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White Noise Approach to Quantum Stochastic Integrals

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Introduction

Over the last decade quantum stochastic calculus on Fock space has been developed considerably into a new field of mathematics as is highlighted in the recent books of Meyer [11] and of Parthasarathy [16]. Therein have been so far discussed two ways of realizing the (Boson) Fock space: one is the straightforward realization by means of the direct sum of symmetric Hilbert spaces; the other is Guichardet's realization which was first adopted by Maassen [9] to study quantum stochastic integrals in terms of *integral-sum kernel operators.* Maassen's approach has been developed by Belavkin [1], Lindsay [7], Lindsay-Maassen [8], and Meyer [10].

As is well known, there is a third realization of Fock space, i.e., as the L^2-space over a Gaussian space through the celebrated Wiener-Itô-Segal isomorphism. This approach enables us to use a strong toolbox of distribution theory. In the recent works [12]–[14] we have established a systemtic theory of operators on Fock space on the basis of white noise calculus, i.e., a Schwartz type distribution theory on Gaussian space. We now believe it very interesting to discuss quantum stochastic integrals within our operator theory using a distribution theory to the full. Moreover, our approach allows taking a quite arbitrary space T as the "time" parameter space. Meanwhile, a white noise approach to quantum stochastic calculus has been proposed also by Huang [4] who reformulated the quantum Itô formula due to Hudson-Parthasarathy [5].

The purpose of this note is to outline the basic idea of how quantum stochastic integrals are generalized by means of white noise calculus. In our operator theory on Fock space a principal role has been played by an integral kernel operator [3]. Developing the idea, we introduce a generalization of an integral kernel operator and derive representation of an arbitrary operator on Fock space. This representation bears a similarity with a quantum stochastic integral against creation and annihilation processes. As a particular case we construct a *quantum Hitsuda-Skorokhod integral* which generalizes a quantum stochastic integral of Itô type to cover non-adapted case. On the other hand, it generalizes a classical Hitsuda-Skorokhod integral (see e.g., [2]) as well. In fact, a classical Hitsuda-Skorokhod integral is recovered as a quantum Hitsuda-Skorokhod integral with values in multiplication operators as the quantum Brownian motion corresponds to the classical one. The full details will be discussed in the forthcoming paper [15].

1 Operator theory on Gaussian space

We adopt exactly the same notations and assumptions used under the name of standard setup of white noise calculus, see e.g., [3], [12]–[14].

Let T be a topological space with a Borel measure $\nu(dt) = dt$ which is thought of as a time parameter space when it is an interval, or more generally as a field parameter space when it is a general manifold (note also that T can be a discrete space as well). Let A be a positive selfadjoint operator on $H = L^2(T, \nu; \mathbb{R})$ with Hilbert-Schmidt inverse and $\inf \operatorname{Spec}(A) > 1$. Then one obtains a Gelfand triple:

$$E \subset H = L^2(T, \nu; \mathbb{R}) \subset E^*$$

in the standard manner, where E and E^* are considered as spaces of test and generalized functions on T. To keep the delta functions δ_t in E^* we need to assume the usual hypotheses (H1)–(H3), see e.g., [3].

Let μ be the Gaussian measure on E^*. Then the complex Hilbert space

$$(L^2) = L^2(E^*, \mu; \mathbb{C})$$

is canonically isomorphic to the Boson Fock space over $H_{\mathbb{C}}$ through the celebrated Wiener-Itô-Segal isomorphism. In fact, each $\phi \in (L^2)$ admits a *Wiener-Itô expansion*:

$$\phi(x) = \sum_{n=0}^{\infty} \left\langle :x^{\otimes n}:, f_n \right\rangle, \qquad x \in E^*, \quad f_n \in H_{\mathbb{C}}^{\widehat{\otimes} n}, \tag{1}$$

with

$$\| \phi \|_0^2 \equiv \int_{E^*} |\phi(x)|^2 \mu(dx) = \sum_{n=0}^{\infty} n! \, | f_n |_0^2 . \tag{2}$$

With each $\xi \in E_{\mathbb{C}}$ we associate the *exponential vector* by

$$\phi_\xi(x) \equiv \sum_{n=0}^{\infty} \left\langle :x^{\otimes n}:, \frac{\xi^{\otimes n}}{n!} \right\rangle = \exp\left(\langle x, \xi \rangle - \frac{1}{2} \langle \xi, \xi \rangle \right), \qquad x \in E^*. \tag{3}$$

In particular, ϕ_0 is called the *Fock vacuum*.

Let $\Gamma(A)$ be the second quantized operator of A, i.e., the unique positive selfadjoint operator on (L^2) such that $\Gamma(A)\phi_\xi = \phi_{A\xi}$. Since $\Gamma(A)$ admits a Hilbert-Schmidt inverse under our assumptions, we obtain a complex Gelfand triple again in the standard manner:

$$(E) \subset (L^2) = L^2(E^*, \mu; \mathbb{C}) \subset (E)^*,$$

where elements in (E) and $(E)^*$ are called a *test (white noise) functional* and a *generalized (white noise) functional*, respectively.

