数理解析研究所講究録
第 875 巻 1994 年 117-128

A SURVEY OF DEFORMATION THEORY OF CR–STRUCTURES

姫路工業大・理　赤堀隆夫（Takao Akahori）

This paper is a survey of deformation theory of CR-structures, which is studied in (A1),(A2),(A3),(Ku),(Mi). Let (V,o) be an n dimensional normal isolated singularity in (C^N, o). We set

$$M = V \cap S_\epsilon^{2N-1}(o)$$

where $S_\epsilon^{2N-1}(o)$ is the ϵ - sphere in C^N. Then we have a real odd dimensional, compact manifold, which is obviously real analytic. Furthermore, over this M, a CR-structure is naturally induced from V. By Rossi(see (R)), this CR-structure $(M, {}^0T")$ determines the normal isolated singularoty (V, o) , uniquely. Kuranishi noted this point, and in order to study deformation theory of isolated singularities, he initiated deformation theory of CR-structures. This method is improved by (A3),(Mi). Namely, in (A3), it is shown that there is a versal family $(M, {}^{\phi(t)}T")$ which satisfies that $\phi(t)$ is a C^k element of $\overline{{}^0T"} \otimes ({}^0T")^*$ valued form, which depends on t , complex analytically, and $\phi(o) = 0$. Later, Miyajima proved that $\phi(t)$ is actually C^∞ in (Mi). Now our $\phi(t)$ satisfies the following non-linear partial differential equation.

$$\overline{\partial}_b\phi(t) + \overline{\partial}_b^* R_2(\phi(t)) = \Box_b \mathcal{L}(\sum_{i=1}^{q} \beta_i t_i)$$

$t = (t_1, .., t_q)$,$\{\beta_i\}_{1 \leq i \leq q}$ is a base of $\mathrm{H}_{T'}^{(1)}$, q $= dim_C \mathrm{H}_{T'}^{(1)}$,(for notations, see (A3)). This non linear equations'principal part is sub-elliptic, and we note that in the non liner term, only $X\phi(t)$,$XY\phi(t)$, where X,Y in ${}^0T" + \overline{{}^0T"}$, terms appear. Of course if there is no non linear term in this equation, the solution must be real analytic(M being real analytic, so real analytic hypo-ellipticity holds)(see (Tar1),(Ko)). In our case, as the non linear term is quite suitable(it doesn't include $XT\phi(t)$ term and $TT\phi(t)$ term, where X in ${}^0T" + \overline{{}^0T"}$ and T is the missing direction), it is natural to expect the same result as in the elliptic case. Hence it is quite natural to follow the Tartakoff's method, which succeeded in the linear sub-elliptic case. Following the Tartakoff's method in the non linear case, we are forced to control $(XY\phi(t))\phi(t)$ term, where X in ${}^0T" + \overline{{}^0T"}$. However, instead of the standard L^2 norm, if we use the $\| \ \|''_{(m)}$ norm(see Sect.1 in this paper), we have

$$\|(XY\phi(t))(\phi(t))\|''_{(m)} \leq C_m \|XY\phi(t)\|''_{(m)} \|\phi(t)\|''_{(m)}, (n \leq m)$$

and moreover, our norm dosen't cause so much problem to control $\|T^r\phi(t)\|''_{(m)}$ and $\|W^I T^r \phi(t)\|''_{(m)}$ where W in ${}^0T" + \overline{{}^0T"}$ (namely, the Tartakoff's method is also valid in our norm). Therefore the real analytic hypo-ellipticity is now trivial.

Sect.1. *CR − structures and E_j structures*

We consider an n dimensional isolated singularity (V, o) in (C^N, o), and study this singularity from the point of view of CR-geometry. For this, we set a real analytic function on C^N,

$$r(z) = \sum_{i=1}^{N} |z_i|^2 - \epsilon,$$

where $\epsilon > 0$ is chosen sufficiently small. Set

$$M \ = \ \{ \ x \ ; x \ in \ M \ , r(x) = 0 \ \}.$$

On M, a CR structure is naturally induced from C^N. That is to say,

$$S \ = \ \{ \ X \ ; X \ in \ C \otimes TM \cap T'C^N|_M \ \}.$$

In this paper, instead of $\overline{{}^0T"}$, we use the notation S. Then, our S satisfie

$$1) \ S \cap \overline{S} = 0, f - dim_C(C \otimes TM/(S + \overline{S})) = 1$$
$$2) \ [\Gamma(M, \overline{S}), \Gamma(M, \overline{S})] \subseteqq \Gamma(M, \overline{S})$$

