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ENERGY DISTRIBUTION OF THE SOLUTIONS OF ELASTIC WAVE PROPAGATION PROBLEMS IN STRATIFIED MEDIA $\mathbf{R}^3$

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§ 1. Introduction

Energy distribution of the solutions of various wave propagation problems has been studied by C. H. Wilcox ([10], [11], [12], [13]). He constructed asymptotic wave functions which approximate the solutions in the sense of L^2 for large times and calculated asymptotic energy distributions of the solutions in several domain by making use of these asymptotic wave functions. The construction of asymptotic wave functions is based on an eigenfunction expansion theorem which is proved by the same author and on the method of stationary phase. J.C.Guillot [3] studied a Rayleich surface wave propagating along the free boundary of a transversely isotropic elastic half space and showed that the energy of the Rayleich component of every solution with finite energy is asymptotically concentrated along the boundary.

In this paper we shall derive energy distributions of the solutions of elastic wave propagation problems in plane-stratified media $\mathbf{R}^3$ using methods due to Wilcox. We construct asymptotic wave functions by using spectral integral representations of the solutions and the method of stationary phase. The integral representations are based on an eigenfunction expansion theory which was proved by the author [8] using methods due to S. Wakabayashi [9]. We calculate asymptotic energy of the solutions for large times of the interface problems for elastic waves and show that the energy of the Stoneley components of the solutions with finite energy is asymptotically concentrated along the interface.

We start with the mathematical formulation of the elastic wave propagation problem.

Consider the plane stratified medium $\mathbf{R}^3 = \{x = (x_1, x_2, x_3); x_i \in \mathbf{R}\}$ with the planar interface $x_3 = 0$, which is defined by

$$(\lambda(x_3), \mu(x_3), \rho(x_3)) = \begin{cases} (\lambda_1, \mu_1, \rho_1), & x_3 < 0, \\ (\lambda_2, \mu_2, \rho_2), & x_3 > 0. \end{cases}$$

Here λ_1, λ_2, μ_1, μ_2 are certain quantities called the Lamé constants and ρ_1, $\rho_2 > 0$ are the densities.

We shall denote the lower halfspace $\mathbf{R}^3_- = \{x \in \mathbf{R}^3; x_3 < 0\}$ by *medium I* and the upper halfspace $\mathbf{R}^3_+ = \{x \in \mathbf{R}^3; x_3 > 0\}$ by *medium II*, respectively, as in

Figure 1.

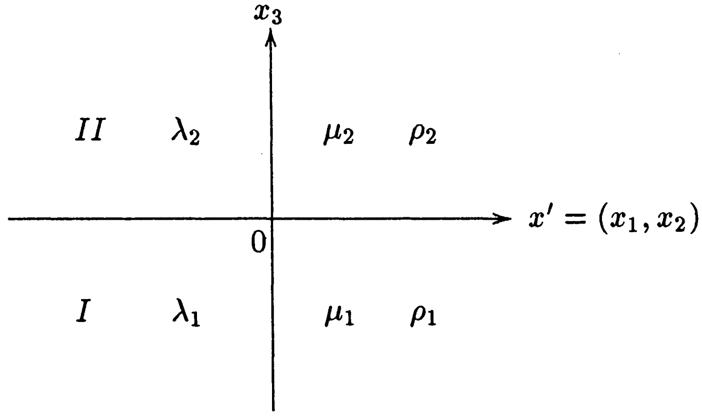

Figure 1 Stratified media I and II

The propagation problem of elastic waves in the stratified medium is formulated as the following mixed initial and interface value problem:

$$\frac{\partial^2 u}{\partial t^2}(t,x) + Mu(t,x) = 0, \tag{1.1}$$

$$u(t,x)|_{x_3=-0} = u(t,x)|_{x_3=+0}, \tag{1.2}$$

$$\sigma_{k3}u(t,x)|_{x_3=-0} = \sigma_{k3}u(t,x)|_{x_3=+0}, \tag{1.3}$$

$$u(0,x) = f(x), \quad \frac{\partial u}{\partial t}(0,x) = g(x), \tag{1.4}$$

where

(1.5)
$$Mu = -\frac{\lambda(x_3)+\mu(x_3)}{\rho(x_3)}\nabla(\nabla\cdot u) - \frac{\mu(x_3)}{\rho(x_3)}\Delta u = \frac{1}{\rho(x_3)}\sum_{k,j=1}^{3} M_{kj}\frac{\partial^2 u}{\partial x_k \partial x_j},$$

(1.6)
$$\sigma_{kj}u = \lambda(x_3)(\nabla\cdot u)\delta_{kj} + 2\mu(x_3)\varepsilon_{kj}u,$$

(1.7)
$$\varepsilon_{kj}u = \frac{1}{2}\left(\frac{\partial u_k}{\partial x_j} + \frac{\partial u_j}{\partial x_k}\right).$$

(1.2) and (1.3) are called interface conditions, and (1.4) is called an initial condition.

