([0,T];\Sigma(3))\|^3.$$

Then, putting together above all estimates, we obtain

$$\|N_1(u);Y_0(T)\| \le C(T+T^{1/2})\|b;L^\infty\|\|u;X\|^3.$$

Secondly, we consider Y_a-norm. In this norm, we have to estimate $\|\partial_{x,y}N_1(u); L^1(\mathbf{R}_x;L^2([0,T]\times\mathbf{R}_y))\|$ and $\|\partial_{x,y}N_1(u);L^1(\mathbf{R}_y;L^2([0,T]\times\mathbf{R}_x))\|$. For the same reason in the previous estimates, we only consider The derivation with respect to x, and we omit the estimation for the terms whose first number is zero. First, (0,0,4,0)-term is estimated as following:

$$\begin{aligned}
&\left\| bu\int_x^\infty \partial_y^4 u\bar{u}d\xi; L^1\left(\mathbf{R}_x;L^2([0,T]\times\mathbf{R}_y)\right)\right\| \\
\le&\|bu;L^1\left(\mathbf{R}_x;L^\infty([0,T]\times\mathbf{R}_y)\right)\|\|\partial_y^4u\bar{u};L^1\left(\mathbf{R}_x;L^2([0,T]\times\mathbf{R}_y)\right)\| \\
\le&C\|b;L^\infty\|\|\langle x\rangle^{1+\epsilon}u;L^\infty([0,T]\times\mathbf{R}^2)\|\|\partial_y^4u\cdot\langle x\rangle^{1/2+\epsilon}\bar{u};L^2([0,T]\times\mathbf{R}^2)\| \\
\le&C\|b;L^\infty\|\|u;C([0,T];\Sigma(3))\|\|\partial_y^4u;L^\infty\left(\mathbf{R}_y;L^2([0,T]\times\mathbf{R}_x)\right)\| \\
&\times\|\langle x\rangle^{1/2+\epsilon}u;L^2\left(\mathbf{R}_y;L^\infty([0,T]\times\mathbf{R}_x)\right)\| \\
\le&C\|b;L^\infty\|\|u;C([0,T];\Sigma(3))\|\|\partial_y^4u;L^\infty\left(\mathbf{R}_y;L^2([0,T]\times\mathbf{R}_x)\right)\| \\
&\times\|\langle x\rangle^{1/2+\epsilon}\langle y\rangle^{1/2+\epsilon}u;L^\infty([0,T]\times\mathbf{R}^2)\| \\
\le&C\|b;L^\infty\|\|u;C([0,T];\Sigma(3))\|^2\|\partial_y^4u;L^\infty\left(\mathbf{R}_y;L^2([0,T]\times\mathbf{R}_x)\right)\|,
\end{aligned}$$

and

$$\begin{aligned}
&\left\| bu\int_x^\infty \partial_y^4 u\bar{u}d\xi; L^1\left(\mathbf{R}_y;L^2([0,T]\times\mathbf{R}_x)\right)\right\| \\
\le&C\left\|b\langle y\rangle^{1/2+\epsilon}u\int_x^\infty \partial_y^4u\bar{u}d\xi;L^2([0,T]\times\mathbf{R}^2)\right\| \\
\le&C\|b;L^\infty\|\langle y\rangle^{1/2+\epsilon}u;L^2\left(\mathbf{R}_x;L^\infty([0,T]\times\mathbf{R}_y)\right)\|\|\partial_y^4u\bar{u};L^1\left(\mathbf{R}_x;L^2([0,T]\times\mathbf{R}_y)\right)\| \\
\le&C\|b;L^\infty\|\|\langle x\rangle^{1/2+\epsilon}\langle y\rangle^{1/2+\epsilon}u;L^\infty([0,T]\times\mathbf{R}^2)\|\|\partial_y^4u\cdot\langle x\rangle^{1/2+\epsilon}\bar{u};L^2([0,T]\times\mathbf{R}^2)\| \\
\le&C\|b;L^\infty\|\|u;C([0,T];\Sigma(3))\|\|\partial_y^4u;L^\infty\left(\mathbf{R}_y;L^2([0,T]\times\mathbf{R}_x)\right)\| \\
&\times\|\langle x\rangle^{1/2+\epsilon}u;L^2\left(\mathbf{R}_y;L^\infty([0,T]\times\mathbf{R}_x)\right)\| \\
\le&C\|b;L^\infty\|\|u;C([0,T];\Sigma(3))\|\|\partial_y^4u;L^\infty\left(\mathbf{R}_y;L^2([0,T]\times\mathbf{R}_x)\right)\| \\
&\times\|\langle x\rangle^{1/2+\epsilon}\langle y\rangle^{1/2+\epsilon}u;L^\infty([0,T]\times\mathbf{R}^2)\|
\end{aligned}$$