The pair (M,S) satisfies 1) and 2) is called a CR structure. Now in our case, obviously, M is real analytic and also the induced CR structure is also real analytic. Next we set a supplement real vector field ζ by;

$$\zeta \ = the \ dual \ vector \ of \ the \ real \ 1 \ form \ \sqrt{-1}\partial r$$

So

$$(1.1) \qquad C \otimes TM \ = \ S \ + \ \overline{S} \ + \ C\zeta$$

Next we recall E_j structures, introduced in (A3). For this, we set $T' = S + C\zeta$. And set a first order differential opeator $\overline{\partial}_{T'}$ from $\Gamma(M, T')$ to $\Gamma(M, T' \otimes (\overline{S})^*)$ by ; for u in $\Gamma(M, T')$,

$$\overline{\partial}_{T'} u(X) = [X, u]_{T'}$$

for X in $\Gamma(M, \overline{S})$, where $[X, u]_{T'}$ means the T'-part of [X,u] according to (1,1). And like the case for scalar valued forms, we have

$$\overline{\partial}_{T'}^{(p)} ; \Gamma(M, T' \otimes \wedge^p(\overline{S})*) \longrightarrow \Gamma(M, T' \otimes \wedge^{(p+1)}(\overline{S})*),\ p = 1, 2, ...$$

Now we set

$$\Gamma_p = Ker\ \overline{\partial}_{T'}^{(p)} \cap \Gamma(M, S \otimes \wedge^{(p)}(\overline{S})*) .$$

Then there is a subbundle E_p of $S \otimes \wedge^p(\overline{S})*$ satisfying;

$$E_0 = 0 ,$$
$$\Gamma_p = \Gamma(M, E_p)$$

And

$$\overline{\partial}_p^{(p)} \Gamma(M, E_p) \subset \Gamma(M, E_{(p+1)})$$

$$Ker\ \partial_b^{(1)} \longrightarrow H^{(1)}(M, T') \longrightarrow 0$$

$$\frac{Ker\ \overline{\partial}_b^{(p)}}{Im\ \overline{\partial}_b^{(p-1)}} \simeq H^{(p)}(M, T') ,\ 2 \leq p \leq n-1$$

where $\overline{\partial}_b^{(p)} = \overline{\partial}_{T'}^{(p)}|_{\Gamma(M, E_i)}$, $dim_R M = 2n-1$, $dim_C^0 T" = n-1$. (In (A1),(A2) and (A3), we used different notations. However, in this paper, for the reader's convenience, we dare to use $\overline{\partial}_b^{(p)}$). And if $n \geq 4$,

$$\Gamma(M, E_i) \xrightarrow{\overline{\partial}_1} \Gamma(M, E_2) \xrightarrow{\overline{\partial}_2} \Gamma(M, E_3)$$

is a subelliptic complex and several important estimates are proved in (A3). We recall this. For this, we set a real 1 form θ by

$$\theta|_{S+\overline{S}} = 0$$
$$\theta(\zeta) = 1$$

And we set $\omega = -d\theta$, and then we have the Levi metric. From this metric, we define the volume element dv , and we set the L^2 norm on $\Gamma(M, E_p)$ by

$$(u, v) = \int_M < u, v > dv\ for\ u, v\ in\ \Gamma(M, E_p)$$

where $<\ ,\ >$ means the hermitian inner product induced from $\Gamma(M, S\otimes\wedge^p(\overline{S})^*)$. We denote $\overline{\partial}_b^*$ by the adjoint operator of $\overline{\partial}_b$ on $\Gamma(M, E_p)$ with respect to the above metric. And we set the Laplacian

$$\square_b = \overline{\partial}_b^*\overline{\partial}_b^* + \overline{\partial}_b\,\overline{\partial}_b^*.$$

