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AVANT-PROPOS TO NILPOTENT GEOMETRY AND ANALYSIS

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§1. At the beginning of the colloquim entitled "Nilpotent Geometry and Analysis" I would like to address what I wish to mean by this title.

First of all I introduce the notion of filtered manifold. A tangential filtration F on a differentiable manifold M is a sequence $\{F^p\}_{p\in\mathbb{Z}}$ of subbundles of the tangent bundle TM of M such that the following conditions are satisfied:

i) $F^p \supset F^{p+1}$,

ii) $F^0 = 0, \quad \bigcup_{p\in\mathbb{Z}} F^p = TM$,

iii) $[\underline{F}^p, \underline{F}^q] \subset \underline{F}^{p+q}, \quad$ for all $\quad p, q \in \mathbb{Z}$,

where $\underline{F}^p$ denotes the sheaf of the germs of sections of F^p. A filtered manifold is a differentiable manifold M equipped with a tangential filtration F. We shall often denote by the bold letter $\mathbb{M}$ the filtered manifold (M, F) and by $\{T^p\mathbb{M}\}$ or $\{F^pTM\}$ its tangential filtration. An isomorphism of a filtered manifold $\mathbb{M}$ onto a filtered manifold $\mathbb{M}'$ is a diffeomorphism $\varphi : M \longrightarrow M'$ such that $\varphi_* T^p\mathbb{M} = T^p\mathbb{M}'$ for all $p \in \mathbb{Z}$, where φ_* denotes the differential of φ. Let $\mathbb{M}$ be a filtered manifold. By definition there is an integer $\mu \geq 0$ such that $T^{-\mu}\mathbb{M} = TM$. The minimum of such integers is called the depth of $\mathbb{M}$.

The notion of filtered manifold is not only a generalization of that of manifold but also includes various interesting geometric structures. Let us give some examples.

1) Trivial filtration. A differentiable manifold M itself may be regarded as a filtered manifold equipped with the trivial filtration defined by $F^p_{tr}TM = TM$ for $p < 0$ and $F^q_{tr}TM = 0$ for $q \geq 0$.

2) Tangential filtration derived from a regular differential system (Tanaka [15]). Let D be a differential system on a differentiable manifold M, that is, a subbundle of the tangent bundle of M. Then there is associated a sequence of subsheaves $\{\mathcal{D}^p\}_{p<0}$ of $\underline{TM}$, called the derived systems of D, which is defined inductively by:

$$\mathcal{D}^{-1} = \underline{D}, \quad \mathcal{D}^{p-1} = \mathcal{D}^p + [\mathcal{D}^p, \mathcal{D}^{-1}] \qquad (p < 0).$$

It then holds that:

$$[\mathcal{D}^p, \mathcal{D}^q] \subset \mathcal{D}^{p+q} \qquad \text{for} \quad p, q < 0.$$

Suppose that the derived systems $\mathcal{D}^p$ are all vector bundles, that is, there are subbundles $D^p \subset TM$ such that $\underline{D}^p = \mathcal{D}^p$ for all $p < 0$ (in this case the differential system D is called

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regular). Then there exists a minimum integer $\mu \geq 1$ such that $D^p = D^{-\mu}$ for all $p \leq -\mu$. Setting

$$F^pTM = \begin{cases} 0 & (p \geq 0) \\ D^p & (-1 \geq p \geq -\mu) \\ TM & (p \leq -\mu - 1), \end{cases}$$

we have a filtered manifold $\mathbb{M} = (M, F)$ derived from the regular differential system D.

There are two cases to distinguish. If $D^{-\mu} \subsetneqq TM$, then $D^{-\mu}$ is completely integrable and defines a foliation on M. In particular, if D is completely integrable the filtered manifold $\mathbb{M}$ is nothing but a foliated manifold. If $D^{-\mu} = TM$, we say that the tangential filtration F is generated by the differential system D.

