数理解析研究所講究録
第 873 巻 1994 年 196-203

Absence of point spectrum for a class of discrete Schrödinger operators with quasiperiodic potential

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Abstract

Treated in this paper are one-dimensional discrete Schrödinger operators with a quasiperiodic potentials, which are derived from the model proposed by Kohmoto, Kadanoff and Tang in 1983. The aim of this paper is to show the absence of point spectrum of the operators under certain conditions.

Mathematics Subject Classification (1991):47A10, 47B39, 47B80, 47N50

1 Introduction

We consider the following discrete one-dimensional Schrödinger operators on $\ell^2(\mathbb{Z})$ given by

$$(H_\theta\psi)(n) := \psi(n+1) + \psi(n-1) + V_\theta(n)\psi(n), \tag{1}$$

with a potential $V_\theta(n)$ given by

$$V_\theta(n) := \lambda\chi_A(\Phi(\alpha n) + \theta). \tag{2}$$

Here λ is a non-zero constant, χ_A is the characteristic function of an interval A on the torus $\mathbb{R}/\mathbb{Z}$, Φ is the canonical projection from $\mathbb{R}$ onto $\mathbb{R}/\mathbb{Z}$, and

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$\theta \in \mathbb{R}/\mathbb{Z}$. This operator was proposed by Kohmoto, Kadanoff and Tang [5] for the case of $\alpha = (\sqrt{5}-1)/2$, $A = \Phi([1-\alpha, 1))$ and $\theta = 0$. The potential $V_\theta(n)$ with an irrational number α means a quasiperiodic one, and the operator (1) is interpreted by Luck and Petritis [8] as a model describing the phonon spectra in one dimensional quasicrystals. In this case, Sütő ([9],[10]) concluded the spectrum of H_0 was a Cantor set (i.e. nowhere dense closed set without an isolated point) of zero Lebesgue measure and was purely singular continuous. Further Bellissard, Iochum, Scoppola and Testard [1] extended this result for any irrational number α. However, for the author's knowledge, the absence of the point spectrum of H_θ for non-zero θ is not yet known for any irrational number α, and we deal with this probrem in the present paper.

Remark 1 *In the case α is a rational number, (2) yields a periodic sequence and H_θ has purely absolutely continuous spectrum for every θ. In particular, H_θ has no point spectrum for every θ.*

Before stating our result, we introduce some notations which are used through this paper. Let $a_n(\alpha)$ be the n^{th} partial quotient of the continued fraction of α; i.e.,

$$\alpha = a_0(\alpha) + \cfrac{1}{a_1(\alpha) + \cfrac{1}{a_2(\alpha) + \ddots}}.$$

And let p_n/q_n be the n^{th} principal convergent of a irrational number α; i.e.,

$$p_{n+1} = a_n(\alpha)p_n + p_{n-1}, \tag{3}$$

$$q_{n+1} = a_n(\alpha)q_n + q_{n-1}, \tag{4}$$

with $p_0 = 1$, $p_1 = a_0(\alpha)$, $q_0 = 0$, $and q_1 = 1$. Then, it is known (see e.g. Lang [7, p.8]) that

$$\left|\alpha - \frac{p_n}{q_n}\right| < \frac{1}{q_n q_{n+1}} \qquad (n \geq 2). \tag{5}$$

We define the following sets;

$$E(n) := \{\theta | \; V_\theta(m+q_n) = V_\theta(m-q_n) = V_\theta(m) \; (1 \leq m \leq q_n)\},$$

$$M := \{\theta | \sigma_p(H_\theta) = \phi\},$$

where $\sigma_p(H_\theta)$ is the set of the point spectrum of H_θ, and we have (Lemma 3 below):

$$\limsup_{n\to\infty} E(n) \subset M.$$

The aim of this paper is to show the following theorems.

Theorem 1 *Suppose a real number* α *satisfies* $0 < \alpha < 1$, *and let* $A = \Phi([1-\alpha, 1))$. *Then* $\sigma_p(H_\theta) = \phi$ *for almost every* θ *with respect to the Lebesgue measure.*

For any interval A, we have:

Theorem 2 *Suppose* $\limsup_{n\to\infty} a_n(\alpha) \geq 4$, *then* $\sigma_p(H_\theta) = \phi$ *for almost every* θ.

Delyon-Petritis [3] proved the absence of the point spectrum under the condition $\limsup_{n\to\infty} a_n(\alpha) \geq 5$, and they proved directly

$$\mu(\limsup_{n\to\infty} E(n)) = 1,$$

where μ denotes the Lebesgue measure on $\mathbb{R}/\mathbb{Z}$. Instead, we use the following lemma, which is obtained by the theory of random Jacobi matrices.