The $c^I_{ki\ell j}, c^{II}_{ki\ell j} (i,j,k,\ell = 1,2,3)$ are the stress-strain tensors given by

$$\begin{aligned} c^I_{ki\ell j} &= \lambda_1\delta_{ki}\delta_{\ell j} + \mu_1(\delta_{k\ell}\delta_{ij} + \delta_{kj}\delta_{i\ell}), \\ c^{II}_{ki\ell j} &= \lambda_2\delta_{ki}\delta_{\ell j} + \mu_2(\delta_{k\ell}\delta_{ij} + \delta_{kj}\delta_{i\ell}) \end{aligned} \tag{1.8}$$

with the properties

$$\begin{aligned} c^I_{ki\ell j} &= c^I_{ik\ell j} = c^I_{kij\ell} = c^I_{\ell jki}, \\ c^{II}_{ki\ell j} &= c^{II}_{ik\ell j} = c^{II}_{kij\ell} = c^{II}_{\ell jki}, \end{aligned}$$

and δ_{ki} is the Kronecker delta. We assume that the constants $c^I_{ki\ell j}$, $c^{II}_{ki\ell j}$ satisfy the following stability conditions

$$\lambda_i + \mu_i > 0, \quad \mu_i > 0, \quad (i = 1, 2), \tag{1.9}$$

which are equivalent to the conditions

$$\begin{aligned} &\sum_{k,i,\ell,j=1}^{3} c^I_{ki\ell j} s_{\ell j} \overline{s_{ki}} \geq \exists \delta_1 \sum_{k,i=1}^{3} |s_{ki}|^2, \qquad \delta_1 > 0, \\ &\sum_{k,i,\ell,j=1}^{3} c^{II}_{ki\ell j} s_{\ell j} \overline{s_{ki}} \geq \exists \delta_2 \sum_{k,i=1}^{3} |s_{ki}|^2, \qquad \delta_2 > 0, \end{aligned} \tag{1.9'}$$

for all complex symmetric 3×3 matrices (s_{ki}), $s_{ki} = s_{ik} \in \mathbf{C}$ (cf. [4]).

We introduce the Hilbert space

$$\mathcal{H} = L^2(\mathbf{R}^3, \mathbf{C}^3, \rho(x_3)\, dx) \tag{1.10}$$

with inner product

$$(u, v) = \int_{\mathbf{R}^3} u \cdot v \rho(x_3)\, dx,$$

where $u \cdot v$ denotes the usual scalar product in $\mathbf{C}^3$: $u \cdot v = \sum_{i=1}^3 u_i \bar{v}_i$. It was shown in [8 Theorem 1.2] that the operator A on $\mathcal{H}$ with domain

$$\begin{aligned} D(A) = \{u \in H^2(\mathbf{R}^3_-, \mathbf{C}^3) \oplus H^2(\mathbf{R}^3_+, \mathbf{C}^3);& \\ u \text{ satisfies the interface conditions (1.2) and (1.3)}& \\ \text{in the sense of trace on } x_3 = 0\}& \end{aligned}$$

and action defined by

$$Au = Mu, \quad u \in D(A) \tag{1.11}$$

is a selfadjoint operator on $\mathcal{H}$. Here

$$H^2(\mathbf{R}^3_\pm, \mathbf{C}^3) = \{u(x);\ D^\alpha_x u \in L^2(\mathbf{R}^3_\pm) \quad \text{for} \quad 0 \leq \alpha \leq 2\}$$

is a Hilbert space with inner product

$$(u, v)_2 = \int_{\mathbf{R}^3_\pm} \sum_{|\alpha| \leq m} D^\alpha u(x) \cdot D^\alpha v(x)\, dx.$$

Every $u \in D(A)$ satisfies the interface conditions (1.2) and (1.3), so the mixed problem (1.1)-(1.4) may be reformulated as the problem of finding a function $u : \mathbf{R} \to \mathcal{H}$ such that

$$\frac{\mathrm{d}^2 u}{\mathrm{d}t^2} + Au = 0 \quad \text{for } \forall t \in \mathbf{R}, \tag{1.12}$$

$$u(0) = f, \quad \frac{\mathrm{d}u}{\mathrm{d}t}(0) = g. \tag{1.13}$$

The operator A is non-negative [8, Lemma 1.4] and the spectral theorem for selfadjoint operators (cf. [2]) implies that (1.12) and (1.13) has a (generalized) solution given by

$$u(t) = \left(\cos tA^{\frac{1}{2}}\right) f + \left(A^{-\frac{1}{2}} \sin tA^{\frac{1}{2}}\right) g, \quad t \in \mathbf{R} \tag{1.14}$$

for every pair f, $g \in \mathcal{H}$. u has derivatives $\frac{\mathrm{d}u}{\mathrm{d}t}$ and $\frac{\mathrm{d}^2 u}{\mathrm{d}t^2}$ and is a strict solution of (1.12) if and only if $f \in D(A)$, $g \in D(A^{\frac{1}{2}})$.

§ 2. Eigenfunction Expansions for A

The eigenfunction expansion theorem for A was developed in [8]. In this section it is applied to give spectral integral representations of the solutions of the elastic propagation problem. This section begins with a brief review of the structure and properties of the eigenfunctions and the expansion theorem.