$$\le C\|b;L^\infty\|\|u;C([0,T];\Sigma(3))\|^2\|\partial_y^4 u;L^\infty\left(\mathbf{R}_y;L^2([0,T]\times\mathbf{R}_x)\right)\|.$$

Next, the estimate of (0,0,3,1)-term is

$$\begin{aligned}
&\left\|bu\int_x^\infty \partial_y^3 u\partial_y\overline{u}d\xi;L^1\left(\mathbf{R}_x;L^2([0,T]\times\mathbf{R}_y)\right)\right\| \\
&\le\|bu;L^1\left(\mathbf{R}_x;L^\infty([0,T]\times\mathbf{R}_y)\right)\|\|\partial_y^3 u\partial_y\overline{u};L^1\left(\mathbf{R}_x;L^2([0,T]\times\mathbf{R}_y)\right)\| \\
&\le C\|b;L^\infty\|\|\langle x\rangle^{1+\epsilon}u;L^\infty([0,T]\times\mathbf{R}^2)\|\|\partial_y^3 u\partial_y\langle x\rangle^{1/2+\epsilon}\overline{u};L^2([0,T]\times\mathbf{R}^2)\| \\
&\le C\|b;L^\infty\|\|\langle x\rangle^{1+\epsilon}u;C([0,T];H^{1+\epsilon}(\mathbf{R}^2))\|\|\partial_y^3 u;C([0,T];L^2(\mathbf{R}^2))\| \\
&\quad\times\|\langle x\rangle^{1/2+\epsilon}\partial_y u;L^2([0,T];L^\infty(\mathbf{R}^2))\| \\
&\le CT^{1/2}\|b;L^\infty\|\|u;C([0,T];\Sigma(3))\|^3,
\end{aligned}$$

and

$$\begin{aligned}
&\left\|bu\int_x^\infty \partial_y^3 u\partial_y\overline{u}d\xi;L^1\left(\mathbf{R}_y;L^2([0,T]\times\mathbf{R}_x)\right)\right\| \\
&\le C\|b;L^\infty\|\|u;C([0,T];\Sigma(3))\|\|\partial_y^3 u\cdot\partial_y\overline{u};L^1\left(\mathbf{R}_x;L^2([0,T]\times\mathbf{R}^2)\right)\| \\
&\le CT^{1/2}\|b;L^\infty\|\|u;C([0,T];\Sigma(3))\|^3.
\end{aligned}$$

Similarly, the estimate of (0,0,2,2)-term is

$$\begin{aligned}
&\left\|bu\int_x^\infty |\partial_y^2 u|^2 d\xi;L^1(\mathbf{R}_x;L^2([0,T]\times\mathbf{R}_y))\right\| \\
&\le C\|b;L^\infty\|\|u;C([0,T];\Sigma(3))\|\|\partial_y^2 u;L^2(\mathbf{R}_x;L^4([0,T]\times\mathbf{R}_y))\|^2 \\
&\le C\|b;L^\infty\|\|u;C([0,T];\Sigma(3))\|\|\langle x\rangle^{1/4+\epsilon}\partial_y^2 u;L^4([0,T]\times\mathbf{R}^2)\|^2 \\
&\le CT^{1/2}\|b;L^\infty\|\|u;C([0,T];\Sigma(3))\|\|\langle x\rangle^{1/4+\epsilon}\partial_y^2 u;C([0,T];H^{1/2}(\mathbf{R}^2))\|^2 \\
&\le CT^{1/2}\|b;L^\infty\|\|u;C([0,T];\Sigma(3))\|^3,
\end{aligned}$$

and

$$\begin{aligned}
&\left\|bu\int_x^\infty |\partial_y^2 u|^2 d\xi;L^1(\mathbf{R}_y;L^2([0,T]\times\mathbf{R}_x))\right\| \\
&\le CT^{1/2}\|b;L^\infty\|\|u;C([0,T];\Sigma(3))\|^3.
\end{aligned}$$