For any $y \in E^*$ and $\phi \in (E)$ we put

$$D_y \phi(x) = \lim_{\theta \to 0} \frac{\phi(x + \theta y) - \phi(x)}{\theta}, \qquad x \in E^*. \tag{4}$$

It is known that the limit always exists and that $D_y \in \mathcal{L}((E),(E))$. Recall that the delta functions δ_t belong to E^* by hypotheses. Then *Hida's differential operator* is defined by

$$\partial_t = D_{\delta_t}, \qquad t \in T.$$

Obviously, $\partial_t \in \mathcal{L}((E),(E))$ is a rigorously defined *annihilation operator* at a point $t \in T$, i.e., ∂_t is *not* an operator-valued distribution but a continuous operator for itself. The *creation operator* is by definition the adjoint $\partial_t^* \in \mathcal{L}((E)^*,(E)^*)$, where $(E)^*$ is always equipped with the strong dual topology. The so-called canonical commutation relation is written as:

$$[\partial_s, \partial_t] = 0, \qquad [\partial_s^*, \partial_t^*] = 0, \qquad [\partial_s, \partial_t^*] = \delta_s(t)I, \qquad s,t \in T, \tag{5}$$

where the last relation is understood in a generalized sense.

For any $\kappa \in (E_{\mathbb{C}}^{\otimes(l+m)})^*$ there exists a unique operator $\Xi_{l,m}(\kappa) \in \mathcal{L}((E),(E)^*)$ such that

$$\langle\langle \Xi_{l,m}(\kappa)\phi, \psi \rangle\rangle = \left\langle \kappa, \left\langle\left\langle \partial_{s_1}^* \cdots \partial_{s_l}^* \partial_{t_1} \cdots \partial_{t_m}\phi, \psi \right\rangle\right\rangle \right\rangle, \qquad \phi, \psi \in (E).$$

This operator is called the *integral kernel operator* with kernel distribution κ and is denoted descriptively by

$$\Xi_{l,m}(\kappa) = \int_{T^{l+m}} \kappa(s_1, \cdots, s_l, t_1, \cdots, t_m) \partial_{s_1}^* \cdots \partial_{s_l}^* \partial_{t_1} \cdots \partial_{t_m} ds_1 \cdots ds_l dt_1 \cdots dt_m.$$

Let $(E_{\mathbb{C}}^{\otimes(l+m)})^*_{\mathrm{sym}(l,m)}$ be the space of all $\kappa \in (E_{\mathbb{C}}^{\otimes(l+m)})^*$ which is symmetric with respect to the first l and the last m variables independently. Then the kernel distribution is uniquely determined in $(E_{\mathbb{C}}^{\otimes(l+m)})^*_{\mathrm{sym}(l,m)}$.

Proposition 1.1 [3] *Let $\kappa \in (E_{\mathbb{C}}^{\otimes(l+m)})^*$. Then $\Xi_{l,m}(\kappa) \in \mathcal{L}((E),(E))$ if and only if $\kappa \in (E_{\mathbb{C}}^{\otimes l}) \otimes (E_{\mathbb{C}}^{\otimes m})^*$. In particular, $\Xi_{0,m}(\kappa) \in \mathcal{L}((E),(E))$ for any $\kappa \in (E_{\mathbb{C}}^{\otimes m})^*$.*

For $\Xi \in \mathcal{L}((E),(E)^*)$ a function on $E_{\mathbb{C}} \times E_{\mathbb{C}}$ defined by

$$\Theta(\xi, \eta) = \langle\langle \Xi\phi_\xi, \phi_\eta \rangle\rangle, \qquad \xi, \eta \in E_{\mathbb{C}}, \tag{6}$$

is called the *symbol* of Ξ. Since the exponential vectors $\{\phi_\xi;\ \xi \in E_{\mathbb{C}}\}$ spans a dense subspace of (E), the symbol recovers the operator uniquely. For an integral kernel operator, we have

$$\langle\langle \Xi_{l,m}(\kappa)\phi_\xi, \phi_\eta \rangle\rangle = \langle\langle \langle \kappa, \eta^{\otimes l} \otimes \xi^{\otimes m} \rangle \phi_\xi, \phi_\eta \rangle\rangle = \langle \kappa, \eta^{\otimes l} \otimes \xi^{\otimes m} \rangle e^{\langle \xi, \eta \rangle}, \tag{7}$$

where $\xi, \eta \in E_{\mathbb{C}}$ and $\kappa \in E_{\mathbb{C}}^{\otimes(l+m)}$.

Theorem 1.2 [12] *Let Θ be a $\mathbb{C}$-valued function on $E_{\mathbb{C}} \times E_{\mathbb{C}}$. Then, it is the symbol of an operator in $\mathcal{L}((E),(E)^*)$ if and only if*

(O1) For any $\xi, \xi_1, \eta, \eta_1 \in E_{\mathbb{C}}$, the function

$$z, w \mapsto \Theta(z\xi + \xi_1, w\eta + \eta_1), \qquad z, w \in \mathbb{C},$$

is entire holomorphic;

(O2) There exist constant numbers $C \geq 0$, $K \geq 0$ and $p \in \mathbb{R}$ such that

$$|\Theta(\xi,\eta)| \leq C \exp K \left(|\xi|_p^2 + |\eta|_p^2 \right), \qquad \xi,\eta \in E_{\mathbb{C}}.$$

Theorem 1.3 [12] *For any $\Xi \in \mathcal{L}((E),(E)^*)$ there is a unique family of kernel distributions $\kappa_{l,m} \in (E_{\mathbb{C}}^{\otimes(l+m)})^*_{\mathrm{sym}(l,m)}$ such that*

$$\Xi\phi = \sum_{l,m=0}^{\infty} \Xi_{l,m}(\kappa_{l,m})\phi, \qquad \phi \in (E), \tag{8}$$

converges in $(E)^*$.

The expression as in (8) is called the *Fock expansion* of Ξ. There are pararell results for $\mathcal{L}((E),(E)) \subset \mathcal{L}((E),(E)^*)$, see [13] for complete discussion.