Lemma 1 *The set* M *is Lebesgue measurable, and* $\mu(M)$=0 *or* 1.

Theorem 2 includes the result of Delyon-Petritis [3]. The author doesn't know an example of the operators of type (1) with the point spectrum for almost every θ, and whether the assumption in Theorem 2 is best possible is a open probrem, to his knowledge.

Remark 2 *It is known (see e.g. Khinchin [4, p.60]) that for almost every* α *we have*

$$\limsup_{n\to\infty} a_n(\alpha) = +\infty.$$

Remark 3 *Arguments in [1] and [10] are based on Kotani [6]. As a consequence of Kotani's result one has the following theorem concerning spectral properties of* H_θ: *For any interval* $A \neq \mathbb{R}/\mathbb{Z}$ *or* ϕ, *for any irrational number* α *and for almost every* θ *with respect to the Lebesgue measure on* $\mathbb{R}/\mathbb{Z}$, H_θ *has no absolutely continuous spectrum.*

2 Proof of Lemma 1

In this section we prove Lemma 1 with the spectral theory of random Jacobi matrices. We remark that V_θ is an element of $\Omega = \{0, \lambda\}^{\mathbb{Z}}$. Define a shift operator T on Ω by $(Tf)(n) = f(n+1)$, and define a metric on Ω by

$$d(f_1, f_2) := \sum_{n=-\infty}^{\infty} 2^{-|n|} |f_1(n) - f_2(n)| .$$

Then Ω is a compact separable metric space and T is continuous. We denote the Borel field on Ω by $\mathbf{B}$. Let Γ be a map from $\mathbb{R}/\mathbb{Z}$ into Ω defined by $\Gamma(\theta) = V_\theta$, then Γ is measurable. Hence we define a probability measure on $(\Omega, \mathbf{B})$ $P = \mu \circ \Gamma^{-1}$ (i.e. $P(S) = \mu(\Gamma^{-1}(S))$ for any $S \in \mathbf{B}$). It is easy to verify that P is a T-preserving probability measure and (Ω, T, P) is ergodic, that is, $TB = B$ implies $P(B) = 0$ or 1. We have the following lemma by the theory of random Jacobi matrices.

Lemma 2 (Kunz-Souillard) *There exists a closed set* Σ *in* $\mathbb{R}$ *such that*

$$\overline{\sigma_p(H_\theta)} = \Sigma \qquad P-a.s.$$

Proof. See e.g. [2; p.196, Theorem 9.4].

The Lemma 1 is a straightforward adaptation of the above lemma.

Proof of Lemma 1.

Let Σ be the set determined by Lemma 2. Then, there exists a P-measurable null set J such that $\overline{\sigma_p(H_\theta)} = \Sigma$ holds for any $V_\theta \in \Omega - J$. Let $F = \{\theta | \overline{\sigma_p(H_\theta)} = \Sigma\}$, then, we have $\Gamma^{-1}(J) \supset F^c$. From $\mu(\Gamma^{-1}(J)) = 0$ and the completeness of the Lebesgue measure, we have $\mu(F) = 1$. Hence, M coincides with F, or M^c contains F. Therefore, M is a Lebesgue measurable set, and $\mu(M) = 0$ or 1. □

3 Proofs of Theorems 1 and 2

In this section we prove Theorems 1 and 2. The proofs are based upon the improvement of the argument in Deyon-Petritis [3]. We prove Theorem 2 before the proof of Theorem 1.

Lemma 3

$$\limsup_{n\to\infty} E(n) \subset M.$$

Proof. See Delyon-Petritis [3].

Lemma 4 *Let*

$$c(\alpha) = \limsup_{n\to\infty} a_n(\alpha),$$

then we have

$$\limsup_{n\to\infty} \frac{q_{n+1}}{q_n} \geq \frac{c(\alpha) + \sqrt{c(\alpha)^2 + 4}}{2}.$$

Proof. Let

$$\beta = \limsup_{n\to\infty} \frac{q_{n+1}}{q_n}.$$

Since the assertion holds in the case of $\beta = \infty$, we give a proof in the case of $\beta < \infty$. From (4), we have

$$\beta = \limsup_{n\to\infty} (a_n(\alpha) + \frac{q_{n-1}}{q_n}).$$

Hence we have $c(\alpha) < \infty$, and

$$\beta \geq c(\alpha) + \frac{1}{\beta},$$

which implies the assertion. □

Lemma 5 *Suppose* $\limsup_{n\to\infty} a_n(\alpha) = 1$, *then the following holds:*

$$\lim_{n\to\infty} (q_n |q_n\alpha - p_n|) = \frac{1}{\sqrt{5}}.$$

Proof. Let

$$\alpha_n = a_n(\alpha) + \cfrac{1}{a_{n+1}(\alpha) + \cfrac{1}{a_{n+2}(\alpha) + \ddots}}.$$