Moreover, the estimate of (0,1,3,0)-term is

$$\left\|b\partial_y u\int_x^\infty \partial_y^3 u\overline{u}d\xi;L^1\left(\mathbf{R}_x;L^2([0,T]\times\mathbf{R}_y)\right)\right\|$$

$$\leq \|b\partial_y u; L^1(\mathbf{R}_x; L^\infty([0,T] \times \mathbf{R}_y))\| \|\partial_y^3 u\overline{u}; L^1(\mathbf{R}_x; L^2([0,T] \times \mathbf{R}_y))\|$$
$$\leq C\|\langle x\rangle^{2\epsilon} b; L^\infty\| \|\langle x\rangle^{1-\epsilon}\partial_y u; L^\infty([0,T] \times \mathbf{R}^2)\| \|\partial_y^3 u; C([0,T]; L^2(\mathbf{R}^2))\|$$
$$\times \|\langle x\rangle^{1/2+\epsilon} u; L^2([0,T]; L^\infty(\mathbf{R}^2))\|$$
$$\leq CT^{1/2}\|\langle x\rangle^{2\epsilon} b; L^\infty\| \|\langle x\rangle^{1-\epsilon}\partial_y u; C([0,T]; H^{1+\epsilon}(\mathbf{R}^2))\| \|u; C([0,T]; \Sigma(3))\|^2$$
$$\leq CT^{1/2}\|\langle x\rangle^{2\epsilon} b; L^\infty\| \|u; C([0,T]; \Sigma(3))\|^3.$$

In this term we use the decay of b. The estimate of $\|b\partial_y u \int_x^\infty \partial_y^3 u\overline{u}d\xi; L^1(\mathbf{R}_y; L^2([0,T] \times \mathbf{R}_x))\|$ is similar. Besides, the estimate of (0,1,2,1)-term is

$$\left\|b\partial_y u \int_x^\infty \partial_y^2 u\partial_y\overline{u}d\xi; L^1(\mathbf{R}_x; L^2([0,T] \times \mathbf{R}_y))\right\|$$
$$\leq \|b\partial_y u; L^1(\mathbf{R}_x; L^4([0,T] \times \mathbf{R}_y))\| \|\partial_y^2 u\partial_y\overline{u}; L^1(\mathbf{R}_x; L^4([0,T] \times \mathbf{R}_y))\|$$
$$\leq C\|b; L^\infty\| \|\langle x\rangle^{3/4+\epsilon}\partial_y u; L^4([0,T] \times \mathbf{R}^2)\| \|\partial_y^2 u\langle x\rangle^{3/4+\epsilon}\partial_y\overline{u}; L^4([0,T] \times \mathbf{R}^2)\|$$
$$\leq CT^{1/2}\|b; L^\infty\| \|\langle x\rangle^{3/4+\epsilon}\partial_y u; C([0,T]; H^{1/2}(\mathbf{R}^2))\| \|\partial_y^2 u; C([0,T]; L^4(\mathbf{R}^2))\|$$
$$\times \|\langle x\rangle^{3/4+\epsilon}\partial_y u; C([0,T]; L^\infty(\mathbf{R}^2))\|$$
$$\leq CT^{1/2}\|b; L^\infty\| \|u; C([0,T]; \Sigma(3))\|^3,$$

