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The rigidity of universal solvable Lie algebras of Iwasawa subalgebras.

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Introduction.

An n-dimensional Lie algebra $\mathfrak{g}$ is called rigid if all n-dimensional Lie algebras near $\mathfrak{g}$ are isomorphic to $\mathfrak{g}$. Nijenhuis & Richardson proved that $\mathfrak{g}$ is rigid if its Chevalley cohomololgy group $H^2(\mathfrak{g}, \mathfrak{g}) = 0$ ([N-R]). As necessary conditions, Carles proved that rigid Lie algebras over the complex numbers $\mathbb{C}$ have to be algebraic and satisfy several other conditions ([Ca] Prop 4.1).

In this paper we treat the rigidity of solvable Lie algebras $\mathfrak{g}$ over $\mathbb{C}$. In § 1 we show that rigid $\mathfrak{g}$ are isomorphic to the universal solvable Lie algebras $\mathfrak{u}(\mathfrak{n}) = \mathfrak{n} \rtimes T$. Here $\mathfrak{n}$ are the nilradicals of $\mathfrak{g}$, and T are maximal abelian subalgebras of $\mathrm{Der}(\mathfrak{n})$ composed of semi-simple elements. Although the rigidity of low dimensional solvable Lie algebras were obtained([C-D], [Be]), in general dimensions there are not many known examples of rigid solvable Lie algebras except for Borel subalgebras of semi-simple Lie algebras([L-L]). From § 2 we try to compute $H^2(\mathfrak{u}(\mathfrak{n}^{\mathbb{C}}), \mathfrak{u}(\mathfrak{n}^{\mathbb{C}}))$ and check their rigidity when $\mathfrak{n}$ are nilpotent parts of Iwasawa decompositions of semi-simple Lie algebras over $\mathbb{R}$. When the semi-simple Lie algebras are normal real forms, $\mathfrak{u} = \mathfrak{u}(\mathfrak{n}^{\mathbb{C}})$ are isomorphic to Borel subalgebras, and $H^2(\mathfrak{u}, \mathfrak{u}) = 0$ by [L-L]. In this paper we determine $H^2(\mathfrak{u}, \mathfrak{u})$ and the rigidity of $\mathfrak{u}$ when $\mathfrak{n}$ are nilpotent parts of real simple Lie algebras of real rank 1. Those results are shown in Proposition 3.2 and 3.4. For other several cases, $H^2(\mathfrak{u}, \mathfrak{u})$ are given in Remark 3.3 without their proofs. We can conclude that not full but partial generalization of [L-L] is possible.

§ 1 Universal solvable Lie algebras

Let $\mathfrak{n}$ be a nilpotent Lie algebra over $\mathbb{C}$, and let $\mathrm{Der}(\mathfrak{n})$ be its all derivations. Choosing a maximal abelian subalgebra T of $\mathrm{Der}(\mathfrak{n})$ consisting of semi-simple elements (another T' and T are conjugate), we define the universal solvable Lie algebra $\mathfrak{u} = \mathfrak{u}(\mathfrak{n})$ by the semi-direct product $\mathfrak{n} \rtimes T$, where T acts on $\mathfrak{n}$ naturally.

Proposition 1.1. *Let* $\mathfrak{g}$ *be a non-nilpotent solvable Lie algebra. If* $\mathfrak{g}$ *is rigid, then* $\mathfrak{g}$ *is isomorphic to* $\mathfrak{u}(\mathfrak{n})$, *where* $\mathfrak{n}$ *is the largest nilpotent ideal of* $\mathfrak{g}$.

proof). If $\mathfrak{g}$ is rigid, then $\mathfrak{g}$ is algebraic and splittable ([Ca] Proposition 4.1). Then $\mathfrak{g}$ is isomorphic to a subalgebra of $\mathfrak{u}(\mathfrak{n})$ ([Ma] Theorem 7), that is to say, there exists a non-zero subspace T_1 of T and $\mathfrak{g} \simeq \mathfrak{n} \rtimes T_1$. If $T_1 \neq T$, then we can choose a continuous family of subspaces $T_t \subset T$ $(0 \leq t \leq 1)$ such that $\mathfrak{n} \rtimes T_t$ $(0 \leq t < 1)$ is not isomorphic to $\mathfrak{g}$, because there are only finite number of subspaces $T' \subset T$ such that $\mathfrak{n} \rtimes T' \simeq \mathfrak{n} \rtimes T_1$ ([Ma] Theorem 7). Hence we get a non-trivial deformation of $\mathfrak{g}$. This is a contradiction, whence $T_1 = T$ and $\mathfrak{g} \simeq \mathfrak{u}(\mathfrak{n})$.

Remark. In Proposition 1.1 we can remove the word "non-nilpotent" because of Colloraire 4.4 (ii) of [Ca]. In [C-D] and [Be] rigid solvable Lie algebras were determined completely when their dimensions are not more than 8. There is a conjecture "No nilpotent Lie algebra is rigid".

§ 2 Reduction of the computation of $H^2(\mathfrak{u}, \mathfrak{u})$.

Our purpose is to find rigid solvable Lie algebras, and we compute 2-cohomology groups of $\mathfrak{u} = \mathfrak{u}(\mathfrak{n})$ for several types of $\mathfrak{n}$. Let $\mathfrak{u} = \mathfrak{n} \rtimes T$ be a universal Lie algebra for a given $\mathfrak{n}$. Since $\mathfrak{u}$ is the semi-direct product of $\mathfrak{n}$ and T, and the action of T on $\mathfrak{n}$ is semi-simple, we can use the following:

Lemma 2.1 (*Hochschild-Serre* [H-S]).

$$H^i(\mathfrak{u}, \mathfrak{u}) = \sum_{j+k=i} H^j(T, \mathbb{C}) \otimes H^k(\mathfrak{n}, \mathfrak{u})^T \qquad (i \geq 0),$$

where $H^i(\mathfrak{u}, \mathfrak{u})$ *and* $H^k(\mathfrak{n}, \mathfrak{u})^T$ *are the cohomology groups with respect to the adjoint representations, and* $H^j(T, \mathbb{C})$ *are the ones with respect to the trivial representations.*

Remark. By this lemma, as necessary conditions for $H^2(\mathfrak{u}, \mathfrak{u}) = 0$, we get the followings :

$$H^1(\mathfrak{u}, \mathfrak{u}) = H^0(\mathfrak{u}, \mathfrak{u}) = 0 \text{ when } \dim T \geq 2, \text{ and}$$
$$\dim H^1(\mathfrak{u}, \mathfrak{u}) = \dim H^0(\mathfrak{u}, \mathfrak{u}) \text{ when } \dim T = 1,$$

because $H^j(T, \mathbb{C}) \simeq \overset{j}{\Lambda}(T) = 0$ if and only if $j > \dim T$. Here the latter condition is equivalent to the condition $\dim \mathrm{Der}(\mathfrak{g}) = \dim \mathfrak{g}$ since $H^1(\mathfrak{g}, \mathfrak{g}) = \mathrm{Der}(\mathfrak{g}) / \mathrm{ad}(\mathfrak{g})$, $H^0(\mathfrak{g}, \mathfrak{g}) = \mathfrak{z}$, and $\mathfrak{g} / \mathfrak{z} \simeq \mathrm{ad}(\mathfrak{g})$. It is remarkable that the necessary conditions for the rigidity in [Ca] Proposition 4.1(i) are no more than the necessary conditons for $H^2(\mathfrak{u}, \mathfrak{u}) = 0$.