For u in $\Gamma(M, E_p)$, we set

$$\| u \|''^2_{(m)} = \sum_{i=0}^{m} \| \square_b u \|^2,$$

$$(u,v)''_{(m)} = \sum_{i=0}^{m} (\square_b^i u , \square_b^i v) \; for u,v \; in \; \Gamma(M, E_p).$$

Then, we easily have
Lemma 1.1

$$(\overline{\partial}_b u , v)''_{(m)} = (u , \overline{\partial}_b^* v)''_{(m)} \; for \; u,v \; in \; \Gamma(M, E_p)$$

Namly, $\overline{\partial}_b^*$ is also the adjoint operator of $\overline{\partial}_b$ with respect to $\| \ \|''_{(m)}$.Furthermore by the result(see Proposition 3.3 in (A2)) with the standard argument, we have
Lemma 1.2 For u in $\Gamma(M, E_p)$,

$$\| WWu \|''_{(m)} \leq C_m \| u \|''_{(m+1)}$$

We must explain notations. Let $\{U_i, h_i\}_{i\in I}$ be a finite set of local coordinate neighborhoods of M. And let $\{\rho\}_{i\in I}$ be a partition of unity subordinate to this covering. Let $\{Y_{j,k}\}_{1\leq j\leq n-1}$ be an orthonormal frame of $\overline{S}$ over U_k according to the Levi metric defined by (1.1). With this preparation, the above inequality means; for u supported in U_k,

$$\| W_{\alpha,k} W_{\beta,k} u \|''_{(m)} \leq C_m \| u \|''_{(m+1)},$$

where $W_{\alpha,k}$,$W_{\beta,k} = Y_{j,k}$ or $\overline{Y}_{(j,k)}$, $1 \leq j \leq n-1$. And henceforth, for this $W\alpha, k$, we use the abbreviation W. Assume the ;$dim_R M = 2n-1 \geq 7$. Then,
Estimate (I)

$$\| u \| \leq C\{\| \overline{\partial}_b u \| + \| \overline{\partial}_b^* u \| + \| u \|\}, \; for \; u \; in \; \Gamma(M, E_2)$$

(see Theorem 4.1(new estimate) in (A3)). Then by the standard argument, we have the Neumann operator N_b for the above differential complex $(\Gamma_p, \overline{\partial}_b^{(p)})$. And so, we have the Kodairaodge type decomposition theorem for this complex, namely

$$u = H_b u + \square_b N_b u \; , \; for \; u \; in \; \Gamma(M, E_2),$$

where H_b means the projection of u into

$$\{ u \; ; \; u \; in \; \Gamma(M, E_2), \; \overline{\partial}_b u \; = \; 0, \; \overline{\partial}_b^* u \; = \; 0 \}.$$

Estimate (II)

$$\| u \| \; \leq \; C'\{ \| \Box_b u \| \; + \; \| u \| \} \; for \; u \; in \; \Gamma(M, E_2).$$

We note that $\| \|''$ norm is the same as $\| \|''_{(o)}$ norm introduced in this section.
Estimate (III)

$$\| W_{\alpha,k}(\rho u) \|''^2_{(m)} + K \| \rho u \|''^2_{(m)} \leq C_m\{ \| \overline{\partial}_b(\rho u)\|''^2_{(m)} + \overline{\partial}_b^*(\rho u)\|''^2_{(m)} \} + C'_m \| \rho u \|''^2_{(m)},$$

where $W_{\alpha,k} \; = \; Yj,k \; or \; \overline{Y}_{j,k}$, and $\rho \; \in \; C^\infty$ is supported in U_k. Finally in this section, we note that there is a real analytic real vector field T on M satisfying;

$$1) \; T_p \; \notin S_p \; + \; \overline{S}_p \; for \; every \; point \; p \; of \; M,$$

$$2) \; [T, Z] \; \equiv 0 \; mod S + \overline{S} \; for \; all \; Z \; \in \; \Gamma(M, S + \overline{S}),$$

(see Proposition 1 in (Tar)). So, using this T, we newly introduce a C^∞ vector bundle decompostion

$$C \otimes TM \; = \; S \; + \; \overline{S} \; + \; C \otimes T \tag{1.2}$$

and also introduce corresponding operators $\overline{\partial}_{T'}, \overline{\partial}^{(p)}_{T'}$. Then, the complete same results hold, and the same estimates hold. From now on, we adopt these. And following (Tar1), we set $W = S + \overline{S}$.