2 Generalization of integral kernel operators

Let $\{e_j\}_{j=0}^{\infty}$ be the normalized eigenfunctions of the operator A. For simplicity, we put $e(\mathbf{i}) = e_{i_1} \otimes \cdots \otimes e_{i_l}$ for $\mathbf{i} = (i_1, \cdots, i_l)$ and $e(\mathbf{j}) = e_{j_1} \otimes \cdots \otimes e_{j_m}$ for $\mathbf{j} = (j_1, \cdots, j_m)$. Then, for a linear map $L : E_{\mathbb{C}}^{\otimes(l+m)} \to \mathcal{L}((E),(E)^*)$ and $p,q,r,s \in \mathbb{R}$ we put

$$\| L \|_{l,m;p,q;r,s} = \sup \left\{ \sum_{\mathbf{i},\mathbf{j}} |\langle\langle L(e(\mathbf{i}) \otimes e(\mathbf{j}))\phi,\, \psi \rangle\rangle|^2 \, |e(\mathbf{i})|_p^2 \, |e(\mathbf{j})|_q^2 \; ; \; \begin{array}{l} \phi,\psi \in (E) \\ \|\phi\|_{-s} \leq 1 \\ \|\psi\|_{-r} \leq 1 \end{array} \right\}^{1/2}.$$

For brevity we put $\| L \|_p = \| L \|_{l,m;p,p;p,p}$. The next result will be useful.

Proposition 2.1 [14] *For a linear map $L : E_{\mathbb{C}}^{\otimes(l+m)} \to \mathcal{L}((E),(E)^*)$ the following four conditions are equivalent:*
(i) *$L \in \mathcal{L}(E_{\mathbb{C}}^{\otimes(l+m)}, \mathcal{L}((E),(E)^*))$;*
(ii) $\sup \left\{ |\langle\langle L(\eta)\phi,\, \psi\rangle\rangle| \; ; \; \begin{array}{ll} \eta \in E_{\mathbb{C}}^{\otimes(l+m)}, & |\eta|_p \leq 1 \\ \phi,\psi \in (E), & \|\phi\|_p \leq 1, \|\psi\|_p \leq 1 \end{array} \right\} < \infty$ *for some $p \geq 0$;*
(iii) $\| L \|_{-p} < \infty$ *for some $p \geq 0$;*
(iv) $\| L \|_{l,m;p,q;r,s} < \infty$ *for some $p,q,r,s \in \mathbb{R}$.*
In that case, for any $p,q,r,s \in \mathbb{R}$ we have

$$|\langle\langle L(\eta)\phi,\, \psi\rangle\rangle| \leq \| L \|_{l,m;-p,-q;-r,-s} \, |\eta|_{l,m;p,q} \, \|\phi\|_s \, \|\psi\|_r, \tag{9}$$

where $\eta \in E_{\mathbb{C}}^{\otimes(l+m)}$, $\phi,\psi \in (E)$.

Each $L \in \mathcal{L}(E_{\mathbb{C}}^{\otimes(l+m)}, \mathcal{L}((E),(E)^*))$ is justifiably called an *$\mathcal{L}((E),(E)^*)$-valued distribution* on T^{l+m}. In fact,

$$\mathcal{L}(E_{\mathbb{C}}^{\otimes(l+m)}, \mathcal{L}((E),(E)^*)) \cong (E_{\mathbb{C}}^{\otimes(l+m)})^* \otimes \mathcal{L}((E),(E)^*)$$

holds by the kernel theorem.

With each $L \in \mathcal{L}(E_{\mathbb{C}}^{\otimes(l+m)}, \mathcal{L}((E),(E)^*))$ we associate an operator $\Xi \in \mathcal{L}((E),(E)^*)$ by the formula:

$$\langle\langle \Xi\phi_\xi, \phi_\eta \rangle\rangle = \left\langle\left\langle L(\eta^{\otimes l} \otimes \xi^{\otimes m})\phi_\xi, \phi_\eta \right\rangle\right\rangle, \qquad \xi, \eta \in E_{\mathbb{C}}. \tag{10}$$

We must check that the definition works; namely, conditions (O1) and (O2) in Theorem 1.2 are to be verified for

$$\Theta(\xi,\eta) = \left\langle\left\langle L(\eta^{\otimes l} \otimes \xi^{\otimes m})\phi_\xi, \phi_\eta \right\rangle\right\rangle, \qquad \xi, \eta \in E_{\mathbb{C}}. \tag{11}$$

In fact, (O1) is straightforward. As for (O2), by Proposition 2.1 we obtain

$$\begin{aligned} |\Theta(\xi,\eta)| &\le \| L \|_{-p} \, | \eta^{\otimes l} \otimes \xi^{\otimes m} |_p \, \| \phi_\xi \|_p \, \| \phi_\eta \|_p \\ &= \| L \|_{-p} \, | \eta |_p^l \, | \xi |_p^m \exp \frac{1}{2} \left(| \xi |_p^2 + | \eta |_p^2 \right), \end{aligned}$$

and therefore

$$|\Theta(\xi,\eta)| \le C \exp K \left(| \xi |_p^2 + | \eta |_p^2 \right), \qquad \xi, \eta \in E_{\mathbb{C}},$$

for some $K \ge 0$ and $C \ge 0$, which proves (O2). It then follows from the characterization theorem (Theorem 1.2) that Θ is the symbol of an operator $\Xi \in \mathcal{L}((E),(E)^*)$; namely, there exists a unique operator $\Xi \in \mathcal{L}((E),(E)^*)$ satisfying (10). It is reasonable to write

$$\Xi = \int_T \partial_{s_1}^* \cdots \partial_{s_l}^* L(s_1, \cdots, s_l, t_1, \cdots, t_m) \partial_{t_1} \cdots \partial_{t_m} ds_1 \cdots ds_l dt_1 \cdots dt_m. \tag{12}$$