In order to compute $H^k(\mathfrak{n}, \mathfrak{u})^T$, we use the weight space decomposition of $\mathfrak{n}$ with respect to T : $\mathfrak{n} = \bigoplus_{\lambda \in W} \mathfrak{n}_\lambda$, $(W \subset T^*)$. Then we have the following:

Lemma 2.2. *For a positive integer* i, *assume* $\lambda_1 + \lambda_2 + \cdots + \lambda_i \neq 0$ (*not necessarily distinct* $\lambda_1, \cdots, \lambda_i \in W$), *then we get* $C^i(\mathfrak{n}, \mathfrak{u})^T = C^i(\mathfrak{n}, \mathfrak{n})^T$. *For* $i \geq 2$, *we have* $H^i(\mathfrak{n}, \mathfrak{u})^T = H^i(\mathfrak{n}, \mathfrak{n})^T$, *and for* $i = 1$ *we have the followings*:
$H^1(\mathfrak{n}, \mathfrak{u})^T = \{ D \in \mathrm{Der}(\mathfrak{n}) \mid D$ *is nilpotent and* $D\, \mathfrak{n}_\lambda \subset \mathfrak{n}_\lambda$ $(\lambda \in W)\}$ *and*
$H^0(\mathfrak{n}, \mathfrak{u})^T = 0$.

proof). Let us write $c \in C^i(\mathfrak{n}, \mathfrak{u})^T$ as $c = \varphi + \psi$ ($\varphi \in C^i(\mathfrak{n}, \mathfrak{n})$, $\psi \in C^i(\mathfrak{n}, T)$), we get $\psi = 0$ by writing down the conditions

$$\{Y \cdot (\varphi + \psi)\}(X_{\lambda_1}, \cdots, X_{\lambda_i}) = 0 \quad (Y \in T,\ X_{\lambda_1} \in \mathfrak{n}_{\lambda_1}, \cdots, X_{\lambda_i} \in \mathfrak{n}_{\lambda_i}),$$

and using the assumption. Therefore $C^i(\mathfrak{n}, \mathfrak{u})^T = C^i(\mathfrak{n}, \mathfrak{n})^T$, and for $i \geq 2$ we have $H^i(\mathfrak{n}, \mathfrak{u})^T = Z^i(\mathfrak{n}, \mathfrak{n})^T / dC^{i-1}(\mathfrak{n}, \mathfrak{n})^T = H^i(\mathfrak{n}, \mathfrak{n})^T$.

When $i = 1$, $\mathfrak{n}^T = 0$ since $0 \notin W$. Then $C^0(\mathfrak{n}, \mathfrak{u})^T = \mathfrak{u}^T = \mathfrak{n}^T \rtimes T = T$, hence $H^1(\mathfrak{n}, \mathfrak{u})^T = Z^1(\mathfrak{n}, \mathfrak{n})^T / dT$. Here $Z^1(\mathfrak{n}, \mathfrak{n})^T = \{ D \in \mathrm{Der}(\mathfrak{n}) \mid DY = YD \ (Y \in T)\}$ and $dT = T$. Since $\mathrm{Der}(\mathfrak{n})$ is algebraic, for $D \in Z^1(\mathfrak{n}, \mathfrak{n})^T$ D_s and $D_n \in Z^1(\mathfrak{n}, \mathfrak{n})^T$ where $D = D_s + D_n$ is the Jordan decomposition of D. Here $D_s \in T$ by the definition of T. Hence we get

$$H^1(\mathfrak{n}, \mathfrak{u})^T = \{ D \in \mathrm{Der}(\mathfrak{n}) \mid D \text{ is nilpotent and } DY = YD \ (Y \in T)\}$$
$$= \{ D \in \mathrm{Der}(\mathfrak{n}) \mid D \text{ is nilpotent and } D\,\mathfrak{n}_\lambda \subset \mathfrak{n}_\lambda \ (\lambda \in W)\}.$$

Next $H^0(\mathfrak{n}, \mathfrak{u})^T = \{ c \in C^0(\mathfrak{n}, \mathfrak{u})^T = \mathfrak{u}^T \mid dc = 0 \}$. Here $dc = c$ because $c \in \mathfrak{u}^T = \mathfrak{n}^T \rtimes T = T$, and we get $H^0(\mathfrak{n}, \mathfrak{u})^T = 0$.

Remark. About the vanishing of $H^i(\mathfrak{u}, \mathfrak{u})$ $(i = 0, 1)$, there is a similar result in Proposition 4.1 of [L-L] which is obtained by a different method.

Corollary 2.3. *If* $\lambda \neq 0$, $\lambda + \mu \neq 0$, *and* $\dim \mathfrak{n}_\lambda = 1$ $(\lambda, \mu \in W)$, *then* $H^i(\mathfrak{u}, \mathfrak{u}) = 0$ $(i = 0, 1)$ *and* $H^2(\mathfrak{u}, \mathfrak{u}) = H^2(\mathfrak{n}, \mathfrak{n})^T$.

proof). By the assumptions $H^i(\mathfrak{n}, \mathfrak{u})^T = 0$ $(i = 0, 1)$, then we get $H^i(\mathfrak{u}, \mathfrak{u}) = 0$ $(i = 0, 1)$ and $H^2(\mathfrak{u}, \mathfrak{u}) = H^2(\mathfrak{n}, \mathfrak{n})^T$ using Lemma 2.1.

Definition. Let $\bar{\mathfrak{n}}$ be a nilpotent Lie algebra over $\mathbb{C}$. We call $\bar{\mathfrak{n}}$ is a *Iwasawa subalgebra* when there exists a semi-simple Lie algebra $\mathfrak{s}$ over $\mathbb{R}$ and its Iwasawa decomposition; $\mathfrak{s} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ such that $\bar{\mathfrak{n}} = \mathfrak{n}^{\mathbb{C}}$.

Proposition 2.4. *Let* $\mathfrak{n}$ *be an Iwasawa subalgebra, then* $\lambda \neq 0$, $\lambda + \mu \neq 0$, $\dim \mathfrak{n}_\lambda = 1$ $(\lambda, \mu \in W)$, *therefore* $H^i(\mathfrak{u}, \mathfrak{u}) = 0$ $(i = 0, 1)$, *and* $H^2(\mathfrak{u}, \mathfrak{u}) = H^2(\mathfrak{n}, \mathfrak{n})^T$.

proof). There exists an Iwasawa decomposition $\mathfrak{s} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ (over $\mathbb{C}$), and let $\mathfrak{h}$ be a Cartan subalgebra containing $\mathfrak{a}$. Then $(\mathrm{ad}\, \mathfrak{h})|_{\mathfrak{n}} \subset \mathrm{Der}(\mathfrak{n})$ and we can choose T such that $(\mathrm{ad}\, \mathfrak{h})|_{\mathfrak{n}} \subset T$ since $\mathfrak{n}$ are direct sum of some positive root spaces of $\mathfrak{n}$ with respect to $\mathrm{ad}\, \mathfrak{h}$ (see e.g. [He]). Therefore each root space $\mathfrak{s}_\alpha$ in $\mathfrak{n}$ $(\alpha \in \Delta_+)$ is decomposed into some weight spaces $\mathfrak{n}_\lambda$ $(\lambda \in W)$. Since $\dim \mathfrak{s}_\alpha = 1$, for any $\lambda \in W$ there exists $\alpha \in \Delta_+$ such that $\mathfrak{n}_\lambda = \mathfrak{s}_\alpha$ and $\lambda|_{\mathrm{ad}\, \mathfrak{h}} = \alpha$.

Since $\alpha \neq 0$, $\alpha + \beta \neq 0$ $(\alpha, \beta \in \Delta_+)$, and $\dim \mathfrak{s}_\alpha = 1$ $(\alpha, \beta \in \Delta_+)$, Proposition 2.4 follows.