Sect.2 *The canonical versal family*

In this section, we recall the construction of the canonical versal family ((A3)). Namely, we set $\Gamma(M, S \otimes \overline{S}^*)$ valued power series

$$\phi(t) \; = \; \sum_{K=(K_1,..,k_q)} \phi_K t_1^{k_1} ....t_q^{k_q}$$

where $t = (t_1, .., t_q) \; \in \; U \subset C^q$, and U is a neighborhood of the origin, and K is a muli index, $q \; = \; dim_C \mathrm{H}^{(1)}_{T'}$. For brevity, we abbreviate this as follows.

$$\phi(t) \; = \; \sum_K \phi_K t^K \; .$$

Now we recall the construction of $\phi(t)$. By the Banach inverse mapping theorem, we solve $\phi(t)$, namely $\phi(t)$ is a unique solution of the following.

$$\phi(t) \; + \; \overline{\partial}_b^* N_b R_2(\phi(t)) \; = \; \mathcal{L}(\sum_{i=1}^{q} \beta_i t_i \;) \; , \; (t_1, ..., t_q) \; \in \; U \; \subset C^q,$$

where N_b is introduced in (A3), and $\{\beta_i\}_{1\leq i\leq q}$ is a base of $\mathbf{H}^{(1)}_{T'}$. It is better to explain $\mathcal{L}$,introduced in (A2). For v in $\Gamma(M, T' \otimes \overline{S}^*)$, we set

$$\mathcal{L}v(X) = v(X) - \overline{\partial}_{T'}\theta_v(X), \; for \; X \in \Gamma(M, \overline{S}),$$

where θ_v is an element of $\Gamma(M, S)$ defined by;

$$[\theta_v, X]_T = (v(X))_T \; for \; X \; in \Gamma(M, \overline{S}),$$

where $[\theta_v, X]_T$(resp. $(v(X))_T$)means the $C \otimes T$ part of $[\theta_v, X]$(resp. (v(X))) according to (1.2).

Sect.3 *The real analyticity*

As we recalled in Sect.2, $\phi(t)$ satisfies

$$\phi(t) + \overline{\partial}_b^* N_b R_2(\phi(t)) = \mathcal{L}(\sum_{i=1}^{q} \beta_i t_i).$$

Hence we have

$$\Box_b\phi(t) + \Box_b\overline{\partial}_b^* N_b R_2(\phi(t)) = \Box_b\mathcal{L}(\sum_{i=1}^{q} \beta_i t_i),$$

namely

$$\Box_b\phi(t) + \overline{\partial}_b^* R_2(\phi(t)) = \Box_b\mathcal{L}(\sum_{i=1}^{q} \beta_i t_i).$$

We must show that this $\phi(t)$ is real analytic. We follow the Tartakoff's line in (Tar1) and we adopt his notations. Let p_o be the reference point of M. Let $U_1(p_o)$ be a sufficiently small neighborhood of p_o in M, and $U_2(p_o)$ be a neighborhood of p_o satisfying; $U_1(p_o) \Subset U_2(p_o)$.
Now we show that there are constants C_1 and C_2 which satisfy; there is a $\epsilon > 0$, and for every q , there is a C^∞ function ψ_q supported in $U_2(p_o)$ and $\psi_q|_{U_1(p_o)} = 1$ satisfying;

$$(*) \qquad \| \psi_q Op(q)\phi(t) \|''_{(m)} \leq C_1 C_2^q q! \, , \; for \; any \; t \; in \; (0, \epsilon)$$

Here Op(q) denotes the q-th order differential operator formed by T,W_j in W. If this is proved, by the Sobolev lemma, for every q,

$$Sup_{\,U_1(p_o)} | \, Op(q)\phi(t) \, | \leq c \, \| \, Op(q)\phi(t) \, \|''_{(m),U_1(p_o)} \, , \; (m \geq n),$$

where $\| \; \|''_{(m),U_1(p_o)}$ means the corresponding norm over $U_1(p_o)$. So

$$\begin{aligned} Sup_{\,U_1(p_o)} | \, Op(q)\phi(t) \, | &\leq c \, \| \, \psi_q Op(q)\phi(t) \, \|''_{(m)} \\ &\leq c \, C_1 C_2^q q! \end{aligned}$$