Let $\eta' = (\eta_1, \eta_2) \in \mathbf{R}^2$ be the dual variables of $x' = (x_1, x_2)$ and let $F_{x'}$ denote the partial Fourier transformation with respect to x';

$$\hat{u}(\eta', x_3) = (F_{x'}u)(\eta', x_3) = \underset{R\to\infty}{\mathrm{l.i.m.}} \frac{1}{2\pi} \int_{|x'|\le R} e^{-i(x_1\eta_1 + x_2\eta_2)} u(x)\, dx'$$

for u in $\mathcal{H}$. Let

$$D(\hat{A}) = F_{x'}D(A) = \{\hat{u}; u \in D(A)\},$$
$$\hat{A}\hat{u} = F_{x'}AF_{\eta'}^{-1}\hat{u}, \qquad \hat{u} \in D(\hat{A}).$$

For every $\eta' \neq 0$, let

$$\mathrm{U} = \frac{1}{|\eta'|}\begin{pmatrix} \eta_1 & -\eta_2 & 0 \\ \eta_2 & \eta_1 & 0 \\ 0 & 0 & |\eta'| \end{pmatrix}, \quad \mathrm{C} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \tag{2.1}$$

where U and C are unitary matrices and $|\eta'| = (\eta_1^2 + \eta_2^2)^{\frac{1}{2}}$. Then we have

$$Au = F_{\eta'}^{-1}\mathrm{UC}(A_1(\eta') \oplus A_2(\eta'))(\mathrm{UC})^{-1}F_{x'}u \quad \text{for } u \in D(A), \tag{2.2}$$

where $A_1(\eta')$ and $A_2(\eta')$ are non-negative selfadjoint operators (see [8, Proposition 1.7], [1], [3]).

We can get an explicit representation of the Green function $G_1(x_3, y_3, \eta'; \zeta)$ for the operator $A_1(\eta') - \zeta I$ ($\zeta \notin \mathbf{R}$) from the expression of the solution for the following problem:

$$(A_1(\eta', D) - \zeta)v(\eta', x_3) = f(\eta', x_3), \tag{2.3}$$
$$v(\eta', x_3)|_{x_3=-0} = v(\eta', x_3)|_{x_3=+0}, \tag{2.4}$$
$$B_1(\eta')v(\eta', x_3)|_{x_3=-0} = B_1(\eta')v(\eta', x_3)|_{x_3=+0}. \tag{2.5}$$

Here (2.4) and (2.5) are the interface conditions for $A_1(\eta', D)$ corresponding to (1.2) and (1.3). $A_1(\eta', D)$ ($D = \frac{1}{i}\frac{\mathrm{d}}{\mathrm{d}x_3}$) is the differential operators corresponding to the selfadjoint operator $A_1(\eta')$. Since the solution v of (2.3) should satisfy the interface conditions (2.4) and (2.5), the denominator of v has the Lopatinski determinant $\Delta(\eta', \zeta)$ as follows:

$$\Delta(\eta', \zeta) = |\eta'|^6 Dis(z),$$

(2.6)

$$Dis(z) = \left(2(\mu_1 - \mu_2) - \frac{\mu_1 z}{c_{s_1}^2} + \frac{\mu_2 z}{c_{s_2}^2}\right)^2 + 4(\mu_1 - \mu_2)^2 a_1 a_2 b_1 b_2$$

$$- a_1b_1\left(2(\mu_1-\mu_2)+\frac{\mu_2 z}{c_{s_2}^2}\right)^2 - a_2b_2\left(2(\mu_1-\mu_2)-\frac{\mu_1 z}{c_{s_1}^2}\right)^2$$
$$-\frac{\mu_1\mu_2}{c_{s_1}^2c_{s_2}^2}(a_1b_2+a_2b_1)z^2,$$

where

$$z=\frac{\zeta}{|\eta'|^2},$$
$$a_1=\sqrt{1-\frac{z}{c_{p_1}^2}},\quad a_2=\sqrt{1-\frac{z}{c_{p_2}^2}},\quad b_1=\sqrt{1-\frac{z}{c_{s_1}^2}},\quad b_2=\sqrt{1-\frac{z}{c_{s_2}^2}}.$$

The squares of propagation speeds of shear(SV, SH) and pressure(P) waves are given by

$$c_{s_i}^2=\frac{\mu_i}{\rho_i},\quad c_{p_i}^2=\frac{\lambda_i+2\mu_i}{\rho_i},\quad (i=1,2), \tag{2.7}$$

respectively. From the conditions (1.9), the minimum speed of $\{c_{s_1}, c_{p_1}, c_{s_2}, c_{p_2}\}$ is either c_{s_1} or c_{s_2}.

We can see that $Dis(z)$ has the only one real zero when $Dis(z)$ has zeros. Denote by c_{St}^2 its real zero. Then the zero of $\Delta(\eta',\zeta)$ is $c_{St}^2|\eta'|^2$ and is the origin of the Stoneley wave propagating along the interface $x_3=0$ in the elastic space R^3, and c_{St} is its speed.