and

$$\left\|b\partial_y u \int_x^\infty \partial_y^2 u\partial_y\overline{u}d\xi; L^1(\mathbf{R}_y; L^2([0,T] \times \mathbf{R}_x))\right\|$$
$$\leq \|b\langle y\rangle^{1/2+\epsilon}\partial_y u; L^1(\mathbf{R}_x; L^4([0,T] \times \mathbf{R}_y))\| \|\partial_y^2 u\partial_y\overline{u}; L^1(\mathbf{R}_x; L^4([0,T] \times \mathbf{R}_y))\|$$
$$\leq C\|b; L^\infty\| \|\langle x\rangle^{1/4+\epsilon}\langle y\rangle^{1/2+\epsilon}\partial_y u; L^4([0,T] \times \mathbf{R}^2)\| \|\partial_y^2 u\langle x\rangle^{3/4+\epsilon}\partial_y\overline{u}; L^4([0,T] \times \mathbf{R}^2)\|$$
$$\leq CT^{1/2}\|b; L^\infty\| \|\langle r\rangle^{3/4+\epsilon}\partial_y u; C([0,T]; H^{1/2}(\mathbf{R}^2))\| \|\partial_y^2 u; C([0,T]; L^4(\mathbf{R}^2))\|$$
$$\times \|\langle x\rangle^{3/4+\epsilon}\partial_y u; C([0,T]; L^\infty(\mathbf{R}^2))\|$$
$$\leq CT^{1/2}\|b; L^\infty\| \|u; C([0,T]; \Sigma(3))\|^3.$$

Similarly, the estimate of (0,2,2,0)-term is

$$\left\|b\partial_y^2 u \int_x^\infty \partial_y^2 u\overline{u}d\xi; L^1(\mathbf{R}_x; L^2([0,T] \times \mathbf{R}_y))\right\|$$
$$\leq \|b\partial_y^2 u; L^1(\mathbf{R}_x; L^{8/3}([0,T] \times \mathbf{R}_y))\| \|\partial_y^2 u\overline{u}; L^1(\mathbf{R}_x; L^8([0,T] \times \mathbf{R}_y))\|$$
$$\leq C\|b; L^\infty\| \|\langle x\rangle^{5/8+\epsilon}\partial_y^2 u; L^{8/3}([0,T] \times \mathbf{R}^2)\| \|\partial_y^2 u\langle x\rangle^{7/8+\epsilon}\overline{u}; L^8([0,T] \times \mathbf{R}^2)\|$$

$$\leq CT^{1/2}\|b;L^\infty\|\|\langle x\rangle^{5/8+\epsilon}\partial_y^2 u;C([0,T];H^{1/4}(\mathbf{R}^2))\|\|\partial_y^2 u;C([0,T];H^{3/4}(\mathbf{R}^2))\|$$
$$\times\|\langle x\rangle^{7/8+\epsilon}u;C([0,T];H^{1+\epsilon}(\mathbf{R}^2))\|$$
$$\leq CT^{1/2}\|b;L^\infty\|\|u;C([0,T];\Sigma(3))\|^3.$$

The estimate of $\|b\partial_y^2 u\int_x^\infty \partial_y^2 u\bar{u}d\xi;L^1(\mathbf{R}_y;L^2([0,T]\times\mathbf{R}_x))\|$ is similar as above estimate. And the estimate of (0,2,1,1)-term is similar as above term's one. At last, the estimate of (0,3,1,0)-term is

$$\left\|b\partial_y^3 u\int_x^\infty \partial_y u\bar{u}d\xi;L^1(\mathbf{R}_x;L^2([0,T]\times\mathbf{R}_y))\right\|$$
$$\leq\|b\partial_y^3 u;L^1(\mathbf{R}_x;L^2([0,T]\times\mathbf{R}_y))\|\|\partial_y u\bar{u};L^1(\mathbf{R}_x;L^\infty([0,T]\times\mathbf{R}_y))\|$$
$$\leq C\|\langle x\rangle^{1/2+\epsilon}b;L^\infty\|\|\partial_y^3 u;L^2([0,T]\times\mathbf{R}^2)\|\|\partial_y u\cdot\langle x\rangle^{1+\epsilon}\bar{u};L^\infty([0,T]\times\mathbf{R}^2)\|$$
$$\leq CT^{1/2}\|\langle x\rangle^{1/2+\epsilon}b;L^\infty\|\|u;C([0,T];\Sigma(3))\|\|\partial_y u;C([0,T];L^\infty(\mathbf{R}^2))\|$$
$$\times\|\langle x\rangle^{1+\epsilon}u;C([0,T];L^\infty(\mathbf{R}^2))\|$$
$$\leq CT^{1/2}\|\langle x\rangle^{1/2+\epsilon}b;L^\infty\|\|u;C([0,T];\Sigma(3))\|^3.$$