Therefore by Lemma 1 in (Tar1), we have that $\phi(t)$ is real analytic for any t in $(0,\epsilon)$. And by the following lemma, it is shown that $\phi(t)$ is real analytic.
Lemma 3.1 Let u(x,t) is a C^k function on $R^m \times C^n$ $(k \geq 1)$, which is real analytic with respect to x, and complex analytic with respect to t, separately. Then, u(x,t) is real analytic on (x,t).

Proof. We consider the partial complexfication of $R^m \times C^n, C^m \times C^n$. And for a fixed t, we can naturally consider $\widetilde{u(z,t)}$ on $C^m \times C^n$ for u(x,t). By the assumption, our $\widetilde{u(z,t)}$ is complex analytically with respect to respectively z and t. So by Osgood's lemma, our $\widetilde{u(z,t)}$ is complex anulutic with respect to both variables. So u(x,t) must be real analytic. **Q**.*E.D.*

For $(*)$, it suffices to show; there are constants C_1 and C_2 which satisfy; there is a $\epsilon > 0$, and for every q, there is a C^∞ function ψ_q supported in $U_2(po)$ and $\psi_q|_{U_1(p_o)} = 1$ satisfying;

$$(**) \qquad \| \psi_q W^I T^r \phi(t) \|''_{(m)} \leq C_1 C^{|I|+r} |I|! r! \ , \ for\ any\ t\ in\ (0,\epsilon).$$

(see Proposition 1 in (Tar2)). We see the sketch of the proof of $(**)$. In order to see this, we recall several lemmas which were shown in (Tar1), and use his useful notations. Following (Tar1), Op(k,q) denotes a q-th order differential operator formed by concatenating k W's and q-k T's.

Lemma 3.2(Lemma 2 in (Tar1)) For $k \geq 1$, any Op(k,q) may be written sybolically

$$Op(k,q) = WOp(k-1,q-1) + \sum_{j=1}^{q} c^j \binom{q}{j} a_{(j)} Oo(k,q-j) \ , \ i.e.$$

if there is a W, we may commute it to the left modulo the indicated sum of at most $c^j \binom{q}{j}$ terms, c some integer depending only on n, of the form $a_{(j)}Op(k,q-j)$.
Lemma 3.3(Lemma 3 in (Tar)) Let a denote any of a finite number of real analytic functions and Z any of a finite number of real analytic vector field. Let $\{a_{(q)}\}$ be recursively defined by;

$$a_{(1)} = any\ of\ the\ a's$$
$$a_{(q+1)} = a_{(1)} a_{(q)}\ or\ Z a_{(q)},$$

i.e., $a_{(1)}a_{(q)}$ stands for one of the a's timers an expression of the form $a_{(q)}$. Then locally there exists K such that for all α and for all q,

$$| D^\alpha a_{(q)} | \leq K\ K^{(|\alpha|+q)}(|\alpha|+q)! \ .$$

Then, as for our norm, we immediately have

Lemma 3.4

$$\| D^{\alpha} a_{(q)} \|''_{(m)} \leq K' K'^{(|\alpha|+q+m)}(|\alpha|+q+m)!.$$

So by choosing a proper K, we have

$$\| D^{\alpha} a_{(q)} \|''_{(m)} \leq K K^{(|\alpha|+q)}(|\alpha|+q)!.$$

Lemma 3.5(Lemma 4 in (Tar1))

$$[T^r, \Box_b] = \sum_{j=1}^{r} c^j \binom{r}{j} \{ W a_{(j+1)} W + W a_{(j+2)} + a_{(j+3)} \} T^{r-j} .$$

Now we begin by estimating $\| \rho W T^p \phi(t) \|''_{(m)}$ and $\|\rho T^p \phi(t)\|''_{(m)}$. As for ρ, we use a general partition of unity, and we recall the basic estimate(Estimate (III)).