Remark 2.5. For an Iwasawa subalgebra $\mathfrak{n}$, we have $[\mathfrak{n}_\lambda, \mathfrak{n}_\mu] = \mathfrak{n}_{\lambda+\mu}$ $(\lambda, \mu, \lambda + \mu \in W)$ because there exist $\alpha, \beta \in \Delta_+$ such that $\mathfrak{n}_\lambda = \mathfrak{s}_\alpha$, $\mathfrak{n}_\mu = \mathfrak{s}_\beta$, and $[\mathfrak{s}_\alpha, \mathfrak{s}_\beta] \neq 0$.

To compute $H^2(\mathfrak{n}, \mathfrak{n})^T$ we use the following:

Lemma 2.6. *If* $\dim \mathfrak{n}_\lambda = 1$ $(\lambda \in W)$, *then we have*

(1). $\dim B^2(\mathfrak{n}, \mathfrak{n})^T = \dim \mathfrak{n} - \dim T$,

(2). $\dim C^2(\mathfrak{n}, \mathfrak{n})^T = \# \{ (\lambda, \mu) \in W \times W \mid \lambda + \mu \in W$ *and* $\lambda < \mu \}$.

Moreover if an Iwasawa subalgebra $\mathfrak{n}$ *is 2-step* (*i.e.* $[\mathfrak{n}, [\mathfrak{n}, \mathfrak{n}]] = 0$), *then we have*

(3). $C^2(\mathfrak{n}, \mathfrak{n})^T = Z^2(\mathfrak{n}, \mathfrak{n})^T$.

proof). To prove (1), we use the surjective homomorphism $d : C^1(\mathfrak{n}, \mathfrak{n})^T \longrightarrow B^2(\mathfrak{n}, \mathfrak{n})^T$. Since $C^1(\mathfrak{n}, \mathfrak{n})^T = \{ c \in \mathrm{End}\, \mathfrak{n} \mid c\, \mathfrak{n}_\lambda \subset \mathfrak{n}_\lambda \ (\lambda \in W)\}$, and $\dim \mathfrak{n}_\lambda = 1$, we get $\dim C^1(\mathfrak{n}, \mathfrak{n})^T = \dim \mathfrak{n}$. And $\ker d = T$, therefore we have the equation (1). By the definition we have

$C^2(\mathfrak{n}, \mathfrak{n})^T = \{ c \in C^2(\mathfrak{n}, \mathfrak{n}) \mid c(\mathfrak{n}_\lambda, \mathfrak{n}_\mu) \subset \mathfrak{n}_{\lambda+\mu} \ (\lambda, \mu, \lambda + \mu \in W)\}$,

and we get the equation (2) because $\dim \mathfrak{n}_\lambda = 1$ $(\lambda \in W)$.

Next for $c \in C^2(\mathfrak{n}, \mathfrak{n})^T$, using Remark 2.5 we can prove

$$dc(\mathfrak{n}_\lambda, \mathfrak{n}_\mu, \mathfrak{n}_\nu) \subset [\mathfrak{n}_\lambda, [\mathfrak{n}_\mu, \mathfrak{n}_\nu]] + [\mathfrak{n}_\nu, [\mathfrak{n}_\lambda, \mathfrak{n}_\mu]] + [\mathfrak{n}_\mu, [\mathfrak{n}_\nu, \mathfrak{n}_\lambda]].$$

Therefore $dc = 0$ if $\mathfrak{n}$ is 2-step, hence we get (3).

§ 3 The rigidity of $\mathfrak{u}$ when $\mathfrak{n}$ are some Iwasawa subalgebras.

We compute $H^2(\mathfrak{u}, \mathfrak{u})$ for the 2-step Iwasawa subalgebras that appear in the simple Lie algebras of real rank 1. Those simple Lie algebras are $\mathfrak{so}(n + 1, 1)$, $\mathfrak{su}(n + 1, 1)$, $\mathfrak{sp}(n + 1, 1)$, and $\mathfrak{f}_{4,(-20)}$. Then those nilpotent parts are known to be $(\mathbb{K}^n \oplus \mathrm{Im}\, \mathbb{K})^{\mathbb{C}}$ $(\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$, $n = 1$ and $\mathfrak{C}$, and $\mathrm{Im}\, \mathbb{K}$ are imaginary parts of $\mathbb{K})$, where the brackets of $\mathbb{K}^n \oplus \mathrm{Im}\, \mathbb{K}$ are given by

$$[(\alpha, \beta), (\alpha', \beta')] = 2\, \mathrm{Im} \left(\sum_{i=1}^{n} \overline{\alpha}_i \alpha'_i \right) \quad (\alpha, \alpha' \in \mathbb{K}^n, \beta, \beta' \in \mathrm{Im}\, \mathbb{K})$$

(see e.g.[Mo]). For those 2-step nilpotent Lie algebras, we compute T, W, and $H^2(u, u)$ using Lemma 2.6.

Propositition 3.1. *In the derivation algebras* $\mathrm{Der}(\mathbb{K}^n \oplus \mathrm{Im}\,\mathbb{K})^{\mathbb{C}}$, *we can choose* T *as follows*:

(i) $\mathbb{K} = \mathbb{R}$, $T = \left\{\begin{pmatrix} d_1 & & & 0 \\ & d_2 & & \\ & & \ddots & \\ 0 & & & d_n \end{pmatrix}\right\}$,

(ii) $\mathbb{K} = \mathbb{C}$, $T = \left\{\left(\begin{array}{c:c:c} D + sI & 0 & \\ \hdashline 0 & -D + sI & 0 \\ \hdashline \multicolumn{2}{c:}{0} & 2s \end{array}\right) \;\middle|\; D = \begin{pmatrix} d_1 & & & 0 \\ & d_2 & & \\ & & \ddots & \\ 0 & & & d_n \end{pmatrix}\right\}$,

(iii) $\mathbb{K} = \mathbb{H}$, $T = \left\{\left(\begin{array}{cc:cc:ccc} \multicolumn{2}{c:}{sI} & -D + tI & 0 & & & \\ & & 0 & D + tI & & 0 & \\ \hdashline D - tI & 0 & \multicolumn{2}{c:}{sI} & & & \\ 0 & -D - tI & & & & & \\ \hdashline & & & & 2s & 0 & 2t \\ \multicolumn{4}{c:}{0} & 0 & 2s & 0 \\ & & & & -2t & 0 & 2s \end{array}\right) D\text{:diag.}\right\}$,

(iv) $\mathbb{K} = \mathbb{C}$ $(n = 1)$,

$$T = \left\{\left(\begin{array}{cccc:cccc} sI+R(\theta) & & & & & & & \\ & sI+R(\theta_1) & & & & & 0 & \\ & & sI+R(\theta_2) & & & & & \\ & & & sI+R(\theta_3) & & & & \\ \hdashline & & & & 2s & & & \\ & & 0 & & & 2sI-R(\theta_2+\theta_3) & & \\ & & & & & & 2sI-R(\theta_1+\theta_3) & \\ & & & & & & & 2sI-R(\theta_1+\theta_2) \end{array}\right)\right\},$$

where $\theta = \theta_1 + \theta_2 + \theta_3$, *and* $R(\theta) = \begin{pmatrix} 0 & \theta \\ -\theta & 0 \end{pmatrix}$.

proof). (i) being trivial, in order to prove (ii) we see the fact

$$\mathrm{Der}(\mathbb{C}^n \oplus \mathrm{Im}\,\mathbb{C})^{\mathbb{C}} = \left\{\left(\begin{array}{c:c} sI + A & 0 \\ \hdashline * & 2s \end{array}\right) \;\middle|\; A \in sp(n, \mathbb{R}),\ s \in \mathbb{R}\right\}^{\mathbb{C}}$$

$$\simeq \{\, \mathbb{R}^{2n} \rtimes (\mathbb{R} \oplus sp(n, \mathbb{R}))\}^{\mathbb{C}}.$$