By virtue of principle of the argument, the conditions for the existence of zeros of the Lopatinski determinant $\Delta(\eta',\zeta)=|\eta'|^6 Dis(z)$ (the existence of the Stoneley waves) are given as follows:
If $c_{s_1}<c_{s_2}$, then

(i) $Dis(c_{s_1}^2)>0 \Longrightarrow$ The zero $\zeta=c_{St}^2|\eta'|^2$ of $\Delta(\eta',\zeta)$ in ζ exists in $[0,\ c_{s_1}^2|\eta'|^2)$ with order 1. More precisely, we shall prove in the proof of [8, Theorem 6.5] that $c_{St}\neq 0$.

(ii) $Dis(c_{s_1}^2)=0 \Longrightarrow c_{St}=c_{s_1}$ and we shall consider this case under some restricted conditions (cf. [8, Lemma 6.4]).

(iii) $Dis(c_{s_1}^2)<0 \Longrightarrow \Delta(\eta',\zeta)$ has no zero.

If $c_{s_2}<c_{s_1}$, then we must replace $Dis(c_{s_1}^2)$ by $Dis(c_{s_2}^2)$.

We also obtain an explicit representation of the Green function $G_2(x_3,y_3,\eta';\zeta)$ for the operator $A_2(\eta')-\zeta I$ $(\zeta\notin\mathbf{R})$ by the same method as $G_1(x_3,y_3,\eta';\zeta)$. The Lopatinski determinant corresponding to the operator $A_2(\eta')-\zeta I$ $(\zeta\notin\mathbf{R})$ has no zero. By using the Green functions $G_1(x_3,y_3,\eta';\zeta)$ and $G_2(x_3,y_3,\eta';\zeta)$, we define

$$\psi_{1j}(x_3,\eta;\zeta)=F_{y_3}^{-1}[G_1(x_3,y_3,\eta';\zeta)](\xi)(\lambda_j(\eta)-\zeta)P_j(\eta)\rho(x_3)^{-1},\quad j\in M,$$
$$\psi_{1j}^{St}(x_3,\eta;\zeta)=\frac{\zeta-c_{St}^2|\eta'|^2}{\zeta-\lambda_j(\eta)}\psi_{1j}(x_3,\eta;\zeta),\quad j\in M,$$
$$\psi_{2k}(x_3,\eta;\zeta)=F_{y_3}^{-1}[G_2(x_3,y_3,\eta';\zeta)](\xi)(\lambda_j(\eta)-\zeta)\rho(x_3)^{-1},\quad k\in N.$$

Here $\eta = (\eta_1, \eta_2, \xi) = (\eta', \xi)$, $\lambda_j(\eta) = c_j^2|\eta|^2$ are the eigenvalues of $A_1(\eta')$, $P_j(\eta)$ are mutually orthogonal projections for $A_1(\eta')$, $\lambda_k(\eta) = c_k^2|\eta|^2$ are the eigenvalues of $A_2(\eta')$, $M = \{s_1, p_1, s_2, p_2\}$ and $N = \{s_1, s_2\}$. When $\zeta \to \lambda_j(\eta) \pm i0$, $\zeta \to c_{St}^2|\eta|^2$, and $\zeta \to \lambda_k(\eta) \pm i0$, the limits $\psi_{1j}^{\pm}(x_3, \eta)$, $\psi_{1j}^{St}(x_3, \eta)$, and $\psi_{2k}^{\pm}(x_3, \eta)$ exist and these limit functions are generalized eigenfunctions for $A_1(\eta')$, $A_2(\eta')$, respectively.

Using these generalized eigenfunctions for $A_1(\eta')$, $A_2(\eta')$, we define generalized eigenfunctions for A as follows:

$$\psi_{1j}^{\pm}(x, \eta) = \frac{1}{2\pi} e^{i(x_1\eta_1 + x_2\eta_2)} \mathrm{UC}(\psi_{1j}^{\pm}(x_3, \eta) \oplus O_{1\times 1}), \quad j \in M, \tag{2.8}$$

$$\psi_{1j}^{St}(x, \eta) = \frac{1}{2\pi} e^{i(x_1\eta_1 + x_2\eta_2)} \mathrm{UC}(\psi_{1j}^{St}(x_3, \eta) \oplus O_{1\times 1}), \quad j \in M, \tag{2.9}$$

$$\psi_{2k}^{\pm}(x, \eta) = \frac{1}{2\pi} e^{i(x_1\eta_1 + x_2\eta_2)} \mathrm{UC}(O_{2\times 2} \oplus \psi_{2k}^{\pm}(x_3, \eta)), \quad k \in N. \tag{2.10}$$

where $O_{n\times n}$ denotes the $n \times n$ zero matrix.