In fact, comparing (7) and (10), we understand that the above Ξ is a generalization of an integral kernel operator. From definition we see that the adjoint operator of Ξ given as in (12) is

$$\Xi^* = \int_T \partial_{t_1}^* \cdots \partial_{t_m}^* L^*(s_1, \cdots, s_l, t_1, \cdots, t_m) \partial_{s_1} \cdots \partial_{s_l} ds_1 \cdots ds_l dt_1 \cdots dt_m,$$

where $L^* \in \mathcal{L}(E_{\mathbb{C}}^{\otimes(l+m)}, \mathcal{L}((E),(E)^*))$ is defined by $L^*(\zeta) = L(\zeta)^*$, $\zeta \in E_{\mathbb{C}}^{\otimes(l+m)}$.

Generalized integral kernel operators occur in an integral kernel operator. Consider an integral kernel operator $\Xi_{l,m}(\kappa)$ with $\kappa \in (E_{\mathbb{C}}^{\otimes(l+m)})^*_{\mathrm{sym}(l,m)}$. For integers $0 \le \alpha \le l$ and $0 \le \beta \le m$,

$$L_0(\eta_1, \cdots, \eta_\alpha, \xi_1, \cdots, \xi_\beta) = \Xi_{l-\alpha, m-\beta}\left((\kappa \otimes_\beta (\xi_1 \otimes \cdots \otimes \xi_\beta)) \otimes^\alpha (\eta_1 \otimes \cdots \otimes \eta_\alpha) \right)$$

becomes a continuous $(\alpha+\beta)$-linear map from $E_{\mathbb{C}}$ into $\mathcal{L}((E),(E)^*)$, where $\otimes_\beta$ and $\otimes^\alpha$ denote the right and left contractions with respect to the last β and first α coordinates, respectively. The proof is straightforward from a norm estimate of an integral kernel operator. Therefore there exists $L \in \mathcal{L}(E_{\mathbb{C}}^{\otimes(\alpha+\beta)}, \mathcal{L}((E),(E)^*))$ such that

$$L(\eta_1 \otimes \cdots \otimes \eta_\alpha \otimes \xi_1 \otimes \cdots \otimes \xi_\beta) = L_0(\eta_1, \cdots, \eta_\alpha, \xi_1, \cdots, \xi_\beta).$$

In other words,

Lemma 2.2 *Let* $0 \le \alpha \le l$ *and* $0 \le \beta \le m$. *For any* $\kappa \in (E_{\mathbb{C}}^{\otimes(l+m)})^*$ *there exists* $L \in \mathcal{L}(E_{\mathbb{C}}^{\otimes(\alpha+\beta)}, \mathcal{L}((E),(E)^*))$ *such that*

$$\begin{aligned} &L(\eta_1 \otimes \cdots \otimes \eta_\alpha \otimes \xi_1 \otimes \cdots \otimes \xi_\beta) \\ &\quad = \Xi_{l-\alpha,m-\beta}\left((\kappa \otimes_\beta (\xi_1 \otimes \cdots \otimes \xi_\beta)) \otimes^\alpha (\eta_1 \otimes \cdots \otimes \eta_\alpha)\right). \end{aligned} \tag{13}$$

Theorem 2.3 (Fubini type) *Fix integers* $0 \le \alpha \le l$ *and* $0 \le \beta \le m$. *Given* $\kappa \in (E_{\mathbb{C}}^{\otimes(l+m)})^*$ *let* $L \in \mathcal{L}(E_{\mathbb{C}}^{\otimes(\alpha+\beta)}, \mathcal{L}((E),(E)^*))$ *be defined as in (13). Then,*

$$\Xi_{l,m}(\kappa) = \int_{T^{\alpha+\beta}} \partial_{s_1}^* \cdots \partial_{s_\alpha}^* L(s_1, \cdots, s_\alpha, t_1, \cdots, t_\beta) \partial_{t_1} \cdots \partial_{t_\beta} ds_1 \cdots ds_\alpha dt_1 \cdots dt_\beta.$$

The verification is simple. We only need to compute the symbols of both sides according to the definitions.

3 A stochastic integral-like representation

Given an operator $\Xi \in \mathcal{L}((E),(E)^*)$ let

$$\Xi = \sum_{l,m=0}^{\infty} \Xi_{l,m}(\kappa_{l,m}), \qquad \kappa_{l,m} \in (E_{\mathbb{C}}^{\otimes(l+m)})^*,$$

be the Fock expansion. We now devide the Fock expansion into three parts:

$$\Xi = \sum_{l \ge 0, m \ge 1} \Xi_{l,m}(\kappa_{l,m}) + \sum_{l \ge 1} \Xi_{l,0}(\kappa_{l,0}) + \Xi_{0,0}(\kappa_{0,0}). \tag{14}$$

Since $\Xi_{0,0}(\kappa_{0,0})$ is a scalar operator, say cI, we obtain immediately,

$$\Xi_{0,0}(\kappa_{0,0}) = cI, \qquad c = \langle\langle \Xi \phi_0, \phi_0 \rangle\rangle .$$

In other words, c is the vacuum expectation of Ξ.