Using a standard Cartan subalgebra of $sp(n, \mathbb{R})$, we have the above expression of T. Next let us prove (iii). We use the fact

$$\mathrm{Der}(\mathbb{H}^n \oplus \mathrm{Im}\,\mathbb{H})^{\mathbb{C}} = \left\{\left(\begin{array}{c:ccc} X + Y & & 0 & \\ \hdashline & 2a & 2b & 2c \\ * & -2b & 2a & -2d \\ & -2c & 2d & 2a \end{array}\right) \;\middle|\; X = \begin{pmatrix} A & -B & -C & -D \\ B & A & -D & C \\ C & D & A & -B \\ D & -C & B & A \end{pmatrix},\right.$$

$$\left. Y = \begin{pmatrix} aI & dI & cI & -bI \\ -dI & aI & bI & cI \\ -cI & -bI & aI & -dI \\ bI & -cI & dI & aI \end{pmatrix},\ A \in \mathrm{Skew}(n),\ B, C, D \in \mathrm{Symm}(n),\ a, b, c, d \in \mathbb{R}\right\}^{\mathbb{C}},$$

$$\simeq \{(\mathbb{R}^{4n} \otimes \mathbb{R}^3) \rtimes (\mathbb{H} \oplus \mathfrak{sp}(n))\}^{\mathbb{C}}.$$

The dimension of maximal abelian subalgebra of $\mathbb{H} \oplus \mathfrak{sp}(n) \simeq \mathbb{R} \oplus \mathfrak{sp}(1) \oplus \mathfrak{sp}(n)$ is $n + 2$. Here the above expression of T is abelian, consisting of semi-simple elements, and $n + 2$ dimension too. Hence we get (iii). In order to prove (iv), we use the fact

$$\mathrm{Der}(\mathfrak{C} \oplus \mathrm{Im}\ \mathfrak{C})^{\mathbb{C}} = \left\{ \left(\begin{array}{c:c} sI + A & 0 \\ \hdashline * & 2sI + B \end{array} \right) \middle| \; A = \left(a_{ij} \right)_{0 \leq i, j \leq 7} \in \mathfrak{so}(8), \right.$$

$$s \in \mathbb{R},\ B = \left(b_{ij} \right)_{1 \leq i, j \leq 7} \in \mathfrak{so}(7), \text{ such that } \begin{array}{l} 2a_{03} = b_{21}+b_{65}+b_{74}, \\ 2a_{47} = b_{21}+b_{65}-b_{74}, \\ 2a_{56} = b_{21}-b_{65}+b_{74}, \\ 2a_{12} = -b_{21}+b_{65}+b_{74}, \end{array}$$

$$\begin{array}{lll} -2a_{57} = b_{31}+b_{64}+b_{75}, & 2a_{01} = b_{32}+b_{54}+b_{76}, & -2a_{14} = b_{41}+b_{63}+b_{72}, \\ -2a_{02} = b_{31}+b_{64}-b_{75}, & 2a_{67} = b_{32}+b_{54}-b_{76}, & -2a_{36} = b_{41}+b_{63}-b_{72}, \\ -2a_{13} = b_{31}-b_{64}+b_{75}, & 2a_{45} = b_{32}-b_{54}+b_{76}, & -2a_{27} = b_{41}-b_{63}+b_{72}, \\ -2a_{46} = -b_{31}+b_{64}+b_{75}, & 2a_{23} = -b_{32}+b_{54}+b_{76}, & -2a_{05} = -b_{41}+b_{63}+b_{72}, \end{array}$$

$$\left. \begin{array}{lll} -2a_{35} = b_{42}+b_{53}+b_{71}, & 2a_{07} = b_{43}+b_{52}+b_{61}, & -2a_{26} = b_{51}+b_{62}+b_{73} \\ -2a_{24} = b_{42}+b_{53}-b_{71}, & 2a_{16} = b_{43}+b_{52}-b_{61}, & -2a_{15} = b_{51}+b_{62}-b_{73} \\ -2a_{06} = b_{42}-b_{53}+b_{71}, & 2a_{25} = b_{43}-b_{52}+b_{61}, & -2a_{04} = b_{51}-b_{62}+b_{73} \\ -2a_{17} = -b_{42}+b_{53}+b_{71}, & 2a_{34} = -b_{43}+b_{52}+b_{61}, & -2a_{37} = -b_{51}+b_{62}+b_{73} \end{array} \right\}^{\mathbb{C}},$$

$$\simeq \{(\mathbb{R}^8 \otimes \mathbb{R}^7) \rtimes (\mathbb{R} \oplus \mathfrak{so}(7))\}^{\mathbb{C}}.$$

Using a standard Cartan subalgebra of $\mathfrak{so}(7)$, we get T in (iv).

Remark. In the above expression of $\mathrm{Der}(\mathfrak{C} \oplus \mathrm{Im}\ \mathfrak{C})$, we can prove that the map $\mathfrak{so}(7) \ni B \longrightarrow A \in \mathfrak{so}(8)$ is the spin representation of $\mathfrak{so}(7)$.

Propositon. 3.2. *When* $\mathfrak{n} = (\mathbb{K}^n \oplus \mathrm{Im}\ \mathbb{K})^{\mathbb{C}}$, $H^2(\mathfrak{u}, \mathfrak{u})$ *are given as follows*:

	(i) $\mathbb{K} = \mathbb{R}$	(ii) $\mathbb{K} = \mathbb{C}$	(iii) $\mathbb{K} = \mathbb{H}$	(iv) $\mathbb{K} = \mathfrak{C}$
$H^2(\mathfrak{u}, \mathfrak{u})$	0	0	$\mathbb{C}^{n-1}$	$\mathbb{C}^5$

proof). First we compute W using T in Proposition 3.1.

(i) $W = \{ d_i \}_{1 \leq i \leq n}$.

(ii) $W = \{ s \pm d_i \ , \ 2s \}_{1 \leq i \leq n}$.

(iii) $W = \{ s \pm (d_i + t)\sqrt{-1},\ s \pm (d_i - t)\sqrt{-1},\ 2s,\ 2s \pm 2t\sqrt{-1} \}_{1 \leq i \leq n}$.

(iv) $W = \{ s \pm \theta\sqrt{-1},\ s \pm \theta_1\sqrt{-1},\ s \pm \theta_2\sqrt{-1},\ s \pm \theta_3\sqrt{-1},$
$2s,\ 2s \pm (\theta_2 + \theta_3)\sqrt{-1},\ 2s \pm (\theta_1 + \theta_3)\sqrt{-1},\ 2s \pm (\theta_1 + \theta_2)\sqrt{-1} \}$.

Next we compute $\dim Z^2(\mathfrak{n}, \mathfrak{n})^T$ using Lemma 2.6 (2) and (3).

(i) $\dim Z^2(\mathfrak{n}, \mathfrak{n})^T = 0$, because $\lambda + \mu \notin W$ $(\lambda, \mu \in W)$.