Now we define the Fourier transform of $f \in \mathcal{H}$ with respect to these generalized eigenfunctions: $f \mapsto (\hat{f}_{1j}^{\pm},\ \hat{f}_{1j}^{St},\ \hat{f}_{2k}^{\pm})$,

$$\hat{f}_{1j}^{\pm}(\eta) = \underset{R\to\infty}{\mathrm{l.i.m.}} \int_{|x|\le \mathbf{R}} \psi_{1j}^{\pm}(x, \eta)^* f(x) \rho(x_3)\, dx, \quad j \in M, \tag{2.11}$$

$$\hat{f}_{1j}^{St}(\eta) = \underset{R\to\infty}{\mathrm{l.i.m.}} \int_{|x|\le \mathbf{R}} \psi_{1j}^{St}(x, \eta)^* f(x) \rho(x_3)\, dx, \quad j \in M, \tag{2.12}$$

$$\hat{f}_{2k}^{\pm}(\eta) = \underset{R\to\infty}{\mathrm{l.i.m.}} \int_{|x|\le \mathbf{R}} \psi_{2k}^{\pm}(x, \eta)^* f(x) \rho(x_3)\, dx, \quad k \in N. \tag{2.13}$$

Theorem 2.1 corresponds to the Parseval and Plancherel formulas.

Theorem 2.1. *We assume that $Dis(c_{s_i}^2) > 0$. Let f, $g \in \mathcal{H}$ and $0 < a < b < \infty$. Then we have*

$$\begin{aligned}(f, g) = &\sum_{j\in M} \left(\int_{\mathbf{R}^3} \hat{f}_{1j}^{\pm}(\eta) \cdot \hat{g}_{1j}^{\pm}(\eta)\, d\eta + \int_{\mathbf{R}^3} \hat{f}_{1j}^{St}(\eta) \cdot \hat{g}_{1j}^{St}(\eta)\, d\eta \right) \\ &+ \sum_{k\in N} \int_{\mathbf{R}^3} \hat{f}_{2k}^{\pm}(\eta) \cdot \hat{g}_{2k}^{\pm}(\eta)\, d\eta.\end{aligned}$$

The first half of Theorem 2.2 expresses the Fourier inversion formula with respect to generalized eigenfunctions. The latter half gives the canonical form for A.

Theorem 2.2. *We assume the same assumption as Theorem 2.1.*

(1) For $f \in \mathcal{H}$,

$$\begin{aligned}f(x) = &\sum_{j\in M} \underset{R\to\infty}{\mathrm{l.i.m.}} \int_{|\eta|\le R} \left(\psi_{1j}^{\pm}(x, \eta) \hat{f}_{1j}^{\pm}(\eta) + \psi_{1j}^{St}(x, \eta) \hat{f}_{1j}^{St}(\eta) \right) d\eta \\ &+ \sum_{k\in N} \underset{R\to\infty}{\mathrm{l.i.m.}} \int_{|\eta|\le R} \psi_{2k}^{\pm}(x, \eta) \hat{f}_{2k}^{\pm}(\eta)\, d\eta.\end{aligned}$$

(2) For $f \in D(A)$,

$$Af(x) = \sum_{j\in M} \underset{R\to\infty}{\mathrm{l.i.m.}} \int_{|\eta|\le R} \left(\lambda_j(\eta)\psi_{1j}^{\pm}(x,\eta)\hat{f}_{1j}^{\pm}(\eta) + c_{St}^2|\eta'|^2\psi_{1j}^{St}(x,\eta)\hat{f}_{1j}^{St}(\eta)\right) d\eta$$
$$+ \sum_{k\in N} \underset{R\to\infty}{\mathrm{l.i.m.}} \int_{|\eta|\le R} \lambda_k(\eta)\psi_{2k}^{\pm}(x,\eta)\hat{f}_{2k}^{\pm}(\eta)\, d\eta,$$

and

$$(\widehat{Af})_{1j}^{\pm}(\eta) = \lambda_j(\eta)\hat{f}_{1j}^{\pm}(\eta), \quad j \in M,$$
$$(\widehat{Af})_{1j}^{St}(\eta) = c_{St}^2|\eta'|^2\hat{f}_{1j}^{St}(\eta), \quad j \in M,$$
$$(\widehat{Af})_{2k}^{\pm}(\eta) = \lambda_k(\eta)\hat{f}_{2k}^{\pm}(\eta), \quad k \in N.$$

Theorem 2.3 gives an explicit expression of the ranges $R(\Phi^{\pm})$.

Theorem 2.3. *Assume the same assumption as Theorem 2.1. We define the mappings by*

$$\Phi_{1j}^{\pm} : \mathcal{H} \ni f \to \hat{f}_{1j}^{\pm}(\eta) \in L^2(\mathbf{R}_{\pm}^3, \mathbf{C}^3), \ j \in M$$
$$\Phi_{1j}^{St} : \mathcal{H} \ni f \to \hat{f}_{1j}^{St}(\eta) \in L^2(\mathbf{R}^3, \mathbf{C}^3), \ j \in M$$
$$\Phi_{2k}^{\pm} : \mathcal{H} \ni f \to \hat{f}_{2k}^{\pm}(\eta) \in L^2(\mathbf{R}_{\pm}^3, \mathbf{C}^3), \ k \in N,$$

and put

$$\Phi^{\pm} = \sum_{j\in M} \Phi_{1j}^{\pm} \oplus \sum_{j\in M} \Phi_{1j}^{St} \oplus \sum_{k\in N} \Phi_{2k}^{\pm}.$$