If a filtered manifold $\mathbb{M}$ (or $\mathbb{M}'$) is derived from a differential system D on M (resp. D' on M'), then $\mathbb{M}$ and $\mathbb{M}'$ are isomorphic if and only if (M, D) and (M', D') are isomorphic, that is, there is a diffeomorphism $\varphi : M \longrightarrow M'$ such that $\varphi_* D = D'$.

The notion of filtered manifold has stemmed from the geometry of differential systems elaborated by N. Tanaka.

3) Higher order contact manifold. Let $\pi : M \longrightarrow N$ be a fibred manifold. Let $J^k(M, N)$ be the bundle of k-jets of cross-sections of π. On this jet bundle we have a sequence of canonical differential systems $\{D^p\}$ called the higher order contact structure. In local coordinates it is expressed as follows: Let $(x^1, \cdots, x^n)$, $(x^1, \cdots, x^n, y^1, \cdots, y^m)$ be local coordinates systems of N and M respectively. Then $(x^1, \cdots, x^n, \cdots, p^i_\alpha, \cdots)$, where $p^i_\alpha = \dfrac{\partial^{|\alpha|} y^i}{\partial x^\alpha}$ with $\alpha = (\alpha_1, \cdots, \alpha_n)$, $|\alpha| \leq k$, gives a local coordinate system of $J^k(M, N)$ called a canonical coordinates system. Put

$$\omega^i_\alpha = dp^i_\alpha - \sum_{j=1}^{n} p^i_{\alpha+1_j} dx^j$$

for $|\alpha| \leq k - 1$, with $\alpha + 1_j = (\alpha_1, \cdots, \alpha_j + 1, \cdots, \alpha_n)$, and define D^p $(p \leq -1)$ by the following Pfaff equations:

$$D^p : \quad \omega^i_\alpha = 0 \qquad (i = 1, \cdots, n \quad |\alpha| \leq k + p).$$

It is easy to see that D^p are well-defined subbundles of $TJ^k(M, N)$ and satisfy:

i) $\underline{D}^{p-1} = \underline{D}^p + [\underline{D}^p,\ \underline{D}^{-1}]$,

ii) $D^p = TJ^k(M, N) \quad$ for $\quad p \leq -k - 1$.

We thus obtain a canonical tangential filtration $\{D^p\}$ on $J^k(M, N)$ of depth $k+1$ generated by D^{-1}. It should be noted that if $\dim M = n + 1$, $\dim N = n$ and $k = 1$ then $J^1(M, N)$ is a contact manifold having D^{-1} as its contact structure.

Higher order contact structures as well as contact structures play a fundamental role in geometric studies of differential equations (cf. [18], [19]).

4) Standard filtered manifold. Let $\mathfrak{n}$ be a finite-dimensional Lie algebra endowed with a gradation $\mathfrak{n} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{n}_p$ such that

i) $[\mathfrak{n}_p,\ \mathfrak{n}_q] \subset \mathfrak{n}_{p+q}$,

ii) $\mathfrak{n}_p = 0 \qquad p \geq 0$.

Note that $\mathfrak{n}$ is therefore nilpotent. Let N be a Lie group whose Lie algebra is $\mathfrak{n}$. Set $\mathfrak{n}^p = \bigoplus_{i \geq p} \mathfrak{n}_i$ and identify $N \times \mathfrak{n}^p$ with a left invariant subbundle of TN, then $\{N \times \mathfrak{n}^p\}_{p \in \mathbb{Z}}$ is a tangential filtration on N. The filtered manifold $\mathbb{N} = (N, \{N \times \mathfrak{n}^p\})$ is called a standard filtered manifold of type $\mathfrak{n}$.