(ii) dim $Z^2(\mathfrak{n}, \mathfrak{n})^T = n$, because $2s = (s + d_i) + (s - d_i)$.
(iii) dim $Z^2(\mathfrak{n}, \mathfrak{n})^T = 4n$, because

$$2s = (s + (d_i \pm t)\sqrt{-1}) + (s - (d_i \pm t)\sqrt{-1}) ,$$

$$2s \pm 2t\sqrt{-1} = (s + (d_i \pm t)\sqrt{-1}) + (s - (d_i \mp t)\sqrt{-1}) .$$

(iv) dim $Z^2(\mathfrak{n}, \mathfrak{n})^T = 16$, because

$$2s = (s + \theta\sqrt{-1}) + (s - \theta\sqrt{-1}) = (s + \theta_i\sqrt{-1}) + (s - \theta_i\sqrt{-1}),$$

$$2s \pm (\theta_2 + \theta_3)\sqrt{-1} = (s \pm \theta_2\sqrt{-1}) + (s \pm \theta_3\sqrt{-1}) = (s \pm \theta\sqrt{-1}) + (s \mp \theta_1\sqrt{-1}),$$

$$2s \pm (\theta_1 + \theta_3)\sqrt{-1} = (s \pm \theta_1\sqrt{-1}) + (s \pm \theta_3\sqrt{-1}) = (s \pm \theta\sqrt{-1}) + (s \mp \theta_2\sqrt{-1}),$$

and

$$2s \pm (\theta_1 + \theta_2)\sqrt{-1} = (s \pm \theta_1\sqrt{-1}) + (s \pm \theta_2\sqrt{-1}) = (s \pm \theta\sqrt{-1}) + (s \mp \theta_3\sqrt{-1}).$$

Last we use the eqation :

$$\dim H^2(\mathfrak{n}, \mathfrak{n})^T = \dim Z^2(\mathfrak{n}, \mathfrak{n})^T - \dim \mathfrak{n} + \dim T .$$

Computing the right hand side, we get Proposition 3.2.

Remark 3.3. In the above proof we have used $C^2(\mathfrak{n}, \mathfrak{n})^T = Z^2(\mathfrak{n}, \mathfrak{n})^T$. This is not true for any Iwasawa nilpotent Lie algebra $\mathfrak{n}$ such that step $\mathfrak{n} \geq 3$. Then we must compute the rank of system of linear equations : $dc(X_\lambda , X_\mu , X_\nu) = 0$ ($X_\lambda \in \mathfrak{n}_\lambda$, $X_\mu \in \mathfrak{n}_\mu$, $X_\nu \in \mathfrak{n}_\nu$). We report the results $H^2(\mathfrak{u}, \mathfrak{u}) = 0$ for the nilpotent parts of $\mathfrak{so}(n + k, k)$ and $\mathfrak{su}(n + k, k)$, (for any $k \in \mathbb{N}$).

Since the condition $H^2(\mathfrak{u}, \mathfrak{u}) \neq 0$ does not mean the rigidity of $\mathfrak{u}$ ([Ri]), we need the following:

Proposition 3.4. $\mathfrak{u}(\mathfrak{n})$ *is not rigid when* $\mathfrak{n} = (\mathbb{H}^n \oplus \mathrm{Im}\,\mathbb{H})^{\mathbb{C}}$ $(n \geq 2)$ *or* $\mathfrak{n} = (\mathbb{C} \oplus \mathrm{Im}\,\mathbb{C})^{\mathbb{C}}$.

proof). We give the proof when $\mathfrak{n} = (\mathbb{H}^n \oplus \mathrm{Im}\,\mathbb{H})$ $(n \geq 2)$. Let us choose weight vectors of T ; $\{X_i, Y_i, Z_i, W_i, A, B, C\}_{1 \leq i \leq n}$ corresponding to the weights ; $\{s + (d_i + t)\sqrt{-1}, s - (d_i + t)\sqrt{-1}, s + (d_i - t)\sqrt{-1}, s - (d_i - t)\sqrt{-1}, 2s, 2s + 2t\sqrt{-1}, 2s - 2t\sqrt{-1}\}$. Let μ be the Lie bracket of $\mathfrak{u}$, and $\varphi \in C^2(\mathfrak{n}, \mathfrak{n})^T$ ($\subset C^2(\mathfrak{u}, \mathfrak{u})^T$) defined by

$$\begin{cases} \varphi(X_i, Y_i) = p_i A \\ \varphi(Z_i, W_i) = q_i A \qquad (p_i , q_i \in \mathbb{C}). \\ \varphi = 0 \quad (\text{other cases}) \end{cases}$$

We can check that $\mu + \varepsilon\varphi$ ($\varepsilon \in \mathbb{C}$) is also a Lie algebra, so $\mu + \varepsilon\varphi$ is a deformation of μ. Assume that μ is rigid, then the tangent vector $\varphi \in B^2(u, u)$ ([N-R]). Since φ is T-invariant $\varphi \in B^2(u, u)^T = d(C^1(u, u)^T)$, therefore there exists $f \in C^1(u, u)^T$ such that $\varphi = df$. As $f(n_\lambda) \subset n_\lambda$ ($\lambda \in W$), and $\varphi = 0$ on $n \times T$ and $T \times T$, $\varphi = d(f|_n)$ and

$$f|_n = \mathrm{diag}(x_i, y_i, z_i, w_i, a, b, c)$$

with respect to $\{X_i, Y_i, Z_i, W_i, A, B, C\}_{1 \le i \le n}$.

Since

$$\begin{cases} df(X_i, W_i) = 0 \\ df(Y_i, Z_i) = 0 \end{cases} \quad \text{and} \quad \begin{cases} df(X_i, Y_i) = p_i A \\ df(Z_i, W_i) = q_i A \end{cases},$$

we have

$$\begin{cases} x_i + w_i - b = 0 \\ y_i + z_i - c = 0 \end{cases} \quad \text{and} \quad \begin{cases} r_i(x_i + y_i - a) = p_i \\ s_i(z_i + w_i - a) = q_i \end{cases},$$

where r_i and s_i are non-zero number defined by $\begin{cases} \mu(X_i, Y_i) = r_i A \\ \mu(Z_i, W_i) = s_i A \end{cases}$.

Computing $x_i + y_i + z_i + w_i$, we get

$$2a + \frac{p_i}{r_i} + \frac{q_i}{s_i} = b + c \quad (1 \le i \le n).$$

As $n \ge 2$, this equation has no solution when we put $p_i = i\, r_i$ and $q_i = i\, s_i$. This is a contradiction, hence μ is not rigid.

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補足（$\mathfrak{U}$と$H^2(\mathfrak{U},\mathfrak{U})$のより効率的な計算法）

初めに

「論文」では、実単純Lie環$\mathfrak{G}$の巾零部$\mathfrak{N}$が$\mathbb{R}\mathrm{rank}\,\mathfrak{G}=1$の時、見易い形で知られている事を用いて$\mathfrak{U}$を定義通りに構成している。即ち、$\mathrm{Der}\,\mathfrak{N}^{\mathbb{C}}$を求め、その中からTを選び、$\mathfrak{U}=\mathfrak{N}^{\mathbb{C}}\rtimes T$を構成している。そして$\mathfrak{N}$が2stepである事を利用し$H^2(\mathfrak{U},\mathfrak{U})$を求めている。

この方法をこのまま一般の実単純Lie環$\mathfrak{G}$の巾零部$\mathfrak{N}$に適用するのは困難である。そこで次の3段階を踏んでより効率的に$\mathfrak{U}$と$H^2(\mathfrak{U},\mathfrak{U})$を計算する方法を示す。具体例として

例1　$\mathfrak{G}=\mathfrak{su}(n+k,k)$　の場合

例2　$\mathfrak{G}=\mathfrak{so}(n+k,k)$　〃

をあげ、Remark 3.3で触れた$H^2(\mathfrak{U},\mathfrak{U})=0$を示す。

これから示す方法では、佐武図形の性質（ルート系とCartan対合の性質）から$\mathrm{Der}\,\mathfrak{N}^{\mathbb{C}}$を経由せずに$\mathfrak{U}$が求めれるのだが、「論文」では$\mathfrak{N}$が、$\mathbb{K}^n\oplus\mathrm{Im}\,\mathbb{K}$（$\mathbb{K}=\mathbb{R},\mathbb{C},\mathbb{H},\mathfrak{C}\,(n=1)$）と同じような表示であるにもかかわらず、$\mathfrak{U}$の表示や$H^2(\mathfrak{U},\mathfrak{U})$がまるで異なる事がより具体的に調べられている。また「論文」中の$\mathfrak{N}$についての$\mathrm{Der}\,\mathfrak{N}$の表示（Prop. 3.1の証明中）には、C. Riehmの論文 " Explicit spin representations and Lie algebras of Heisenberg type. J. London