Then we have

$$R(\Phi^{\pm}) = \sum_{j\in M} \oplus(P_j(\eta) \oplus O_{1\times1})L^2(\mathbf{R}_{\pm}^3, \mathbf{C}^3) \oplus \sum_{j\in M} \oplus(P_j(\eta) \oplus O_{1\times1})L^2(\mathbf{R}^3, \mathbf{C}^3)$$
$$\oplus \sum_{k\in N} \oplus(O_{2\times2} \oplus 1)L^2(\mathbf{R}_{\pm}^3, \mathbf{C}^3).$$

This implies that $\Phi^{\pm}$ are unitary operators in $\mathcal{H}$, and that the systems of generalized eigenfunctions $\{\psi_{1j}^{+}, \psi_{1j}^{St}, \psi_{2k}^{+}\}$ and $\{\psi_{1j}^{-}, \psi_{1j}^{St}, \psi_{2k}^{-}\}$ are complete, respectively.

The next theorem shows the utility of the eigenfunction expansion theorem for the operator A.

Theorem 2.4. *Let $\Psi(\lambda)$ be a complex-valued bounded Lebesgue measurable function on $\sigma(A) = \{\lambda : \lambda \ge 0\}$ and let $\Psi(A)$ be the corresponding operator defined by means of the spectral theorem.*

Then we have

$$(\widehat{\Psi(A)f})_{1j}^{\pm}(\eta) = \Psi(c_j^2|\eta|^2)\hat{f}_{1j}^{\pm}(\eta) \in (P_j(\eta) \oplus 0_{1\times1})L^2(\mathbf{R}_{\pm}^3, \mathbf{C}^3), \quad j \in M,$$
$$(\widehat{\Psi(A)f})_{1j}^{St}(\eta) = \Psi(c_{St}^2|\eta'|^2)\hat{f}_{1j}^{St}(\eta) \in (P_j(\eta) \oplus 0_{1\times1})L^2(\mathbf{R}^3, \mathbf{C}^3), \quad j \in M,$$
$$(\widehat{\Psi(A)f})_{2k}^{\pm}(\eta) = \Psi(c_k^2|\eta|^2)\hat{f}_{2k}^{\pm}(\eta) \in (0_{2\times2} \oplus 1)L^2(\mathbf{R}_{\pm}^3, \mathbf{C}^3), \quad k \in N.$$

It will be convenient to rewrite the solution (1.12)-(1.13) in the following form.

Theorem 2.5. *Let f and g be real-valued functions such that*

$$f \in \mathcal{H}, \quad g \in D(A^{-\frac{1}{2}}), \tag{2.14}$$

and define

$$h = f + iA^{-\frac{1}{2}} g \in \mathcal{H}. \tag{2.15}$$

Then the solution in $\mathcal{H}$ defined by (1.14) satisfies

$$u(t,x) = Re\{v(t,x)\}, \tag{2.16}$$

where $v(t,x)$ is the complex-valued solution in $\mathcal{H}$ defined by

$$v(t,\cdot) = e^{-itA^{\frac{1}{2}}} h. \tag{2.17}$$

The proof of Theorem 2.5 is due to Wilcox [10, Theorem 2.3]. This theorem implies that the solution $u(t,x)$ of (1.12) and (1.13) is determined by $v(t,x)$.

Combining Theorem 2.4 and Theorem 2.5, we have the following:

Theorem 2.6. *We assume that*

$$f \in \mathcal{H}, \quad g \in D(A^{-\frac{1}{2}}), \quad Dis(c_{s_i}^2) > 0.$$

Then the solution of the elastic wave propagation problem, defined by (1.12) and (1.13) has the representation

$$v(t,x) = \sum_{j \in M} v_{1j}^{\pm}(t,x) + \sum_{j \in M} v_{1j}^{St}(t,x) + \sum_{k \in N} v_{2k}^{\pm}(t,x) \in \mathcal{H}, \tag{2.18}$$

where

(2.19)
$$v_{1j}^{\pm}(t,x) = \underset{R\to\infty}{\mathrm{l.i.m.}} \int_{|\eta|\le R} e^{-itc_j|\eta|} \psi_{1j}^{\pm}(x,\eta) \hat{h}_{1j}^{\pm}(\eta)\, d\eta, \ j \in M,$$

(2.20)
$$v_{1j}^{St}(t,x) = \underset{R\to\infty}{\mathrm{l.i.m.}} \int_{|\eta|\le R} e^{-itc_{St}|\eta'|} \psi_{1j}^{St}(x,\eta) \hat{h}_{1j}^{St}(\eta)\, d\eta, \ j \in M,$$