For the first term in (14) we have the following

Lemma 3.1 *There exists* $L \in \mathcal{L}(E_{\mathbb{C}}, \mathcal{L}((E),(E)^*)$ *such that*

$$\int_T L(t) \partial_t dt = \sum_{l \ge 0, m \ge 1} \Xi_{l,m}(\kappa_{l,m}).$$

Proof. For $l \ge 0$, $m \ge 1$ we put

$$L_{l,m}(\xi) = \Xi_{l,m-1}(\kappa_{l,m} \otimes_1 \xi), \quad \xi \in E_{\mathbb{C}}.$$

It follows from Lemma 2.2 and Theorem 2.3 that $L_{l,m} \in \mathcal{L}(E_{\mathbb{C}}, \mathcal{L}((E),(E)^*))$ and

$$\Xi_{l,m}(\kappa_{l,m}) = \int_T L_{l,m}(t) \partial_t dt. \tag{15}$$

With the help of some precise norm estimates established in [12] and [13] we can prove that

$$\sum_{l\geq 0, m\geq 1} L_{l,m}(\xi)\phi, \qquad \xi \in E_{\mathbb{C}}, \quad \phi \in (E),$$

converges in $(E)^*$ and defines a continuous bilinear map from $E_{\mathbb{C}} \times (E)$ into $(E)^*$. Since $\mathcal{B}(E_{\mathbb{C}}, (E); (E)^*) \cong \mathcal{L}(E_{\mathbb{C}}, \mathcal{L}((E), (E)^*))$ by the kernel theorem, there exists $L \in \mathcal{L}(E_{\mathbb{C}}, \mathcal{L}((E), (E)^*))$ such that

$$L(\xi)\phi = \sum_{l\geq 0, m\geq 1} L_{l,m}(\xi)\phi, \qquad \xi \in E_{\mathbb{C}}, \quad \phi \in (E).$$

It is then straightforward to see that the above L has the desired property. qed

In a similar manner we have the following

Lemma 3.2 *There exists $M \in \mathcal{L}(E_{\mathbb{C}}, \mathcal{L}((E), (E)))$ such that*

$$M(\xi)\phi = \sum_{l\geq 1} \Xi_{0,l-1}(\kappa_{l,0} \otimes_1 \xi)\phi, \qquad \phi \in (E),$$

where the righthand side converges in (E). Moreover, $[M(\xi), \partial_t] = 0$ for all $\xi \in E_{\mathbb{C}}$ and $t \in T$.

The last part of the assertion follows from the next result of which proof is a simple application of Fock expansion.

Lemma 3.3 *$\Xi \in \mathcal{L}((E), (E))$ commutes with all ∂_t, $t \in T$, if and only if the Fock expansion of Ξ is of the form:*

$$\Xi = \sum_{m\geq 0} \Xi_{0,m}(\kappa_{0,m}).$$

For any $L \in \mathcal{L}(E_{\mathbb{C}}, \mathcal{L}((E), (E)^*))$, we write $L^*(\xi) = L(\xi)^*$ for $\xi \in E_{\mathbb{C}}$. Then $L^* \in \mathcal{L}(E_{\mathbb{C}}, \mathcal{L}((E), (E)^*))$ again. If $M \in \mathcal{L}(E_{\mathbb{C}}, \mathcal{L}((E), (E)))$, we have

$$M^* \in \mathcal{L}(E_{\mathbb{C}}, \mathcal{L}((E)^*, (E)^*)) \subset \mathcal{L}(E_{\mathbb{C}}, \mathcal{L}((E), (E)^*)).$$

Then, a straightforward argument with operator symbols leads us to the following

Lemma 3.4 *Notations being the same as in Lemma 3.2, we have*

$$\int_T \partial_s^* M^*(s) ds = \sum_{l\geq 1} \Xi_{l,0}(\kappa_{l,0}).$$

This corresponds to the second term in (14). In view of Lemmas 3.1 and 3.4, we obtain

Theorem 3.5 *Every $\Xi \in \mathcal{L}((E), (E)^*)$ admits a representation of the form:*

$$\Xi = \int_T L(t)\partial_t dt + \int_T \partial_t^* M^*(t) dt + cI, \tag{16}$$

where $c \in \mathbf{C}$, $L \in \mathcal{L}(E_{\mathbb{C}}, \mathcal{L}((E), (E)^*))$ and $M \in \mathcal{L}(E_{\mathbb{C}}, \mathcal{L}((E), (E)))$ such that $[M(\xi), \partial_t] = 0$ for any $\xi \in E_{\mathbb{C}}$ and $t \in T$.

Since $M(\xi)$ commutes with ∂_t for all $t \in T$, expression (16) may be written as

$$\Xi = \int_T L(t)\partial_t dt + \int_T M^*(t)\partial_t^* \, dt + cI. \tag{17}$$

Then (17) is regarded as (a sort of) quantum stochastic integral against the creation and annihilation processes when T is an interval, see e.g., [11], [16].

There is a simple modification of Theorem 3.5. For the proof we need only to consider the adjoint Ξ^* in the above theorem.

Theorem 3.6 *Every $\Xi \in \mathcal{L}((E),(E)^*)$ admits a representation of the form:*

$$\Xi = \int_T L(t)\partial_t dt + \int_T \partial_t^* M^*(t) dt + cI, \tag{18}$$

where $c \in \mathbb{C}$, $M \in \mathcal{L}(E_{\mathbb{C}}, \mathcal{L}((E),(E)^*))$ and $L \in \mathcal{L}(E_{\mathbb{C}}, \mathcal{L}((E),(E)))$ satisfying $[L(\xi), \partial_t] = 0$ for any $\xi \in E_{\mathbb{C}}$ and $t \in T$.