Let $\mathbb{M}$ be a filtered manifold. The tangential filtration $\{T^p\mathbb{M}\}$ defines on each tangent space $T_xM, x \in M$, the induced filtration $\{T^p_x\mathbb{M}\}$. We denote by $T_x\mathbb{M}$ this filtered vector space $(T_xM, \{T^p_x\mathbb{M}\})$. Now by setting

$$gr_p T_x\mathbb{M} = T^p_x\mathbb{M}/T^{p+1}_x\mathbb{M},$$

we form a graded vector space:

$$grT_x\mathbb{M} = \bigoplus_{p \in \mathbb{Z}} gr_p T_x\mathbb{M}.$$

This vector space carries a natural bracket operation induced from the Lie bracket of vector fields: For $\xi \in gr_pT_x\mathbb{M}$, $\eta \in gr_qT_x\mathbb{M}$, take local cross-sections X, Y of $T^p\mathbb{M}, T^q\mathbb{M}$ respectively such that $\xi \equiv X_x \,(\mathrm{mod}\, T^{p+1}_x\mathbb{M}), \eta \equiv Y_x \,(\mathrm{mod}\, T^{q+1}_x\mathbb{M})$, and define $[\xi, \eta] \equiv [X, Y]_x \,(\mathrm{mod}\, T^{p+q+1}_x\mathbb{M})$. It is then easy to see that this bracket operation is well-defined and makes $grT_x\mathbb{M}$ a Lie algebra. Clearly we have:

i) $[gr_pT_x\mathbb{M},\ gr_qT_x\mathbb{M}] \subset gr_{p+q}T_x\mathbb{M}$,
ii) $gr_pT_x\mathbb{M} = 0 \qquad$ for $p \geq 0$.

This graded Lie algebra $grT_x\mathbb{M}$ is called the symbol algebra of $\mathbb{M}$ at x ([15]), and may be considered as the tangent space (algebra) at x of the filtered manifold $\mathbb{M}$ or the first order approximation (in some weighted sense) at x of the filtered manifold $\mathbb{M}$.

We wish to study geometric objects and differential equations on filtered manifolds by letting the symbol algebras (nilpotent Lie algebras) play the fundamental role that the tangent spaces (abelian Lie algebras) usually have played for manifolds. This involves , in particular, a generalization from the abelian to the nilpotent, and from the polynomial ring to the enveloping algebra of a nilpotent Lie algebra. We wish to call nilpotent geometry and analysis attempts in these directions and related topics.

Now I would like to discuss what has interested me and has been done toward this.

§2. Geometry on filtered manifolds.

One of the fundamental problems in geometry is the equivalence problem of geometric structures. It is to find criteria to decide whether or not two geometric structures are (locally) equivalent.

Let us briefly mention the history. The general equivalence problem has been studied by many geometers since S. Lie. In particular, E. Cartan, in his study of infinite groups [1], invented a general method to treat the equivalence problem on the basis of the method of moving frames and the theory of Pfaff systems in involution, and found important applications in various domains of his work. However, his method was rather of the nature of a general heuristic principle not settled in precise mathematical concepts.

As was brought to light by C. Ehresmann and others, one of the fundamental concepts underlying his method is that of principal fibre bundle and G-structure. The extensive works which followed, in particular, I.M. Singer - S. Sternberg [12] and S. Sternberg [14], gave a rigorous foundation to deal with the general equivalence problem as that of G-structures and clarified important aspects of Cartan's ideas.

But the theory of G-structures as achieved there did not seem adequate to treat the equivalence problem in full generality: Even if one confines oneself to the equivalence problem of G-structures (the first order geometric structures), one has to deal with higher order geometric structures in a way suitable to find the higher order invariants of G-structures, and moreover it is necessary to develop a theory including the intransitive structures.

In answer to this, we developed in [8] a general scheme to treat the equivalence problem on the basis of the higher order "non-commutative" frame bundles, and gave a method to solve the general equivalence problem in a neighbourhood of every generic point in the analytic category.

On the other hand, in applications to various geometric problems the general method of G-structures is not always effective. For instance, the deep work of Cartan on *les systèmes de Pfaff à cinq variables* [2] is far from being well understood merely by the usual approach of G-structures. Here he elaborated a more refined method fitting in with the structures considered: a method of reduction by using Pfaff systems and of constructing what is now called a Cartan connection.

In his series of papers (in particular, [15],[17]), N. Tanaka developed this aspect extensively as the geometry of differential systems, and found various applications, especially in CR geometry [16] and in the geometric study of ordinary differential equations [18]. Of particular importance are the prolongation method based on the algebraic prolongation of fundamental graded Lie algebras and the construction of Cartan connections for geometric structures associated with simple graded Lie algebras.