(2.21)
$$v_{2k}^{\pm}(t,x) = \underset{R\to\infty}{\mathrm{l.i.m.}} \int_{|\eta|\le R} e^{-itc_k|\eta|} \psi_{2k}^{\pm}(x,\eta) \hat{h}_{2k}^{\pm}(\eta)\, d\eta, \ k \in N,$$

and

(2.22)
$$\hat{h}_{1j}^{\pm}(\eta) = \hat{f}_{1j}^{\pm}(\eta) + i\frac{1}{c_j|\eta|}\hat{g}_{1j}^{\pm}(\eta) \in (P_j(\eta) \oplus 0_{1\times1}) L^2(\mathbf{R}_{\pm}^3, \mathbf{C}^3),$$

(2.23)
$$\hat{h}_{1j}^{St}(\eta) = \hat{f}_{1j}^{St}(\eta) + i\frac{1}{c_{St}|\eta'|}\hat{g}_{1j}^{St}(\eta) \in (P_j(\eta) \oplus 0_{1\times1}) L^2(\mathbf{R}^3, \mathbf{C}^3),$$

(2.24)
$$\hat{h}_{2k}^{\pm}(\eta) = \hat{f}_{2k}^{\pm}(\eta) + i\frac{1}{c_k|\eta|}\hat{g}_{2k}^{\pm}(\eta) \in (0_{2\times2} \oplus 1) L^2(\mathbf{R}_{\pm}^3, \mathbf{C}^3).$$

§ 3. The Asymptotic Energy Distributions for Large Times

This section deals with the asymptotic energy distributions for large times.

We define the energy of solution u on a set $K \subset \mathbf{R}^3$ at time t for the elastic wave propagation problem by

$$E(u,K,t) = \int_K \left(\left| \frac{\partial u}{\partial t} \right|^2 \rho(x_3) - \sum_{k,j=1}^{3} M_{kj} \frac{\partial u}{\partial x_j} \cdot \frac{\partial u}{\partial x_k} \right) dx. \tag{3.1}$$

If u is a solution of (1.1)-(1.4), u satisfies the conservation laws of energy:

$$E(u, \mathbf{R}^3, t) = E(u, \mathbf{R}^3, 0) = \text{const.} \quad \text{for } \forall t \in \mathbf{R},$$

where the constant may be finite or infinite. If one defines a sesquilinear form B in $\mathcal{H}$ by

$$D(B) = H^1(\mathbf{R}^3, \mathbf{C}^3) \subset \mathcal{H}$$

and

$$B(u,v) = -\sum_{k,j=1}^{3} \int_{\mathbf{R}^3} M_{kj} \frac{\partial u}{\partial x_j} \cdot \frac{\partial u}{\partial x_k} dx,$$

then it is easy to verify that B is closed and non-negative, and that A is the unique selfadjoint non-negative operator in $\mathcal{H}$ associated with B (cf. [5]). Then $D(A^{\frac{1}{2}}) = H^1(\mathbf{R}^3, \mathbf{C}^3)$ and for all $u \in D(A^{\frac{1}{2}})$ one has

$$\left\| A^{\frac{1}{2}} u \right\|^2 = B(u,u) = -\sum_{k,j=1}^{3} \int_{\mathbf{R}}^{3} M_{kj} \frac{\partial u}{\partial x_j} \cdot \frac{\partial u}{\partial x_k} dx,$$

where $\|\cdot\|$ is the norm in $\mathcal{H}$. It follows that

$$E(u, \mathbf{R}^3, t) = \left\| \frac{\mathrm{d}u}{\mathrm{d}t} \right\|^2 + \left\| A^{\frac{1}{2}} u \right\|^2 = \|u\|_{\mathcal{E}}^2 . \tag{3.2}$$

Here the norm $\|u\|_{\mathcal{E}}$ is called the energy norm. If $f \in D(A^{\frac{1}{2}})$, $g \in \mathcal{H}$, then $u(t) \in D(A^{\frac{1}{2}})$, $\frac{\mathrm{d}u}{\mathrm{d}t} \in \mathcal{H}$ for all $t \in \mathbf{R}$ and $u(t)$ satisfies

$$\|u(t)\|_{\mathcal{E}}^2 = \|u(0)\|_{\mathcal{E}}^2 < \infty \quad \text{for } \forall\, t \in \mathbf{R}. \tag{3.3}$$

Therefore a necessary and sufficient condition for u to have this property is that the initial state f, g has finite energy:

$$f \in D(A^{\frac{1}{2}}), \quad g \in \mathcal{H}. \tag{3.4}$$

Hereafter we consider only solutions with finite total energy.

When

$$f \in \mathcal{H}, \quad g \in D(A^{-\frac{1}{2}}),$$

the solution u of the elastic wave propagation problem in $\mathcal{H}$, defined by (1.12) and (1.13), satisfies

$$u(t,x) = Re\{v(t,x)\},$$

where

$$v(t,\cdot) = e^{-itA^{\frac{1}{2}}}h, \quad h = f + iA^{-\frac{1}{2}}g,$$

then $v(t,x)$ has the following representation (see Section 2):

$$v(t,x) = \sum_{j\in M} v_{1j}^{\pm}(t,x) + \sum_{j\in M} v_{1j}^{St}(t,x) + \sum_{k\in N} v_{2k}^{\pm}(t,x) \in \mathcal{H}.$$