For the uniqueness we only mention the following

Proposition 3.7 *Let $L \in \mathcal{L}(E_{\mathbb{C}}, \mathcal{L}((E),(E)^*))$, $M \in \mathcal{L}(E_{\mathbb{C}}, \mathcal{L}((E),(E)))$ and $c \in \mathbb{C}$. Assume that $[M(\xi), \partial_t] = 0$ for any $\xi \in E_{\mathbb{C}}$ and $t \in T$. If*

$$\int_T L(t)\partial_t dt + \int_T \partial_t^* M^*(t) dt + cI = 0,$$

then

$$\int_T L(t)\partial_t dt = 0, \qquad \int_T \partial_t^* M^*(t) dt = 0, \qquad c = 0.$$

Remark. Note that

$$\int_T L(t)\partial_t dt = 0, \qquad L \in \mathcal{L}(E_{\mathbb{C}}, \mathcal{L}((E),(E)^*)),$$

does not imply $L = 0$. For example, consider $L(t) = \xi(t)D_\eta - \eta(t)D_\xi$ with some $\xi, \eta \in E_{\mathbb{C}}$.

4 Quantum Hitsuda-Skorokhod integrals

In this section we consider a particular case where

$$T = \mathbb{R}, \qquad A = 1 + t^2 - \frac{d^2}{dt^2}, \qquad E = \mathcal{S}(\mathbb{R}).$$

According to the general theory established in the previous section one obtains a generalized integral kernel operator:

$$\int_T \partial_t^* L(t)\, dt, \qquad L \in \mathcal{L}(E_{\mathbb{C}}, \mathcal{L}((E),(E)^*)).$$

In this section we should like to introduce a "stochastic integral" of the form:

$$\int_0^t \partial_s^* L(s)\, ds, \qquad t \geq 0.$$

For that purpose L should possess a stronger property that L is continuously extended to a linear map from $E_{\mathbb{C}}^*$ into $\mathcal{L}((E),(E)^*)$. Note the natural inclusion relation

$$\mathcal{L}(E_{\mathbb{C}}^*,\mathcal{L}((E),(E)^*)) \subset \mathcal{L}(E_{\mathbb{C}},\mathcal{L}((E),(E)^*)).$$

For $L \in \mathcal{L}(E_{\mathbb{C}}^*,\mathcal{L}((E),(E)^*))$ we write $L_s = L(\delta_s)$ for simplicity. Then $\{L_s\}$ is regarded as a quantum stochastic process with values in $\mathcal{L}((E),(E)^*)$.

Lemma 4.1 *Let $L \in \mathcal{L}(E_{\mathbb{C}}^*,\mathcal{L}((E),(E)^*))$. Then for any $f \in E_{\mathbb{C}}^*$ there exists an operator $M_f \in \mathcal{L}((E),(E)^*)$ such that*

$$\langle\langle M_f\phi_\xi,\, \phi_\eta\rangle\rangle = \langle\langle L(f\eta)\phi_\xi,\, \phi_\eta\rangle\rangle\,, \qquad \xi,\eta \in E_{\mathbb{C}}.$$

Moreover, $f \mapsto M_f$ is continuous, i.e., $M \in \mathcal{L}(E_{\mathbb{C}}^*,\mathcal{L}((E),(E)^*))$.

PROOF. First note that, for any $p \geq 0$ there exist $q > 0$ and $A_{p,q} \geq 0$ such that

$$|\,\xi\eta\,|_p \leq A_{p,q}\,|\,\xi\,|_{p+q}\,|\,\eta\,|_{p+q}\,, \qquad \xi,\eta \in E_{\mathbb{C}}.$$

Then, by duality we obtain

$$|\,f\eta\,|_{-(p+q)} \leq A_{p,q}\,|\,f\,|_{-p}\,|\,\eta\,|_{p+q}\,, \qquad \eta \in E_{\mathbb{C}}, \quad f \in E_{\mathbb{C}}^*. \tag{19}$$

On the other hand, using the canonical isomorphism

$$\mathcal{L}(E_{\mathbb{C}}^*,\mathcal{L}((E),(E)^*)) \cong \mathcal{L}(E_{\mathbb{C}}^*,((E)\otimes(E))^*),$$

which comes from the kernel theorem, we find $L^* \in \mathcal{L}((E)\otimes(E),E_{\mathbb{C}})$ such that

$$\langle\langle L(f)\phi,\, \psi\rangle\rangle = \langle f,\, L^*(\phi\otimes\psi)\rangle\,, \qquad f \in E_{\mathbb{C}}^*, \quad \phi,\psi \in (E).$$

By continuity, for any $p \geq 0$ there exist $q \geq 0$ and $B_{p,q} \geq 0$ such that

$$|\,L^*(\phi\otimes\psi)\,|_p \leq B_{p,q}\,\|\,\phi\,\|_{p+q}\,\|\,\psi\,\|_{p+q}\,, \qquad \phi,\psi \in (E). \tag{20}$$

We now consider

$$\Theta_f(\xi,\eta) = \langle\langle L(f\eta)\phi_\xi,\, \phi_\eta\rangle\rangle = \langle f\eta,\, L^*(\phi_\xi\otimes\phi_\eta)\rangle\,.$$