In [11], by integrating the Tanaka theory and our general method [8], we introduced the notion of filtered manifold and developed a unified method to study the geometric structures on filtered manifolds.

In particular, we have introduced the concept of involutive geometric structure on filtered manifold by using the generalized Spencer cohomology group,to study the geometric structures of infinite type.

As another achievement of our method, we have obtained a general criterion to construct a Cartan connection associated with a geometric structure, generalizing and simplifying the construction of Tanaka.

For a treatise on these subjects refer to [11]. Here we only explain just the entrance of our formulation.

Let $\mathbb{M}$ be a filtered manifold. Choose a graded vector space $\mathfrak{v} = \bigoplus \mathfrak{v}_i$ isomorphic to the graded Lie algebra $grT_x\mathbb{M}$ filtered vector space $T_x\mathbb{M}$ for some and hence all $x \in M$. Let $\mathcal{R}^{(0)}(\mathbb{M})$ be the totality of the isomorphisms as graded vector spaces $z : \mathfrak{v} \longrightarrow grT_x\mathbb{M}$ for any $x \in M$. Then $\mathcal{R}^{(0)}(\mathbb{M})$ is a principal fiber bundle over M with structure group $Aut(\mathfrak{v})$ and called the first order frame bundle of $\mathbb{M}$.

A subbundle of $\mathcal{R}^{(0)}(\mathbb{M})$ is called a first order geometric structure on $\mathbb{M}$. It is a generalization of G-structure and most of interesting geometric structures can be formulated in this way. It is for these geometric structures that the studies mentioned above are developed. It should be noted that a filtered manifold $\mathbb{M}$ is in general not locally trivial and that the local equivalence problem of $\mathcal{R}^{(0)}(\mathbb{M})$ itself is already a difficult and interesting problem.

§3. Differential equations on filtered manifold.

For a differential operator on a filtered manifold (M, F), it is often more natural and better to use a weighted ordering induced from the tangential filtration F rather than the usual ordering: The weighted order is defined for a vector field X to be $\leq k$ if X is a section of $F^{-k}TM$ and is extended to any differential operator in the obvious manner. We wish to study differential equations systematically by using this weighted ordering and see to what extent the usual theory can be naturally generalized and on the other hand what phenomena different from the usual occur.

First we consider the formal theory, that is, to obtain a criterion for the formal integrability of a given differential equations. As well-known as Cartan-Kähler theorem, Cartan formulated the formal integrability as systems in involution in terms of Pfaff systems. The algebraic aspect of the notion of involutivity is later clarified by Kuranishi, Sternberg and others. In terms of jet bundles the formal theory is then settled by Spencer, Quillen, and Goldschmidt. (See, e.g., [3], [4], [13].)

The formal theory can be generalized quite naturally to differential equations on filtered manifolds. For a vector bundle E on a filtered manifold $\mathbb{M} = (M, F)$ we introduce the k-th weighted jet bundle $\hat{J}^k E$. Let $\underline{E}_x$ be the stalk at x of the sheaf $\underline{E}$ and let $F^{k+1}\underline{E}_x$ be the subspace of $\underline{E}_x$ consisting of $u \in \underline{E}_x$ such that $D_x(\langle \alpha, u \rangle) = 0$ for for any differential operator D of weighted order $\leq k$ and for any section α of E^*. We then set $\hat{J}^k_x E = \underline{E}_x / F^{k+1}\underline{E}_x$ and $\hat{J}^k E = \bigcup_{x \in M} \hat{J}^k_x E$, which is called the weighted k-th jet bundle.

Now a system of differential equations on $\mathbb{M}$ of weighted order $\leq k$ is defined by a subbundle (submanifold if nonlinear)R^k of $\hat{J}^k E$. A section u of E is a solution of R^k if the weighted jet $\hat{j}^k u$ is contained in R^k.