$$\| W \rho u \|''^2_{(m)} + K \| \rho u\|''^2_{(m)} \leq c_m \{ \| \overline{\partial}_b(\rho u) \|''^2_{(m)} + \| \overline{\partial}^*_b(\rho u) \|''^2_{(m)} \} + C_{K,m} \| \rho u \|^2$$

namely,

$$\leq c_m \{ \sum_{i=0}^{m} (\Box_b^i \overline{\partial}_b(\rho u), \Box_b^i \overline{\partial}_b(\rho u)) + \sum_{i=0}^{m} (\Box_b^i \overline{\partial}^*_b(\rho u), \Box_b^i \overline{\partial}^*_b(\rho u))\} + C_{K,m} \| \rho u\|^2.$$

So in the place of u in this equality, we put $u = T^p \phi(t)$. Then, we have

$$(***) \quad \begin{aligned} &\| W \rho T^p \phi(t) \|''^2_{(m)} + K \| \rho T^p \phi(t) \|''^2_{(m)} \\ &\leq c_m \{ \sum_{i=0}^{m} (\Box_b^i \overline{\partial}_b(\rho\phi(t)), \Box_b^i \overline{\partial}_b(\rho\phi(t))) \\ &+ \sum_{i=0}^{m} (\Box_b^i \overline{\partial}^*_b(\rho\phi(t)), \Box_b^i \overline{\partial}^*_b(\rho\phi(t)))\} + C_{K,m} \| \rho T^p \phi(t) \|^2 \\ &\leq c_m \{ \sum_{i=0}^{m} (\Box_b^i \Box_b(\rho T^p \phi(t)), \Box_b^i(\rho T^p \phi(t)))\} + C_{K,m} \| \rho T^p \phi(t) \|^2. \end{aligned}$$

The commutator $[\rho, \Box_b]$ does not make troubles so much. In fact, the above can be estimated as follows.

$$\begin{aligned} &\leq c'_m \|\rho' T^p \phi(t)\|''^2_{(m)} + (\frac{\epsilon}{C}) \|\rho W T^p \phi(t)\|''^2_{(m)} \\ &+ C_{\epsilon} \sum_{j=1}^{p} \tilde{c}^j \binom{p}{j} \{ \|\rho a_{(j+1)} W T^{p-j} \phi(t)\|''^2_{(m)} + \|\rho a_{(j+3)} T^{p-j} \phi(t)\|''^2_{(m)}\} \\ &+ \|j^{(1)}(\rho)\|''_{(m)} \|W T^p \phi(t)\|''_{(m)} + \|j^{(2)}(\rho)\|''_{(m)} \|T^p \phi(t)\|''_{(m)} \\ &+ \cdots\cdots\cdots \\ &+ \|j^{(2k+1)}(\rho)\|''_{(m)} \|W T^p \phi(t)\|''_{(m-k-1)} + \|j^{(2k+2)}(\rho)\|''_{(m)} \|T^p \phi(t)\|''_{(m-k-1)} \\ &+ \cdots\cdots\cdots \\ &+ \|j^{(2m-1)}(\rho)\|''_{(m)} \|W T^p \phi(t)\|''_{(0)} + \|j^{(2m)}(\rho)\|''_{(m)} \|T^p \phi(t)\|''_{(0)}. \end{aligned}$$

while these are estimated by

$$(large\ constant)\|j^{(2k+1)}(\rho)\|''^2_{(m)} + (small\ constant)\|WT^p\phi(t)\|''^2_{(m-k-1)}$$

and

$$(large\ constant)\|j^{(2k+2)}(\rho)\|''^2_{(m)} + (small\ constant)\|T^p\phi(t)\|''^2_{(m-k-1)},$$

(for the notation, see (A2) , (A3)). Therefore at most, they are estimated by

$$(\frac{C}{\epsilon})\|j^{(2m)}(\rho)\|''^2_{(m)} + \epsilon\{ \|WT^p\phi(t)\|''^2_{(m)} + \|T^p\phi(t)\|''^2_{(m)} \}.$$