Math. Soc. (2) 29 (1984) 49-62." 等との関連も有るようなので、「論文」をこれから述べる方法で書き直す事はしなかった。

第一段階 佐武図形からのIwasawa部分環の構成

教科書[He]に従って$\mathfrak{n}^{\mathbb{C}}$をルート部分空間の直和で表す。

$\mathfrak{G}$：実単純Lie環

$\mathfrak{G} = \mathfrak{k} \oplus \mathfrak{m}$：Cartan分解（$\theta$をそのCartan対合とする）

$\mathfrak{a}$：$\mathfrak{m}$内の1つの極大可換部分環

$\mathfrak{h}$：$\mathfrak{a}$を含む$\mathfrak{G}$の　〃　（Cartan部分環）

Δ_+：$\mathfrak{h}^{\mathbb{C}}$についての$\mathfrak{G}^{\mathbb{C}}$の正ルート全体（ある順序について）

$P_+ = \{\alpha \in \Delta_+ \mid \alpha \circ \theta \neq \alpha \ (\Leftrightarrow \alpha|_{\mathfrak{a}} \neq 0)\}$

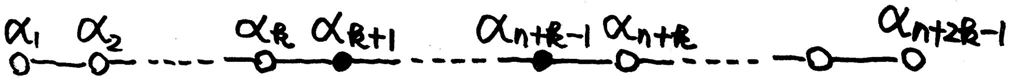
$P_- = \{$　〃　$= \alpha$ (　〃　$= 0)\}$

すると
$$\mathfrak{n}^{\mathbb{C}} = \bigoplus_{\alpha \in P_+} (\mathfrak{G}^{\mathbb{C}})_\alpha$$

佐武図形の黒丸単純ルートだけを使って表せるルートがP_-で、白丸単純ルートを1つ以上使って表せるルートがP_+になる。というのは $\alpha, \beta \in \Delta_+$ s.t $\alpha + \beta \in \Delta_+$ に対し
$$\alpha \text{ or } \beta \in P_+ \Leftrightarrow \alpha + \beta \in P_+$$
$$(\alpha \text{ and } \beta \in P_- \Leftrightarrow \alpha + \beta \in P_-)$$
が成立するからである。そして $\mathfrak{n}^{\mathbb{C}}/[\mathfrak{n}^{\mathbb{C}}, \mathfrak{n}^{\mathbb{C}}]$ の基底として、P_+の元を単純ルートの和として表した場合の白丸が1つ

であるルートのルートベクトルが選べ、$\mathfrak{n}^{\mathbb{C}}$のStep数は最高ルートを単純ルートの和として表した場合の白丸の個数で与えられる。

例1



$\mathfrak{g}^{\mathbb{C}} \simeq \mathfrak{sl}(n+2k, \mathbb{C})$, $\mathfrak{h}^{\mathbb{C}} \simeq \mathrm{diag}(\lambda_1, \cdots, \lambda_{n+2k})$で同一視すると

$\Delta_+ = \{\lambda_i - \lambda_j\}_{1 \leqq i < j \leqq n+2k}$ $\quad (\alpha_i = \lambda_i - \lambda_{i+1})$

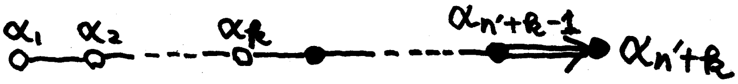
$P_- = \{ \quad " \quad \}_{k+1 \leqq i < j \leqq n+k-1}$

$\lambda_i - \lambda_j$のルートベクトルはE_{ij}に取れるから

$$\mathfrak{n}^{\mathbb{C}} \simeq \left\{ \left(\begin{array}{c|c|c} \begin{matrix} 0 & * \\ 0 & 0 \end{matrix} & * & * \\ \hline O & O & * \\ \hline O & O & \begin{matrix} 0 & * \\ 0 & 0 \end{matrix} \end{array} \right) \right\}$$
$$\leftarrow k \rightarrow \leftarrow n \rightarrow \leftarrow k \rightarrow$$

例2

① $n = 2n' + 1$の時



$$\mathfrak{g}^{\mathbb{C}} \simeq \mathfrak{so}(n+2k, \mathbb{C}) \simeq \left\{ \left(\begin{array}{c|c|c} 0 & a & b \\ \hline -{}^t b & X & Y \\ \hline -{}^t a & Z & -{}^t X \end{array} \right) \right\} \begin{array}{l} a, b \in \mathbb{C}^{n'+k} \\ {}^tY = -Y, \ {}^tZ = -Z \end{array}$$
$$\leftarrow n'+k \rightarrow \leftarrow \ " \ \rightarrow$$

$\mathfrak{h}^{\mathbb{C}} \simeq \mathrm{diag}(0, \lambda_1, \cdots \lambda_{n'+k}, -\lambda_1, \cdots, -\lambda_{n'+k})$

$\Delta_+ = \{\lambda_i \pm \lambda_j, \lambda_i\}_{1 \leqq i < j \leqq n'+k,\ 1 \leqq i \leqq n'+k}$

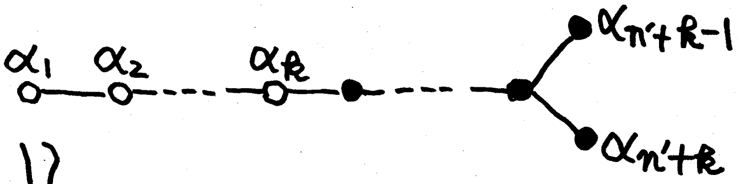
$\alpha_i = \lambda_i - \lambda_{i+1} \ (1 \leqq i \leqq n'+k-1)$, $\alpha_{n'+k} = \lambda_{n'+k}$

$P_+ = \{\lambda_i \pm \lambda_j,\ \lambda_i\}_{1 \leqq i \leqq k}$

$\overset{\leftarrow k \rightarrow \leftarrow n' \rightarrow}{\left\{\begin{pmatrix} \begin{smallmatrix} a & * \\ & 0 \end{smallmatrix} & * \\ 0 & 0 \end{pmatrix}\right\}}$

$$\mathfrak{n}^{\mathbb{C}} \simeq \left\{\begin{pmatrix} 0 & 0 & b \\ -{}^t b & X & Y - {}^t Y \\ 0 & 0 & -{}^t X \end{pmatrix}\right\} \quad b_{k+1} = \cdots = b_{n'+k} = 0,\ X, Y \in \mathfrak{n}_{k,n'}$$

② $n = 2n'$ の時



$$\mathfrak{g}^{\mathbb{C}} \simeq \left\{\begin{pmatrix} X & Y \\ Z & -{}^t X \end{pmatrix}\right\}_{{}^t Y = -Y,\ {}^t Z = -Z}$$

$\Delta_+ = \{\lambda_i \pm \lambda_j\}_{1 \leqq i < j \leqq n'+k}$

$\alpha_i = \lambda_i - \lambda_{i+1}\ (1 \leqq i \leqq n'+k-1),\ \alpha_{n'+k} = \lambda_{n'+k-1} + \lambda_{n'+k}$

$P_+ = \{\lambda_i \pm \lambda_j\}_{1 \leqq i \leqq k}$

$$\mathfrak{n}^{\mathbb{C}} \simeq \left\{\begin{pmatrix} X & Y - {}^t Y \\ 0 & -{}^t X \end{pmatrix}\right\} \quad X, Y \in \mathfrak{n}_{k,n'}$$