The formal theory for differential equations based on weighted jet bundles can be developed analogously to the one based on usual jet bundle, in which the following exact sequence plays an important role:

$$(*) \qquad 0 \to U_k(grT\mathbb{M})^* \otimes E \to \hat{J}^k E \to \hat{J}^{k-1} E \to 0,$$

where $U_k(grT\mathbb{M})^*$ is the bundle over M whose fibre $U_k(grT_x\mathbb{M})^*$ at x is the dual of the k-th homogeneous component (in the weighted sense) of the universal enveloping algebra $U(grT_x\mathbb{M})$ of the graded nilpotent Lie algebra $grT_x\mathbb{M}$. Note that in the weighted version $U_k(grT\mathbb{M})^*$ takes the the place that the the k-times symmetric tensor of the cotangent bundle $S^k(T^*M)$ does in the usual jet. We have also the notion of involutivity in the weighted sense (roughly speaking, it is a criterion for formal solvability in terms of weighted formal expansion) by using the generalized Spencer cohomology group. The detail will be developed in [10].

Next let us consider the problem of convergence. Cartan-Kähler theorem asserts that if a system of differential equations is involutive and analytic then it has an analytic solution.

We then ask whether it remains true for a weighted involutive system. For that, as a simple example, consider the following differential equation in a neighbourhood of 0 in $\mathbb{R}^2$:

$$(a(x,t)\frac{\partial^2}{\partial^2 x} + \frac{\partial}{\partial t})u = f(x,t),$$

where a and f are analytic functions. If $a(0) = 0$ the equation is not involutive at 0. But if we regard $\mathbb{R}^2$ as a filtered manifold with a tangential filtration F defined by: $F^{-1} = \langle \frac{\partial}{\partial x} \rangle, F^{-2} = T\mathbb{R}^2$ then the left hand side of the equation is homogeneous of weighted order 2 and therefore the equation is involutive in the weighted sense even if a vanishes at 0. Since the equation is not Kowalevskian if $a(0) = 0$, it does not in general admit analytic solution. In fact, we can choose a and f so that any formal solution at 0 is divergent. Thus it turns out that there does not necessarily exist convergent solutions for a weighted involutive system.

However, it holds and is not hard to prove that the equation always possesses a formal solution u satisfying the following estimate:

$$\left|((\frac{\partial}{\partial x})^p (\frac{\partial}{\partial t})^q u)(0)\right| \leq Cp!(q!)^2 \rho^{p+q}$$

for any non-negative integers p, q with some positive constant C, ρ.

We take another example. Consider the following equation in a neighbourhood of 0 in $\mathbb{R}^3$:

$$(**) \qquad (Z + aX^2 + bXY + cY^2)u = f,$$

where X, Y, Z are vector fields on $\mathbb{R}^3$ defined by

$$X = \frac{\partial}{\partial x} - \frac{1}{2}y\frac{\partial}{\partial z}, \quad Y = \frac{\partial}{\partial y} + \frac{1}{2}x\frac{\partial}{\partial z}, \quad Z = \frac{\partial}{\partial z},$$

and a, b, c are analytic functions which vanish at the origin and f is any analytic function.

This equation is degenerate second order differential equation and is not involutive in the usual sense, so there is not necessarily analytic solution.

But since $[X,Y] = Z, [X,Z] = [Y,Z] = 0$ the vectors X, Y, Z generate a nilpotent graded Lie algebra (3-dimensional Heisenberg Lie algebra) $\mathfrak{n} = \mathfrak{n}_{-1} \oplus \mathfrak{n}_{-2}$ with $\mathfrak{n}_{-1} = \langle X, Y \rangle, \mathfrak{n}_{-2} = \langle Z \rangle$ we can identify $\mathbb{R}^3$ with the Heisenberg Lie group. If we regard $\mathbb{R}^3$ as a standard filtered manifold in this way, then the equation is weighted involutive because the weighted order of Z is equal to 2 while those of X, Y are 1. We then expect there always exists a formal solution u of (**) at the origin satisfying the following estimate:

$$|X^p Y^q Z^r u(0)| \leq Cp!q!(r!)^2 \rho^{p+q+r}$$

for any non-negative integers p, q, r with some positive constant C, ρ.