Suppose $p \geq 0$ is given arbitrarily. Take $q > 0$ with property (19). In view of (20) we may find $r \geq 0$ such that

$$\begin{aligned} |\Theta_f(\xi,\eta)| &\leq |\,f\eta\,|_{-(p+q)}\,|\,L^*(\phi_\xi\otimes\phi_\eta)\,|_{p+q} \\ &\leq A_{p,q}\,|\,f\,|_{-p}\,|\,\eta\,|_{p+q}\,B_{p+q,r}\,\|\,\phi_\xi\,\|_{p+q+r}\,\|\,\phi_\eta\,\|_{p+q+r} \\ &\leq A_{p,q}B_{p+q,r}\rho^r\,|\,f\,|_{-p}\,|\,\eta\,|_{p+q+r}\exp\frac{1}{2}\left(|\,\xi\,|_{p+q+r}^2 + |\,\eta\,|_{p+q+r}^2\right). \end{aligned}$$

Consequently, for any $p \geq 0$ we have found constants $C \geq 0$, $K \geq 0$ and $s \geq 0$ such that

$$|\Theta_f(\xi,\eta)| \leq C\,|\,f\,|_{-p}\exp K\left(|\,\xi\,|_{p+s}^2 + |\,\eta\,|_{p+s}^2\right), \qquad f \in E_{\mathbb{C}}^*, \quad \xi,\eta \in E_{\mathbb{C}}. \tag{21}$$

Hence by the characterization theorem (Theorem 3.1), for any $f \in E_{\mathbb{C}}^*$ there exists an operator $M_f \in \mathcal{L}((E),(E)^*)$ such that

$$\langle\langle M_f\phi_\xi,\, \phi_\eta\rangle\rangle = \Theta_f(\xi,\eta) = \langle\langle L(f\eta)\phi_\xi,\, \phi_\eta\rangle\rangle\,, \qquad \xi,\eta \in E_{\mathbb{C}}.$$

Obviously, $f \mapsto M_f$ is linear. Inequality (21) implies the continuity on the Hilbert space $\{f \in E_{\mathbb{C}}^*; |f|_{-p} < \infty\}$. Since $E_{\mathbb{C}}^*$ is the inductive limit of such Hilbert spaces, we conclude that $M \in \mathcal{L}(E_{\mathbb{C}}^*, \mathcal{L}((E),(E)^*))$. qed

The operator M_f constructed above is denoted by

$$M_f = \int_T f(s)\partial_s^* L_s ds.$$

In particular, for $f = 1_{[0,t]}$ we write

$$\Omega_t \equiv \int_0^t \partial_s^* L_s ds, \qquad t \geq 0,$$

which form a one-parameter family of operators in $\mathcal{L}((E),(E)^*)$. This is called a *quantum Hitsuda-Skorokhod integral* with values in $\mathcal{L}((E),(E)^*)$. To be sure we rephrase the definition:

$$\langle\langle \Omega_t\phi_\xi,\, \phi_\eta\rangle\rangle = \langle\langle L(1_{[0,t]}\eta)\phi_\xi,\, \phi_\eta\rangle\rangle, \qquad \xi,\eta \in E_{\mathbb{C}}. \tag{22}$$

It is interesting to observe how our operator-valued process $\{\Omega_t\}$ generalizes the classical Hitsuda-Skorokhod integral of which definition we shall review after [2] quickly. Let $\Phi_t \in (E)^*$, $t \geq 0$, be given. Since $\partial_t^* \in \mathcal{L}((E)^*,(E)^*)$ for any t, for any $\phi \in (E)$ one obtains a function: $t \mapsto \langle\langle \partial_t^*\Phi_t,\, \phi\rangle\rangle$. Assume that the function is measurable and

$$\int_0^t |\langle\langle \partial_s^*\Phi_s,\, \phi\rangle\rangle|\, ds < \infty, \qquad t \geq 0.$$

Then there exists $\Psi_t \in (E)^*$, $t \geq 0$, uniquely such that

$$\langle\langle \Psi_t,\, \phi\rangle\rangle = \int_0^t \langle\langle \partial_s^*\Phi_s,\, \phi\rangle\rangle\; ds, \qquad \phi \in (E).$$

The above obtained Ψ_t is denoted by

$$\Psi_t = \int_0^t \partial_s^*\Phi_s\, ds$$

and is called the *Hitsuda-Skorokhod integral.* The Hitsuda-Skorokhod integral coincides with the usual Itô integral when the integrand $\{\Phi_t\}$ is an adapted L^2-function with respect to the filtration generated by the Brownian motion

$$B_t(x) = \left\langle x,\, 1_{[0,t]}\right\rangle, \qquad x \in E^*, \quad t \geq 0.$$

We need one more remark. Each $\Phi \in (E)^*$ gives rise to a continuous operator in $\mathcal{L}((E),(E)^*)$ by multiplication since $(\phi,\psi) \mapsto \phi\psi$ is a continuous bilinear map from $(E) \times (E)$ into (E). This identification extends to a natural inclusion relation:

$$\mathcal{L}(E_{\mathbb{C}}^*,(E)^*) \subset \mathcal{L}(E_{\mathbb{C}}^*, \mathcal{L}((E),(E)^*)).$$

Given $\Phi \in \mathcal{L}(E^*_{\mathbb{C}}, (E)^*)$ let $\tilde{\Phi}$ denote the corresponding element in $\mathcal{L}(E^*_{\mathbb{C}}, \mathcal{L}((E),(E)^*))$. Then one has a quantum Hitsuda-Skorokhod integral:

$$\Omega_t = \int_0^t \partial_s^* \tilde{\Phi}_s ds, \qquad t \geq 0, \tag{23}$$

as well as the classical Hitsuda-Skorokhod integral:

$$\Psi_t = \int_0^t \partial_s^* \Phi_s ds, \qquad t \geq 0. \tag{24}$$

In fact, since the both maps $t \mapsto \delta_t \in E^*$ and $t \mapsto \partial_t \phi \in (E)$ are continuous, so is $t \mapsto \langle\!\langle \partial_t^* \Phi_t, \phi \rangle\!\rangle$. Therefore Ψ_t is well defined.