Namely, it does not bother us. Hence we can neglect this. As X is compact, and T is globally defined, for $\rho_i \in C_o^\infty$ of small support,

$$\|\rho' T^p\phi(t)\|''_{(m)} \leq \sum_{i=1}^{N} C_\rho \|\rho_i T^p\phi(t)\|''_{(m)}.$$

And if $K \geq 2(C_\rho C + C'')N$ and ρ itself is one of the ρ_i, upon summing this over i , then this error term will be absorbed on the left. Furthermore

$$\begin{aligned}
&(\Box_b^m \rho \Box_b T^p\phi(t), \Box_b^m \rho \Box_b T^p\phi(t)) \\
=&(\Box_b^m \rho T^p \Box_b\phi(t), \Box_b^m \rho T^p \Box_b\phi(t)) + (\Box_b^m \rho[\Box_b, T^p]\phi(t), \Box_b^m \rho T^p\phi(t)) \\
=&(\Box_b^m \rho T^p \Box_b\phi(t), \Box_b^m \rho T^p\phi(t)) \\
&+(\Box_b^m \rho W(\sum_{j=1}^{p} a_{(j+1)} W T^{p-j}\phi(t)), \Box_b^m \rho T^p\phi(t)) \ . \ (by\ Lemma\ 3.5)
\end{aligned}$$

By the same way as in (Tar1), we can handle the second term. So we omit this. We will control the first term.

$$T^p\Box_b\phi(t) + T^p\overline{\partial}_b^* R_2(\phi(t)) = T^p\Box_b\mathcal{L}(\sum_{i=1}^{q} \beta_i t_i)\ .$$

And so this term can be estimated by;

$$\begin{aligned}
&|(\Box_b^m \rho_i T^p \Box_b \mathcal{L}(\sum_{i=1}^{q} \beta_i t_i), \Box_b^m \rho_i T^p\phi(t))| \\
&+ |(\Box_b^m \rho_i T^p \overline{\partial}_b^* R_2(\phi(t)), \Box_b^m \rho_i T^p\phi(t))|.
\end{aligned}$$

The first term was already handled by (Tar1). We see the second term which didn't appear in (Tar1). The second term becomes

$$\begin{aligned}
&|(\Box_b^m \rho_i T^p R_2(\phi(t)), \Box_b^m \rho_i \overline{\partial}_b T^p\phi(t))| + commutator\ terms \\
&\leq (large\ constant)\|\Box_b^m \rho_i T^p R_2(\phi(t))\|^2 + (small\ constant)\|\Box_b^m \rho_i \overline{\partial}_b T^p\phi(t)\|^2 \\
&+ commutator\ terms.
\end{aligned}$$

To contro; commutator terms is tedious. But the method is standard. So we omit this. For the non-linear term $T^pR_2(\phi(t))$,
Lemma3.6 If we choose C_1,C_2 sufficiently large, we have

$$\|T^pR_2(\phi(t))\|''_{(m)} \leq (\frac{1}{4})C_1C_2^pp!\ for\ every\ p.$$

Proof To estimate $T^pR_2(\phi(t))$, we must estimate $T^p\{(W\phi(t))\phi(t)\}$. Namely,

$$\begin{aligned} T^p\{(W\phi(t))\phi(t)\} &= (T^p(W\phi(t)))\phi(t) + \binom{p}{1}(T^{p-1}(W\phi(t))(T\phi(t)) \\ &+ \binom{p}{p-1}(T(W\phi(t)))(T^{p-1}\phi(t)) + (W\phi(t))(T^p\phi(t)) \\ &+ \sum_{r=2}^{p-2}\binom{p}{r}(T^r(W\phi(t)))(T^{p-r}\phi(t))). \end{aligned}$$