It is remarkable that not only this does hold but also we have a much more general theorem as shown in [9]. To state the theorem, we quote a part of it.

Soit $\mathfrak{n}$ une algèbre de Lie de dimension n sur le corps $\mathbb{R}$ des nombres réels munie d'une graduation; $\mathfrak{n} = \bigoplus_{p=1}^{\mu} \mathfrak{n}_p$ telle que $[\mathfrak{n}_p, \mathfrak{n}_q] \subset \mathfrak{n}_{p+q}$ pour tous entiers p, q (on convient que $\mathfrak{n}_p = 0$ pour $p \notin \{1, \ldots, \mu\}$). Donc $\mathfrak{n}$ est nilpotente. Soit N un groupe de Lie dont l'algèbre de Lie est $\mathfrak{n}$. Posons $\mathfrak{n}^p = \bigoplus_{i \leq p} \mathfrak{n}^i$ et identifions $N \times \mathfrak{n}^p$ au sous-fibré invariant à gauche de $TN(= N \times \mathfrak{n})$. Alors $(N, \{N \times \mathfrak{n}^p\})$ est une variété filtrée, dite *standard.* Coisissons une base $\{X_1, \cdots, X_n\}$ de $\mathfrak{n}$ compatible avec la graduation, c'est-à-dire telle que $\{X_{d(p-1)+1}, \cdots, X_{d(p)}\}$ soit une base de $\mathfrak{n}_p$ pour tout p où $d(p) = \dim \mathfrak{n}^p$. On définit une fonction de poids $w : \{1, \cdots, n\} \to \{1, \cdots, \mu\}$ par la propriété: $X_i \in \mathfrak{n}_{w(i)}$ pour tout i. On emploiera les notations suivantes: Pour $I = (i_1, \cdots, i_l) \in \{1, \cdots, n\}^l$, on pose

$$X_I = X_{i_1} \cdots X_{i_l}, \qquad w(X_I) = w(I) = \sum_{a=1}^{l} w(i_a).$$

On identifiera aussi X_i (ou X_I) au champ de vecteurs (resp. à l'opérateur différentiel) invariant à gauche sur N.

Soient V, W des espaces vectoriels de dimensions finies sur $\mathbb{R}$. Soit $\Phi(x, y_I)$ une fonctions à valeurs dans W de $(x, y_I) \in N \times \prod_{w(I) \leq k} V_I$ avec $V_I = V$, et considérons l'équation différentielle d'ordre pondéré $\leq k$, pour une fonction F à valeurs dans V:

$$\text{(E)} \qquad \Phi(x, (X_I F)(x)) = 0 \qquad (w(I) \leq k).$$

Soit $o \in N$ et soit $F^l = \hat{j}_o^l F \in \hat{J}_o^l(N \times V)$ avec $l \geq k$. On dit que F^l est une *l-jet solution* de (E) si

$$X_J \Phi(x, (X_I F)(x)) \,|_{\, x=o} = 0$$

pour tout J tel que $w(J) \leq l - k$. On dit aussi d'après Malgrange [7] qu'une l-jet solution F^l est *fortement prolongeable* si pour toute m-jet solution F^m $(m \geq l)$ telle que $\hat{j}_o^l F^m = F^l$, il existe $(m+1)$-jet solution F^{m+1} tell que $\hat{j}_o^m F^{m+1} = F^m$.

Si $F^k \in \hat{J}_o^k(N \times V)$ est fortement prolongeable, on peut donc trouver une série $\{F^l\}_{l \geq k}$ de l-jet solutions telle que $\hat{j}_o^l F^m = F^l$ pour $m \geq l \geq k$, et par passage à la limite une solution formelle de (E): $F^\infty = \varprojlim F^l$ telle que $\hat{j}_o^k F^\infty = F^k$. On demade alors s'il existe une solution analytique pour l'équation Φ supposée analytique. Lorsque la filtration de N est triviale, i.e., $\mathfrak{n} = \mathfrak{n}_1$, ceci est vrai, car c'est exactement une version du théorème de Cartan-Kähler démontré par Malgrange ([7] Appendice). Mais dans le cas général cela ne reste plus vrai. (Voir les deux exemples précédents.)