In this term, $\langle r\rangle^{1/2+\epsilon}b\in L^\infty(\mathbf{R}^2)$ is needed. The estimate of $\|b\partial_y^3 u\int_x^\infty \partial_y u\bar{u}d\xi;L^1(\mathbf{R}_y;L^2([0,T]\times\mathbf{R}_x))\|$ is similar. Then we finished all estimates.

From above estimates, we obtain

$$\|N_1(u);Y(T)\|\leq C(1+T)\|u;X(T)\|^3.$$

Similar calculation shows

$$\|N_1(u)-N_1(v);Y(T)\|\leq C(1+T)(\|u;X(T)\|^2\vee\|v;X(T)\|^2)\|u-v;X(T)\|.$$

Since the spaces are symmetric with x and y, the estimate of $N_2(u):=bu\int_y^\infty \partial_x(|u(x,\eta)|^2)d\eta$ is same as $N_1(u)$. The estimation of $a|u|^2u$ is easy, so we omit this estimate. Thus we have

$$\|N(u);Y(T)\|\leq C(1+T)\|u;X(T)\|^3,$$

and

$$\|N(u)-N(v);Y(T)\|\leq C(1+T)(\|u;X(T)\|^2\vee\|v;X(T)\|^2)\|u-v;X(T)\|.$$

By virtue of Lemma 2 and 3, we get

$$\|SN(u); X(T)\| \le C_0(1+T^4)\|u; X(T)\|^3,$$

and

$$\|S\big(N(u)-N(v)\big); X(T)\| \le C_0(1+T^4)(\|u; X(T)\|^2 \vee \|v; X(T)\|^2)\|u-v; X(T)\|,$$

for some $C_0 < \infty$.

Moreover, by using Lemma 2, we have

$$\|Uu_0; X_0(T)\| \le C_0(1+T^3)\|u_0; \Sigma(3)\|,$$

and

$$\begin{aligned}\|Uu_0; X_a(T)\| \le & C\|\langle r\rangle^{1/2+\epsilon}\partial_{x,y}^4 Uu_0; L^2(\mathbf{R}^2)\| \\ \le & CT_0^{1/2}(1+T^5)\|u_0; \Sigma(5)\|.\end{aligned}$$

Now, we take $\delta > 0$ sufficiently small such that $4C_0\delta^2 < 1$. Then, if $C_0\|u_0; \Sigma(5)\| < \delta/2$ and $T < 1$, $\Phi(u) := Uu_0 - iS(N(u))$ becomes a contraction in closed ball $B := \{u \in X(T) : \|u; X\| \le \delta\}$. Thus there exists a unique fixed point of Φ in X, and this fixed point u is the solution of (9). This proves the theorem. ∎

§3. Proof of Lemma 3. In this section, we prove the key estimate Lemma 3. We first remark that

$$Sf = -i\mathcal{F}_{t,x,y}^{-1}(\tau+\xi^2+\eta^2-i0)^{-1}\mathcal{F}_{t,x,y}f$$

if $f(t,x,y) \equiv 0$ for $t < 0$. Here, $\mathcal{F}_{t,x,y}$ is Fourier transform with respect to whole space-time variables (t,x,y). In fact, simple calculation shows

$$\mathcal{F}_{t,x,y}^{-1}(\tau+\xi^2+\eta^2-i0)^{-1}\mathcal{F}_{t,x,y}f\big|_{t=0} = -i\int_{-\infty}^{0} U(-s)f(s)ds,$$

and then,

$$-i\mathcal{F}_{t,x,y}^{-1}(\tau+\xi^2+\eta^2-i0)^{-1}\mathcal{F}_{t,x,y}f = Sf - \int_{-\infty}^{0} U(-s)f(s)ds.$$

This shows our claim.