Then, $a_n(\alpha) = 1$ for sufficiently large n,and we have

$$\alpha_n = \frac{1}{\omega},$$

where $\omega = (\sqrt{5} - 1)/2$. By Lang [7, p.8], we have

$$q_n\alpha - p_n = \frac{(-1)^{n+1}}{q_{n+1} + \omega q_n}.$$

and,

$$q_n|q_n\alpha - p_n| = \frac{1}{\frac{q_{n+1}}{q_n} + \omega}. \tag{6}$$

On the other hand, for sufficiently large n we have by (4)

$$q_{n+1} = q_n + q_{n-1},$$

and we have

$$\lim_{n\to\infty} \frac{q_{n+1}}{q_n} = \frac{1}{\omega}. \tag{7}$$

From (6) and (7), we reach the assertion. □

Proof of Theorem 2.

Considering Lemma 2 and Lemma 4, we are sufficient to show

$$\mu(\limsup_{n\to\infty} E(n)) > 0.$$

Let $\theta_1 and \theta_2$ be the two end points of the interval A. We define sets

$$E_i(n) = \{\theta| \min_{1\le m\le q_n} |\Phi(m\alpha) + \theta - \theta_i|_1 > |q_n\alpha - p_n|\} \qquad (i = 1, 2), \tag{8}$$

where $|\cdot|_1$ denotes the distance from 0 in $\mathbb{R}/\mathbb{Z}$. From (5), we have

$$|(\Phi((m \pm q_n)\alpha) + \theta) - (\Phi(m\alpha) + \theta)|_1 = |q_n\alpha - p_n|, \tag{9}$$

and from (8) and (9), we have

$$E_1(n) \cap E_2(n) \subset E(n). \tag{10}$$

By the definition of $E_i(n)$, we have

$$E_i(n)^c = \bigcup_{m=1}^{q_n} \{\theta | |\Phi(m\alpha) + \theta - \theta_i|_1 \leq |q_n\alpha - p_n|\} \qquad (i = 1, 2),$$

thus

$$\mu(E_i(n)^c) \leq 2q_n|q_n\alpha - p_n| \qquad (i = 1, 2). \tag{11}$$

From (5), (10) and (11), we have

$$\mu(E(n)) \geq 1 - 4\frac{q_n}{q_{n+1}},$$

therefore

$$\limsup_{n\to\infty} \mu(E(n)) \geq 1 - \frac{4}{\limsup_{n\to\infty} \frac{q_{n+1}}{q_n}}.$$

By $c(\alpha) \geq 4$ and Lemma 4, we have

$$\mu(\limsup_{n\to\infty} E(n)) \geq \limsup_{n\to\infty} \mu(E(n)) > 0,$$

which concludes the proof. □

Proof of Theorem 1.

By Remark 1, it is sufficient to consider the case where α is irrational. By the hypothesis, choose $\theta_1 = \Phi(1 - \alpha)$ and $\theta_2 = \Phi(1) = 0$ in (8), and we have

$$E_1(n) = \{\theta | \min_{1 \leq m \leq q_n} |\Phi((m+1)\alpha) + \theta|_1 > |q_n\alpha - p_n|\},$$

$$E_2(n) = \{\theta | \min_{1 \leq m \leq q_n} |\Phi(m\alpha) + \theta|_1 > |q_n\alpha - p_n|\}.$$

Therefore,

$$E_1(n) \cap E_2(n) = \{\theta | \min_{1 \leq m \leq q_n+1} |\Phi(m\alpha) + \theta|_1 > |q_n\alpha - p_n|\}.$$

Hence, we obtain

$$\mu(E(n)) \geq 1 - 2(q_n + 1)|q_n\alpha - p_n|. \tag{12}$$

Firstly, consider the case where $\limsup_{n\to\infty} a_n(\alpha) \geq 2$, then, from (5),(12) and Lemma 4, we have

$$\limsup_{n\to\infty} \mu(E(n)) > 0.$$

Secondly, consider the case where $\limsup_{n\to\infty} a_n(\alpha) = 1$, then, from (12) and Lemma 5, we have

$$\limsup_{n\to\infty} \mu(E(n)) \geq 1 - \frac{2}{\sqrt{5}},$$

which concludes the proof. □

Acknowledgements

The author would like to express his sincere thanks to Professor T. Ikebe and Professor A. Iwatsuka for their fruitful suggestions.

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