注意　例12の計算には「Lie群(II)」(岩堀長慶 岩波書店)の中の古典単純Lie環の表示を使った。

第2段階　$\mathrm{ad}\,\mathfrak{h}$を利用する$T$の求め方

Prop.2.4で触れたように、$T \supset \mathrm{ad}\,\mathfrak{h}^{\mathbb{C}}|_{\mathfrak{n}^{\mathbb{C}}}$に選べる。すると$T$と$\mathrm{ad}\,\mathfrak{h}^{\mathbb{C}}|_{\mathfrak{n}^{\mathbb{C}}}$は可換だから、$\mathfrak{n}^{\mathbb{C}}$を構成する各ルート空間($\mathfrak{g}^{\mathbb{C}}_\alpha$: 1次元)を保つ。

$\therefore\ D \in T \Longrightarrow {}^{\exists 1} d_\alpha \in \mathbb{C}$ s.t. $D|_{\mathfrak{g}^{\mathbb{C}}_\alpha} = d_\alpha 1_{\mathfrak{g}^{\mathbb{C}}_\alpha}$ $(\alpha \in P_+)$

ここで、$\{d_\alpha\}_{\alpha \in P_+}$は$D \in \mathrm{Der}\,\mathfrak{n}^{\mathbb{C}}$より連立方程式

$$d_\alpha + d_\beta = d_{\alpha+\beta} \quad (\forall \alpha, \beta, \alpha+\beta \in P_+) \text{ ------(*)}$$

を満たす。$\mathfrak{n}^C$は巾零だから$D\in T$は$\mathfrak{n}^C/[\mathfrak{n}^C,\mathfrak{n}^C]$の上だけで決まる。従って(*)は{$d_\alpha$ | $\alpha\in P_+$は白丸を1つだけ使うルート}を使って表せる。その連立方程式を(*)'とすると

$$\dim T = \dim\mathfrak{n} - \mathrm{rank}(*) = \dim \mathfrak{n}/[\mathfrak{n},\mathfrak{n}] - \mathrm{rank}(*)'$$

他方 $\mathrm{ad}\,\mathfrak{h}^C|_{\mathfrak{n}^C}$の次元については次が成立

<u>命題</u> $\dim \mathrm{ad}\,\mathfrak{h}^C|_{\mathfrak{n}^C} = \dim\mathfrak{h}$　(∴ $\mathrm{rank}\,\mathfrak{g} \leqq \dim T$)

<u>証明</u> $\mathfrak{h}\ni H \longrightarrow \mathrm{ad}\,H|_{\mathfrak{n}}\in \mathrm{Der}\,\mathfrak{n}$ が単射、を言う。

$[H,\mathfrak{n}]=0 \Rightarrow [H, \bigoplus_{\alpha\in P_+}(\mathfrak{g}^C)_\alpha]=0$

P_+には$\dim\mathfrak{a}$個の一次独立なルートがあるから

$\alpha(H)=0$ ($\alpha\in P_+$) より $H\in\mathfrak{h}\cap\mathfrak{k}$

∴ $[H,\mathfrak{n}]=[H,\theta\mathfrak{n}]=0$

∴ $[H,\mathfrak{m}]=0$ (∵ $\mathfrak{n}\oplus\theta\mathfrak{n}\supset\mathfrak{m}$)

$\mathfrak{k}$の中には$\mathfrak{g}$のidealは0しかないので $H=0$ //

これらによって殆どの場合 $T=\mathrm{ad}\,\mathfrak{h}^C|_{\mathfrak{n}^C}$が示せる。

例1

$D\in T$ なら $\mathfrak{g}^C_{\lambda_i-\lambda_j}$の上で定数(=$d_{ij}$とおく)倍, Dは

$$\begin{cases} d_{i\,i+1} & (1\leqq i\leqq k-1 \text{ or } n+k+1\leqq i\leqq n+2k-1) \\ d_{ki},\ d_{i\,n+k+1} & (k+1\leqq i\leqq n+k) \end{cases}$$

で決まるが

$$d_{ki} + d_{i\,n+k+1} = d_{k\,n+k+1} \quad (k+1 \leqq i \leqq n+k)$$

を満たす。よって $\mathrm{rank}(※)' \geqq n-1$

$$\therefore \dim \mathrm{ad}\,\mathfrak{g}^{\mathbb{C}}|_{\mathfrak{n}^{\mathbb{C}}} \leqq \dim T \leqq 2n+2(k-1)-(n-1)$$

（共に $n+2k-1$）

$$\therefore \mathrm{ad}\,\mathfrak{g}^{\mathbb{C}}|_{\mathfrak{n}^{\mathbb{C}}} = T$$

例2.

$$D \in T \Rightarrow \begin{cases} \mathfrak{G}^{\mathbb{C}}{}_{\lambda_i-\lambda_j} \text{の上で } d_{ij} \text{倍},\ \mathfrak{G}^{\mathbb{C}}{}_{\lambda_i+\lambda_j} \text{の上で } d'_{ij} \text{倍} \\ \mathfrak{G}^{\mathbb{C}}{}_{\lambda_i} \text{の上で } d_i \text{倍}\ (n=2n'+1 \text{の時のみ}) \end{cases}$$

とおける。

① $n=2n'+1$ の時　Dは

$$\begin{cases} d_{i\,i+1}\ (1 \leqq i \leqq k-1),\ d_{kj}\ (k+1 \leqq j \leqq n'+k) \\ d'_{kj}\ (k+1 \leqq j \leqq n'+k),\ d_k \end{cases}$$

で決まるが

$k=1$ の時 ----- $\mathfrak{n}^{\mathbb{C}}$ 可換

$$\therefore \mathrm{rank}(※)' = 0 \qquad \mathrm{ad}\,\mathfrak{g}^{\mathbb{C}}|_{\mathfrak{n}^{\mathbb{C}}} \subsetneqq T$$

（左辺は $n'+1$ 次元、T は $2n'+1$ 次元）

$k \geqq 2$ の時 ----- $d'_{k-1\,k}$ の表し方より

$$d_{k-1\,j} + d'_{kj} = (d_{k-1\,k} + d_k) + d_k \quad (k+1 \leqq j \leqq n'+k)$$

$$\therefore \mathrm{rank}(※)' \geqq n'$$

$$\therefore n'+k \leqq \dim T \leqq 2n'+k-n'$$

$$\therefore \mathrm{ad}\,\mathfrak{g}^{\mathbb{C}}|_{\mathfrak{n}^{\mathbb{C}}} = T$$

② $n=2n'$ の時　Dは

$d_{i\,i+1}$ $(1 \leqq i \leqq k-1)$, d_{kj}, d'_{kj} $(k+1 \leqq j \leqq n'+k)$

で決まるが①の場合と同様に

$k=1$ の時　　$\mathrm{ad}\,\mathfrak{h}^{\mathbb{C}}|_{\mathfrak{n}^{\mathbb{C}}} \subsetneqq T$

$k \geqq 2$ の時

$$d_{k-1\,j} + d'_{kj} = d'_{k-1\,k} \quad (k+1 \leqq j \leqq n'+k)$$

$\therefore\ \mathrm{rank}\,(※)' \geqq n'-1$

$\therefore\ n'+k \leqq \dim T \leqq 2n'+k-1-(n'-1)$

$\therefore\ \mathrm{ad}\,\mathfrak{h}^{\mathbb{C}}|_{\mathfrak{n}^{\mathbb{C}}} = T$

注意　古典型の実単純Lie環では、$\mathfrak{so}(n+1,1)$以外では $\mathrm{ad}\,\mathfrak{h}^{\mathbb{C}}|_{\mathfrak{n}^{\mathbb{C}}} = T$ に取れる事が上の方法で証明できる。例外型でも可能と思われる。