Pour traiter le problème de convergence, nous introduisons une classe de fonctions formelles Gevrey sur la variété firtrée $(N, \{N \times \mathfrak{n}^p\})$. Soit V, comme en haut, un espace vectoriel de dimension finie, et le munissons d'une norme quelconque $|\cdot|$. Désignons par E le fibré trivial $N \times V$ sur N. Posons: $\hat{J}^\infty E = \varprojlim \hat{J}^l E$; son fibré $\hat{J}_x^\infty E$ est donc l'ensemble des fonctions formelles à valeurs dans V en $x \in N$. Soit $r = (r_1, \cdots, r_n)$ avec $r_i > 0$. Définissons une pseudo-norme $|\cdot|_r$ de $\hat{J}_x^\infty E$, en posant pour $F \in \hat{J}_x^\infty$,

$$|F|_r = \sum_{k=0}^{\infty} \frac{1}{k!} \sup_{w(K)=k} |\,(X_K F)(x)\,|\; r^K,$$

où $r^K = r_{k_1} \cdots r_{k_l}$ pour $K = (k_1, \cdots, k_l)$. On dit que $F \in \widehat{J}_x^\infty E$ est une *fonction formelle Gevrey* en x s'il existe r et $C > 0$ tels que $|F|_r \leq C$. On voit que cette définition ne dépend que de la filtration $\{N \times \mathfrak{n}^p\}$. On notera $\mathcal{G}_x(N) \otimes V$ l'ensemble des fonctions formelles Gevrey en x à valeurs dans V. Remarqueons que lorsque la filtration de N est trivale, une fonction formelle Gevrey n'est pas autre qu'une fonction analytique.

En revenant maintenant à l'équation différentielle (E), nous disons que $\Phi(x, y_I)$ *est une fonction formelle Gevrey par rapport à* x *et analytique par rapport aux* y_I *en* (o, y_I^0) si

$$\Phi \in \mathcal{G}_{(o,y_I^0)}(\widetilde{N}) \otimes W,$$

où on pose $\widetilde{N} = N \times \prod V_I$, et le regarde comme un groupe de Lie filtré dont l'algèbre de Lie $\widetilde{\mathfrak{n}}$ est graduée par $\widetilde{\mathfrak{n}} = \bigoplus \widetilde{\mathfrak{n}}_p$ avec

$$\widetilde{\mathfrak{n}}_1 = \mathfrak{n}_1 \oplus \bigoplus_{m(I)\leq k} V_I, \qquad \widetilde{\mathfrak{n}}_p = \mathfrak{n}_p \quad (p \geq 2), \qquad [\widetilde{\mathfrak{n}}, \oplus V_I] = 0.$$

Alors le théorème principal s'énonce comme suit:

Théorème. *Supposons que* $\Phi(x, y_I)$ *est une fonction formelle Gevrey par rapport à* x *et analytique par rapport aux* y_I *en* (o, y_I^0)*. Si* $F^k = (y_I^0) \in \widehat{J}_o^k(N \times V)$ *est une* k*-jet solution de (E) et fortement prolongeable, alors il existe une solution formelle Gevrey* $F \in \mathcal{G}_o(N) \otimes V$ *de (E) telle que* $\widehat{J}_o^k F = F^k$.

Pour démontrer ce théorème nous suivons de près Malgrange (loc.cit.) qui établit ce théorème dans le cas analytique, c'est-à-dire où $\mathfrak{n} = \mathfrak{n}_1$. La clef de la démonstration est d'employer le théorème des voisinages privilégiés (une version géneralisée pour l'algèbre enveloppante d'une algèbre de Lie nilpotente, un anneau non-commutatif) pour choisir par récurrence une série formelle de sorte qu'elle satisfasse la majoration voulue.