$v_{1j}^{\pm}(t,x)(j \in \{p_1,p_2\})$ are called Pressure (P) components, $v_{1j}^{\pm}(t,x)(j \in \{s_1,s_2\})$ are called Shear Vertical (SV) components, $v_{1j}^{St}(t,x)(j \in M = \{s_1,p_1,s_2,p_2\})$ are called Stoneley components and $v_{2k}^{\pm}(t,x)(k \in N = \{s_1,s_2\})$ are called Shear Horizontal (SH) components. We remark that if

$$Dis(c_{s_\iota}^2) > 0, \tag{3.5}$$

then the Stoneley components exist. Here $c_{s_\iota} = \min\{c_{s_1}, c_{s_2}\}$ and $Dis(z)$ is defined by (2.6) (cf. Section 2, [8, Section 3]). This condition is determined by Lamé constants λ_i, μ_i and densities ρ_i $(i = 1,2)$.

Our main results are the following theorems. Theorem 1.1 shows that the energy of the Stoneley components $v_{1j}^{St}(t,x)(j \in M)$ of v is asymptotically concentrated along the interface $x_3 = 0$.

Theorem 1.1. *We assume that*

$$f \in D(A^{\frac{1}{2}}) \cap \mathcal{H}, \quad g \in \mathcal{H} \cap D(A^{-\frac{1}{2}}), \quad Dis(c_{s_\iota}^2) > 0,$$

then

$$\lim_{t\to\infty} E(v_{1j}^{St}, (C^-(\theta) \cup C^+(\theta)) \cap B(t,\vartheta(t)), t) = E(v_{1j}^{St}, \mathbf{R}^3, 0), \quad j \in M,$$

where

$$\begin{aligned}
&C^-(\theta) = \{x \in \mathbf{R}_-^3;\ -\theta(|x'|) < x_3 < 0\},\\
&C^+(\theta) = \{x \in \mathbf{R}_-^3;\ 0 < x_3 < \theta(|x'|)\},\\
&B(t,\vartheta(t)) = \{x \in \mathbf{R}^3;\ c_{St}t - \vartheta(t) \le |x'| \le c_{St}t + \vartheta(t),\ x_3 \in \mathbf{R}\},\\
&\vartheta(t) : \lim_{t\to\infty} \vartheta(t) = \infty,\ |\vartheta(t)| < 2c_{St}t,\\
&\theta(|x'|) : \lim_{|x'|\to\infty} \theta(|x'|) = \infty, \text{ monotone increasing function},\\
&c_{St} : \text{propagation speed of Stoneley wave.}
\end{aligned}$$

The next theorem shows that the P, SV, SH components $v_{1j}^{\pm}(t,x)(j \in M)$, $v_{2k}^{\pm}(t,x)(k \in N)$ behave like free waves.

Theorem 1.2. *We assume that*

$$f \in D(A^{\frac{1}{2}}) \cap \mathcal{H}, \quad g \in \mathcal{H} \cap D(A^{-\frac{1}{2}}),$$

then

$$\lim_{t\to\infty} E(v_{1j}^{\pm},\ S_{s_1}(t,\vartheta) \cup S_{p_1}(t,\vartheta) \cup S_{s_2}(t,\vartheta) \cup S_{p_2}(t,\vartheta), t) = E(v_{1j}^{\pm}, \mathbf{R}^3, 0), \quad j \in M,$$

$$\lim_{t\to\infty} E(v_{2k}^{\pm},\ S_{s_1}(t,\vartheta) \cup S_{s_2}(t,\vartheta), t) = E(v_{2k}^{\pm}, \mathbf{R}^3, 0), \qquad k \in N,$$

where

$$S_{s_1}(t,\vartheta(t)) = \{x \in \mathbf{R}^3_-;\ c_{s_1}t - \vartheta(t) \le |x| \le c_{s_1}t + \vartheta(t)\}$$

$$S_{p_1}(t,\vartheta(t)) = \{x \in \mathbf{R}^3_-;\ c_{p_1}t - \vartheta(t) \le |x| \le c_{p_1}t + \vartheta(t)\}$$

$$S_{s_2}(t,\vartheta(t)) = \{x \in \mathbf{R}^3_+;\ c_{s_2}t - \vartheta(t) \le |x| \le c_{s_2}t + \vartheta(t)\}$$

$$S_{p_2}(t,\vartheta(t)) = \{x \in \mathbf{R}^3_+;\ c_{p_2}t - \vartheta(t) \le |x| \le c_{p_2}t + \vartheta(t)\}$$

$$\vartheta(t) : \lim_{t\to\infty} \vartheta(t) = \infty,$$

c_{p_1}, c_{p_2} : propagation speeds of P waves,

c_{s_1}, c_{s_2} : propagation speeds of SV and SH waves.

These theorems are obtained calculating the energy of the asymptotic wave functions $v_{1j}^{St\infty}(t,x)$, $v_{1j}^{\pm\infty}(t,x)$ $(j \in M)$, $v_{2k}^{\pm\infty}(t,x)$ $(k \in N)$ which defined by means of the stationary phase method.

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