才3段階　$H^2(\mathfrak{n},\mathfrak{n}) = H^2(\mathfrak{n},\mathfrak{n})^T$ の計算

$\mathfrak{n}$のT不変な部分Lie環$\mathfrak{a}$で、$\mathrm{Der}\,\mathfrak{a}$の同時対角化可能極大集合が同じTに取れるものを考える。するとT不変コホモロジーの定義とLemma 2.6(1)より、

$$\dim H^2(\mathfrak{n})^T = \dim H^2(\mathfrak{a})^T + \dim C^2(\mathfrak{n})^T/C^2(\mathfrak{a})^T - \dim \mathfrak{n}/\mathfrak{a}$$

（$H^2(\mathfrak{n})^T$：$H^2(\mathfrak{n},\mathfrak{n})^T$の略）　$-\mathrm{rank}\{\mathfrak{a}$のに含まれぬ$\mathfrak{n}$のコサイクル条件$\}$

が成り立つ。$\mathfrak{a}$をうまく取り帰納的に$H^2(\mathfrak{n})^T$を計算する。

例1　このままでも示せるが、次のように一般化して示す。
　$n \in \mathbb{N}$ と $k, l \in \mathbb{Z}$ s.t. $k+l \geqq 1$ に対し

$$\mathfrak{n}_{k,n,l}=\left\{\begin{pmatrix} \begin{smallmatrix}0 & * \\ & 0\end{smallmatrix} & * & * \\ 0 & 0 & * \\ 0 & 0 & \begin{smallmatrix}0 & * \\ & 0\end{smallmatrix}\end{pmatrix}\right\} \begin{matrix} \updownarrow k \\ \updownarrow n \\ \updownarrow l \end{matrix}$$ とおく。

対角行列のadjoint表現が共通のTに取れ、Corollary 2.3の仮定が満たされ $H^2(\mathfrak{u})=H^2(\mathfrak{n})^T$ である。

$\mathfrak{n}_{1,n,0}$ は可換なので $C^2(\mathfrak{n})^T=0$ $\therefore H^2(\mathfrak{n})^T=0$

$\mathfrak{n}_{k,n,0}$ $(k\geqq 2)$ では $\mathfrak{a}=\mathfrak{n}_{k-1,n+1,0}$ に取る。

Lemma 2.6(2)より

$$\dim C^2(\mathfrak{n})^T/C^2(\mathfrak{a})^T=\#\{(E_{ik},E_{kj})\}_{\substack{1\leqq i\leqq k-1\\ k+1\leqq j\leqq n+k}}$$
$$=(k-1)n$$

そして $\mathfrak{a}$ のに含まれぬ $\mathfrak{n}$ のコサイクル条件のrankは

$$\geqq \#\{(E_{i\,k-1},E_{k-1\,k},E_{kj})\}_{\substack{1\leqq i\leqq k-2\\ k+1\leqq j\leqq n+k}}$$
$$\geqq (k-2)n$$

$$\therefore \dim H^2(\mathfrak{n})^T\leqq 0+(k-1)n-n-(k-2)n \quad (=0)$$
$$\therefore H^2(\mathfrak{n})^T=0$$

$\mathfrak{n}_{k,n,l}$ $(l\geqq 1)$ では $\mathfrak{a}=\mathfrak{n}_{k,n+1,l-1}$ に取る。同様に

$$\dim C^2(\mathfrak{n})^T/C^2(\mathfrak{a})^T=\#\{(E_{ij},E_{j\,n+k+1})\}_{\substack{1\leqq i\leqq k\\ k+1\leqq j\leqq n+k}}$$
$$+\#\{(E_{j\,n+k+1},E_{n+k+1\,i})\}_{\substack{k+1\leqq j\leqq n+k\\ n+k+2\leqq i\leqq n+k+l}}$$
$$=(k+l-1)n$$

$$(\mathfrak{a}\text{のに含ま}\cdots\text{rank})\geqq\#\{(E_{ik},E_{kj},E_{j\,n+k+1})\}_{\substack{1\leqq i\leqq k-1\\ k+1\leqq j\leqq n+k}}$$
$$+\#\{(E_{kj},E_{j\,n+k+1},E_{n+k+1\,i})\}_{\substack{k+1\leqq j\leqq n+k\\ n+k+2\leqq i\leqq n+k+l}}$$
$$\geqq(k+l-2)n$$

$$\therefore \dim H^2(\mathfrak{n})^T\leqq 0+(k+l-1)n-n-(k+l-2)n \quad (=0)$$

$\therefore\ H^2(\mathfrak{U}) = H^2(\mathfrak{n})^T = 0$

例2　$n = 2n' + 1$ の時だけ示す。オ1段階より

$$\mathfrak{n}^c = \left\{ \begin{pmatrix} 0 & \boldsymbol{0} & \boldsymbol{b} \\ -{}^t\boldsymbol{b} & X & Y - {}^tY \\ \boldsymbol{0} & 0 & -{}^tX \end{pmatrix} \right\} \quad \begin{array}{l} \boldsymbol{b}_{k+1} = \cdots = \boldsymbol{b}_{k+n'} = 0 \\ X, Y \in \mathfrak{n}_{k,n'} \end{array}$$

を $\mathfrak{n}(k, n')$ と書く。$k \geqq 2$ の時 T が共通に取れるので帰納法で示す。$k = 2$ では Lemma 2.6 (1) と

上の表示で Y の $(2,j)$ 成分だけが1の行列

$$\dim C^2(\mathfrak{n})^T = \#\left\{ \begin{array}{l} (X_{12}, X_{2j}), (X_{12}\ Y_{2j}), (X_{1j}, Y_{2j}) \\ (X_{2j}, Y_{1j}), (X_{12}, \boldsymbol{b}_2), (\boldsymbol{b}_1, \boldsymbol{b}_2) \end{array} \right\}_{3 \leqq j \leqq n'+2} = 4n' + 2$$

$$\mathrm{rank}\{T\text{-不変コサイクル条件}\} \geqq \#\{(X_{12}, X_{2j}, Y_{2j})\}_{3 \leqq j \leqq n'+2} \geqq n'$$

より $\dim H^2(\mathfrak{n})^T \leqq 0$ が示せ $H^2(\mathfrak{U}) = 0$。$k \geqq 3$ では

$\mathfrak{n}(k, n') \supset \mathfrak{a} = \mathfrak{n}(k-1, n'+1)$ とすると

$$\dim C^2(\mathfrak{n})^T / C^2(\mathfrak{a})^T = \#\left\{ \begin{array}{l} (X_{ik}, Y_{kj}), (X_{ij}, Y_{kj}) \\ (X_{ik}, \boldsymbol{b}_k) \end{array} \right\}_{\substack{1 \leqq i \leqq k-1 \\ k+1 \leqq j \leqq n'+k}} = (k-1)(2n'+1)$$

$$\dim \mathfrak{n}/\mathfrak{a} = 2n' + 1$$

$$(\mathfrak{a}\text{ の}\text{に}\cdots\mathrm{rank}) \geqq \#\left\{ \begin{array}{l} (X_{i\,k-1}, X_{k-1\,k}, Y_{kj}), \\ (X_{i\,k-1}, X_{k-1\,j}, Y_{kj}) \end{array} \right\}_{\substack{1 \leqq i \leqq k-2 \\ k+1 \leqq j \leqq n'+k}} + \#\{(X_{i\,k-1}, X_{k-1\,k}, \boldsymbol{b}_k)\}_{1 \leqq i \leqq k-2} \geqq (k-2)(2n'+1)$$

これら3式と $H^2(\mathfrak{a})^T = 0$ より例1と同様に $H^2(\mathfrak{U}) = 0$