As shown above, the class of formal Gevrey functions seems well-fitted to differential equations on filtered manifolds. However, at present we do not know very well the nature of formal Gevrey functions. Though they are only formal functions (in general divergent), they seem to have interesting features related with geometric properties of the underlying filtered manifolds. For example, even in the simplest case of Heisenberg Lie group it would be interesting to study formal Gevrey functions by analytic continuation along integral curves of the contact structure.

Finally we mention some topics or problems seeming to relate to our approach.

1. Hörmander has shown that if a system of vector fields $X_1, \cdots, X_{k+1}$ on a differential manifold M satisfies the Hörmander condition, that is, it generates the tangent space at each point, and if $a \in C^\infty(M)$, the differential operator $\sum_{i=1}^k X_i^2 + X_{k+1} + a$ is hypoelliptic.

This theorem is closely related with nilpotent graded Lie algebras, and after Hörmander many people have elaborated to generalize it to differential operators on graded nilpotent Lie groups. For a detailed exposition see Helffer and Nourrigat [6].

It seems that these theories become more transparent at least in the geometric aspect if reformulated as a theory of differential operators on filtered manifolds. The exact sequence (*) will play a fundamental role providing us with a simple way to associate with a differential operators on a filtered manifold $\mathbb{M}$ an invariant differential operator on the graded nilpotent Lie group (algebra)$grT_x\mathbb{M}$ for each $x \in M$ as the symbol at x.

2. A CR-structure on a differential manifold M is defined by giving a subbundle D of the tangent bundle and a complex structure on D satisfying certain integrability conditions. So CR-manifold may be viewed as a first order geometric structure on a filtered manifold. In particular, a strongly pseudo-convex CR- manifold has a contact structure as underlying structure. The extensive studies on CR-manifolds are a rich example of nilpotent geometry and analysis. It is interesting to exploit some other nice geometric structures on filtered manifolds to which geometry is relatively developed and analysis, relating to the geometry, is fertile. To develop an analysis on Cartan's space of five variables would be interesting.

3. In the C^∞-category the equivalence problem often involves very delicate problems in analysis.

If a geometric structure is formally transitive one can associate with it a transitive Lie algebra which formally fulfills the invariants of the structure: Given two geometric structures formally transitive, they are formally equivalent if and only if the corresponding Lie algebras are conjugate equivalent.

It then naturally arises the following question:

If two formally transitive geometric structures have a same transitive Lie algebra L associated with them, are they equivalent?

To obtain a map which gives an equivalence one has to solve a certain differential equation. If L is finite dimensional, the equation is of finite type and one can obtain a solution by solving a Frobenius type equation. Thus the problem is the case when L is infinite dimensional. In the analytic category one can find a solution by Cartan-Kähler theorem. In the C^∞-category it is called the C^∞-integrability problem.

Darboux's theorem and Newlander-Nirenberg theorem are typical examples to which the C^∞-integrability problem is affirmatively solved. Sternberg and Conn gave counter-examples based on the Lewy equation. Guillemin found the Jordan-Hölder decomposition of a transitive Lie algebra L, and proposed a general program to study the C^∞-integrability problem according to the decomposition as a sort of Galois theory for differential equations. Spencer, Kumpera, and Goldschmidt elaborated a general machine to attack the problem in terms of Lie equations. Malgrange proved the conjecture of Spencer that if a structure comes from an elliptic analytic Lie equation then the problem is affirmative. At present, one of the known best results is the one proved by Goldschmidt and Molino, which asserts that if L contains the translations then the C^∞-integrability problem is affirmative. (For a short survey and references on these topics see Goldschmidt [5].)

However, there are many examples which are not covered by the above results. We therefore propose to study the C^∞-integrability problem in the case when L is graded, that is L is isomorphic to the completion of a graded Lie algebra $\bigoplus_{i=-\mu}^{\infty} \mathfrak{g}_i$ (note that if μ is 1 then it is contained in the case studied by Goldschmidt and Molino), and we conjecture the problem is affirmative in this case. This problem is closely related to the C^∞-solvability of invariant differential operators on nilpotent Lie groups.

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