Since $\phi(o) = 0$, we can assume that $\|\phi(t)\|''_{(m)}$,$\|W\phi(t)\|''_{(m)}$ are sufficiently small if we choose ϵ sufficiently small. So the first term, the second term, the third term, and the fourth term can be absorbed in the left of $(***)$. Now we see how to control the other term. By induction, we see

$$\|T^k(W\phi(t))\|''_{(m)}\ ,\ \|T^k\phi(t)\|''_{(m)}\ le\ C_1C_2^{k-2}(k-2)!\ if\ k \geq 2.$$

k=2 case is OK, if we choose C_1 sufficiently large. We assume k=p-1 case. Now we see p case. By $m \geq n$

$$\begin{aligned} &\|\sum_{r=2}^{p-2}\binom{p}{r}(T^r(W\phi(t)))(T^{p-r}\phi(t))\|''_{(m)} \\ &\leq \sum_{r=2}^{p-2}\binom{p}{r}\|T^r(W\phi(t))\|''_{(m)}\|T^{p-r}\phi(t)\|''_{(m)} \\ &\leq \sum_{r=2}^{p-2}\binom{p}{r}C_1C_2^{r-2}(r-2)!C_1C_2^{p-r-2}(p-r-2)! \\ &\leq \sum_{r=2}^{p-2}\binom{p}{r}C_1^2C_2^{p-4}(r-2)!(p-r-2)! \end{aligned} \tag{3.6.1}$$

And

$$\sum_{r=2}^{p-2}\binom{p}{r}(r-2)!(p-r-2)! = \sum_{r=2}^{p-2}\frac{p(p-1)(p-2)!}{r(r-1)(p-r)(p-r-1)}$$

While if $r \leq [\frac{p}{2}]$, $p - r \geq \frac{p}{2}$, $p - r - 1 \geq \frac{p}{2}$. Hence

$$\begin{aligned} &\sum_{r=2}^{p-2} \frac{p(p-1)(p-2)!}{r(r-1)(p-r)(p-r-1)} \\ = &\sum_{r=2}^{[\frac{p}{2}]} \frac{2p(p-1)(p-2)!}{r(r-1)(p-r)(p-r-1)} \\ \leq &\sum_{r=2}^{[\frac{p}{2}]} \frac{8}{r(r-1)p^2} \times p(p-1)(p-2)! \\ \leq &8\frac{p-1}{p}(p-2)! \\ \leq &8(p-2)! \end{aligned}$$

Hence

$$(3.6.1) \ \leq \ (\frac{8C_1}{C^2})C_1 C_2^{p-2}(p-2)!$$

So if we choose $\frac{8C_1}{C_2^2} \leq 1$, then we have our estimate. So we can control

$$\|T^p\phi(t)\|''_{(m)} \ , \ \|WT^p\phi(t)\|''_{(m)} \ .$$

For $\|W^I T^p \phi(t)\|''_{(m)}$, $|I| \geq 2$, following the Tartakoff's method, namely using Ehrenpreis's localizing function with careful study of the non-linear term as in Lemma 3.6, we have our estimate.

References

[A1] T. Akahori, *Intrinsic formula for Kuranishi's $\overline{\partial}_b^{\phi}$*, Publ.RIMS **14** (1978), 615–641.

[A2] T. Akahori, *Complex analytic construction of the Kuranishi family on a normal strongly pseudo convex manifold*, Publ.RIMS **14** (1978), 789–847.

[A3] T.Akahori, *The new estimate for the subbundles E_j and its application to the deformation of the boundaries of strongly pseudo convex domains*, Invent.math. **63** (1981), 311–334.

[Ku] M.Kuranishi, *Deformations of isolated singularities and $\overline{\partial}_b$*, preprint.

[Ko] G.Komatsu, *Global analytic hypoellipticity of the $\overline{\partial}$ - Neumann problem*, Tohoku Math. J. Ser.2 **28** (1976), 145–156.

[Mi] K.Miyajima, *Completion of Akahori's construction of the versal family of strongly pseudo convex CR structures*, Trans. Amer. Math. Soc. **277** (1980), 162–172.

[R] H.Rossi, *Isolated singularities*, Proceedings of KATATA conference, 1966, pp. 96–102.

[Tar1] D.Tartakoff, *On the global real analyticity of the solution to $\Box_b$ on compact manifolds*, Comm. in partial differential equations **4** (1976), 283–311.

[Tar2] D.Tartakoff, *The local real analytic solutions to* $\Box_b$ *and the* $\bar{\partial}$ *- Neumann problem*, Acta. math. **145** (1980), 117–204.

SHOSYA 2167,HIMEJI 671-22,JAPAN

E-mail: akahorit@ sci.himeji-tech.ac.jp