First, we prove that $\mathcal{F}_{t,x,y}^{-1}\xi(\tau+\xi^2+\eta^2-i0)^{-1}\mathcal{F}_{t,x,y}$ is bounded operator from $L^1(\mathbf{R}_x; L^2(\mathbf{R}_t\times\mathbf{R}_y))$ to $L^\infty(\mathbf{R}_x; L^2(\mathbf{R}_t\times\mathbf{R}_y))$. By Plancherel's equality, we have

$$\begin{aligned}
&\sup_{x\in\mathbf{R}}\|\mathcal{F}_{t,x,y}^{-1}\xi(\tau+\xi^2+\eta^2-i0)^{-1}\mathcal{F}_{t,x,y}f; L^2(\mathbf{R}_t\times\mathbf{R}_y)\|\\
=&\sup_{x\in\mathbf{R}}\|\mathcal{F}_x^{-1}\xi(\tau+\xi^2+\eta^2-i0)^{-1}\mathcal{F}_x f; L^2(\mathbf{R}_\tau\times\mathbf{R}_\eta)\|\\
=&\sup_{x\in\mathbf{R}}\left\|\int_{\mathbf{R}} G(x-\tilde{x};\tau,\eta)(\mathcal{F}_{t,y}f)(\tau,\tilde{x},\eta)d\tilde{x}; L^2(\mathbf{R}_\tau\times\mathbf{R}_\eta)\right\|.
\end{aligned}$$

Here,

$$G(z;\tau,\eta) := \int_{\mathbf{R}}\frac{\xi\exp[iz\xi]}{\xi^2+(\tau+\eta^2)-i0}d\xi.$$

Direct calculation shows G is uniformly bounded with respect to z,τ,η, then we get

$$\begin{aligned}
&\sup_{x\in\mathbf{R}}\|\mathcal{F}_{t,x,y}^{-1}\xi(\tau+\xi^2+\eta^2-i0)^{-1}\mathcal{F}_{t,x,y}f; L^2(\mathbf{R}_t\times\mathbf{R}_y)\|\\
\leq&\sup_{x\in\mathbf{R}}\int_{\mathbf{R}}\|G(x-\tilde{x};\tau,\eta)(\mathcal{F}_{t,y}f)(\tau,\tilde{x},\eta); L^2(\mathbf{R}_\tau\times\mathbf{R}_\eta)\|d\tilde{x}\\
\leq& C\int_{\mathbf{R}}\|(\mathcal{F}_{t,y}f)(\cdot,x,\cdot); L^2(\mathbf{R}_\tau\times\mathbf{R}_\eta)\|dx\\
=&C\int_{\mathbf{R}}\|f(\cdot,x,\cdot); L^2(\mathbf{R}_t\times\mathbf{R}_y)\|dx\\
=&C\|f; L^1(\mathbf{R}_x; L^2(\mathbf{R}_t\times\mathbf{R}_y))\|.
\end{aligned}$$

This shows our claim.

Now, take $\phi\in C_0^\infty(\mathbf{R})$ such that $\phi\equiv 1$ on some neighborhood of 0, and decompose $Sf = S\phi(-i\partial_x)f + S(1-\phi(-i\partial_x))f$. Then, since $1-\phi(\xi)=0$ on the neighborhood of $\xi=0$, we have similar estimate for this term. On the other hand, since

$$S\phi(-i\partial_x)f = \int_0^t \exp[i(t-s)\Delta_{x,y}]\phi(-i\partial_x)f(s)ds,$$

we have

$$\begin{aligned}
&\|S\phi(-i\partial_x)f; L^2([0,T]\times\mathbf{R}_y)\|\\
=&\left\|\int_0^t \exp[i(t-s)\partial_x^2]\phi(-i\partial_x)\exp[i(t-s)\partial_y^2]f(s)ds; L^2([0,T]\times\mathbf{R}_y)\right\|.
\end{aligned}$$