Theorem 4.2 *For any $\Phi \in \mathcal{L}(E^*_{\mathbb{C}}, (E)^*)$ let Ω_t be the quantum Hitsuda-Skorokhod integral defined as in (23) and let Ψ be the classical Hitsuda-Skorokhod integral defined as in (24). Then, it holds that*

$$\Psi_t = \Omega_t \phi_0, \qquad t \geq 0,$$

where ϕ_0 is the Fock vacuum.

PROOF. By definition (22) we have

$$\langle\!\langle \Omega_t \phi_0, \phi_\eta \rangle\!\rangle = \left\langle\!\!\left\langle \tilde{\Phi}(1_{[0,t]}\eta)\phi_0, \phi_\eta \right\rangle\!\!\right\rangle = \left\langle\!\!\left\langle \Phi(1_{[0,t]}\eta), \phi_\eta \right\rangle\!\!\right\rangle.$$

In terms of the adjoint operator $\Phi^* \in \mathcal{L}((E), E_{\mathbb{C}})$ the last expression becomes

$$\left\langle\!\!\left\langle \Phi(1_{[0,t]}\eta), \phi_\eta \right\rangle\!\!\right\rangle = \left\langle 1_{[0,t]}\eta, \Phi^*\phi_\eta \right\rangle = \int_0^t \eta(s)(\Phi^*\phi_\eta)(s) ds.$$

Moreover, note that

$$\begin{aligned} \eta(s)(\Phi^*\phi_\eta)(s) &= \eta(s) \langle \delta_s, \Phi^*\phi_\eta \rangle = \eta(s) \langle\!\langle \Phi(\delta_s), \phi_\eta \rangle\!\rangle \\ &= \langle\!\langle \Phi(\delta_s), \partial_s \phi_\eta \rangle\!\rangle = \langle\!\langle \partial_s^* \Phi_s, \phi_\eta \rangle\!\rangle. \end{aligned}$$

Consequently,

$$\langle\!\langle \Omega_t \phi_0, \phi_\eta \rangle\!\rangle = \int_0^t \langle\!\langle \partial_s^* \Phi_s, \phi_\eta \rangle\!\rangle ds = \langle\!\langle \Psi_t, \phi_\eta \rangle\!\rangle,$$

and we come to $\Omega_t \phi_0 = \Psi_t$ as desired. qed

References

[1] V. P. Belavkin: *A quantum nonadapted Ito formula and stochastic analysis in Fock scale*, J. Funct. Anal. **102** (1991), 414–447.

[2] T. Hida, H.-H. Kuo, J. Potthoff and L. Streit: "White Noise," Kluwer Academic, 1993.

[3] T. Hida, N. Obata and K. Saitô: *Infinite dimensional rotations and Laplacians in terms of white noise calculus*, Nagoya Math. J. **128** (1992), 65–93.

[4] Z.-Y. Huang: *Quantum white noises – White noise approach to quantum stochastic calculus*, Nagoya Math. J. **129** (1993), 23–42.

[5] R. L. Hudson and K. R. Parthasaraty: *Quantum Ito's formula and stochastic evolutions*, Commun. Math. Phys. **93** (1984), 301–323.

[6] I. Kubo and S. Takenaka: *Calculus on Gaussian white noise I–IV*, Proc. Japan Acad. **56A** (1980), 376–380; 411-416; **57A** (1981), 433–437; **58A** (1982), 186–189.

[7] J. M. Lindsay: *On set convolutions and integral-sum kernel operators*, in "Probability Theory and Mathematical Statistics, Vol.2 (B. Grigelionis et al. eds.)," pp. 105–123, Mokslas, Vilnius, 1990.

[8] J. M. Lindsay and H. Maassen: *An integral kernel approach to noise*, in "Quantum Probability and Applications III (L. Accardi and W. von Waldenfels eds.)," pp. 192–208, Lect. Notes in Math. Vol. 1303, Springer-Verlag, 1988.

[9] H. Maassen: *Quantum Markov processes on Fock space described by integral kernels*, in "Quantum Probability and Applications II (L. Accardi and W. von Waldenfels eds.)," pp. 361–374, Lect. Notes in Math. Vol. 1136, Springer-Verlag, 1985.

[10] P. A. Meyer: *Elements de probabilités quantiques VI*, in "Séminaire de Probabilités XXI (J. Azéma et al. eds.)," pp. 34–49, Lect. Notes in Math. Vol. 1247, Springer-Verlag, 1987.

[11] P. A. Meyer: "Quantum Probability for Probabilists," Lect. Notes in Math. Vol. 1538, Springer-Verlag, 1993.

[12] N. Obata: *An analytic characterization of symbols of operators on white noise functionals*, J. Math. Soc. Japan **45** (1993), 421–445.

[13] N. Obata: "White Noise Calculus and Fock Space," Lect. Notes in Math. Vol. 1577, Springer-Verlag, to be published.

[14] N. Obata: *Operator calculus on vector-valued white noise functionals*, J. Funct. Anal. **121** (1994), 185–232.

[15] N. Obata: *Integral kernel operators on Fock space and quantum Hitsuda-Skorokhod integrals*, preprint, 1994.

[16] K. R. Parthasarathy: "An Introduction to Quantum Stochastic Calculus," Birkhäuser, 1993.

[17] K. R. Parthasarathy and K. B. Sinha: *Stochastic integral representation of bounded quantum martingales in Fock space*, J. Funct. Anal. **67** (1986), 126–151.