Now, we put $H(r,z)$ the integral kernel of the operator $\exp[ir\partial_x^2]\phi(-i\partial_x)$. Then, we have

$$\begin{aligned}|H(r,z)| =&\frac{1}{2\pi}\left|\int_{\mathbf{R}}\exp[i(z\xi - r\xi^2)]\phi(\xi)d\xi\right| \\ \leq&\frac{1}{2\pi}\int_{\mathbf{R}}|\phi(\xi)|d\xi < \infty,\end{aligned}$$

and

$$\begin{aligned}&\sup_{x\in\mathbf{R}}\|S\phi(-i\partial_x)f; L^2([0,T]\times\mathbf{R}_y)\| \\ \leq&\sup_{x\in\mathbf{R}}\left\|\int_0^t ds\int_{\mathbf{R}} d\tilde{x}\|H(t-s,x-\tilde{x})\exp[i(t-s)\partial_y^2]f(s,\tilde{x},\cdot); L^2(\mathbf{R}_y)\|; L^2(t\in[0,T])\right\| \\ \leq&\sup_{x\in\mathbf{R}}\left\|\int_0^t ds\int_{\mathbf{R}} d\tilde{x}|H(t-s,x-\tilde{x})|\,\|f(s,\tilde{x},\cdot); L^2(\mathbf{R}_y)\|; L^2(t\in[0,T])\right\| \\ \leq&\sup_{x\in\mathbf{R}}\left\|\int_{\mathbf{R}} d\tilde{x}\left(\int_0^t|H(t-s,x-\tilde{x})|^2ds\right)^{1/2}\|f(\cdot,\tilde{x},\cdot); L^2([0,t]\times\mathbf{R}_y)\|; L^2(t\in[0,T])\right\| \\ \leq&\sup_{x\in\mathbf{R}}\int_{\mathbf{R}}\left(\int_0^T t\sup_{s\in[0,t]}|H(s,x-\tilde{x})|^2dt\right)^{1/2}\|f(\cdot,\tilde{x},\cdot); L^2([0,T]\times\mathbf{R}_y)\|d\tilde{x} \\ \leq&\sup_{x\in\mathbf{R}}\int_{\mathbf{R}} T\sup_{s\in[0,T]}|H(s,x-\tilde{x})|\,\|f(\cdot,\tilde{x},\cdot); L^2([0,T]\times\mathbf{R}_y)\|d\tilde{x} \\ \leq&T\sup_{(s,x)\in[0,T]\times\mathbf{R}}|H(s,x)|\,\|f; L^1(\mathbf{R}_x; L^2([0,T]\times\mathbf{R}_y))\|.\end{aligned}$$

This means our desired result. ∎

References

[AbFo] Ablowitz M.J. and Fokas A.S., *On the inverse scattering transform of multidimensional nonlinear equation*, J. Math. Phys. **25** (1984), p. 2494–2505.

[AnFr] Anker D. and Freeman N. C., *On the solitons of the Davey-Stewartson equation for long waves*, Proc. R. Soc. **A 360** (1978), p. 529–540.

[BC] Beals R. and Coifman R.R., *The spectral problem for the Davey-Stewartson and Ishimori hierarchies*, in "Proc. Conf. on Nonlinear Evolution Equations: Integrability and Spectral Methods," Manchester, U. K., 1988.

[DS] Davey A. and Stewartson K., *On three-dimensional packets of surface waves*, Proc. R. Soc. A **338** (1974), p. 101–110.

[FS] Fokas A.S. and Santini P.M., *Recursion operators and bi-Hamiltonian structures in multidimensions*, Commun. Math. Phys. **115** (1988), p. I:375–419, II:449–474.

[GS] Ghidaglia J.M. and Saut J.C., *On the initial value problem for the Davey-Stewartson System*, Nonlinearity **3** (1990), p. 475–506.
[LP] Linares F. and Ponce G., *On the Davey-Stewartson systems*, (preprint).
[Oh1] Ohta M., *Stability of standing waves for the generalized Davey-Stewartson system*, (preprint).
[Oh2] Ohta M., *Instability of standing waves for the generalized Davey-Stewartson system*, (preprint).
[TM] Tsutsumi M., *Decay of weak solutions for D-S system*, (to appear J. Math. Anal. Appl.), Report of Science and Engineering Research Lab. Waseda University (1991).