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SOME PROBLEMS ON THE MODULI OF FIRST ORDER PDE WITH COMPLETE INTEGRALS AND WEB GEOMETRY OF THEIR SOLUTIONS

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This note is prepaired for the talk in the work shop on Nilpotent geometry at Kyoto RIMS November 25-28 1993. The main ingredients are first order PDE with complete integral, singularity theory of functions on varieties, Web geometry, affine connection of solutions of PDE, residue of 1-dimensional dynamics, oscillatory integral, etc. Many creatures come into the stage underlit of clasical mechanics.

The study of the oscillatory integral

$$I_\lambda(q) = \int \exp \frac{i\ s(p,q)}{\lambda}\ dp, \qquad p \in \mathbb{R}^{n+k},\ q \in \mathbb{R}^n$$

is one of the main subjects in the quantum physics. It is known that the integral is asymptotic to 0 in order $\lambda^a\ (\log\ \lambda)^b$ (Nilsson class) as the wave length λ tends to 0 at each q [29]. The stationary phase method suggests that the principal term of the asymptotic expantion is determined by the geometry of the family of the wave front $D_d \subset \mathbb{R}^n$ in the configuration space parametrized by $d \in \mathbb{R}$ defined as follows: $q \in D_d$ if and only if d is one of the critical values of the function $s(*, q)$. For example the Airy-Fock function

$$Ai_\lambda(q) = \int_{-\infty}^{\infty} \exp \frac{i(p^3/3 + pq)}{\lambda}\ dp$$

($Ai_1 = Ai$) has the asymptotic expantion

$$Ai_\lambda(q) = \begin{cases} \frac{1}{\sqrt{\pi}} \frac{\lambda^{1/2}}{|q|^{1/4}} \sin\left(\frac{2}{3}\frac{q^{3/2}}{\lambda} + \frac{\pi}{4}\right) + C\ \lambda^{3/2} + \cdots & (q < 0) \\ \frac{1}{3^{3/2}\sqrt{\pi}} \Gamma(\frac{1}{3}) \sin(\frac{2}{3}\pi) \lambda^{1/3} & (q = 0) \\ \frac{1}{2\sqrt{\pi}} \frac{\lambda^{1/2}}{q^{1/4}} \frac{1}{\exp\ \frac{2}{3}\frac{q^{3/2}}{\lambda}} + C'\ \lambda^{3/2} + \cdots & (q > 0) \end{cases}$$

as $\lambda \to 0$, where $\frac{2}{3}\frac{q^{3/2}}{\lambda}$ is the critical value of the potential function $\frac{(p^3/3+pq)}{\lambda}$ [18]. This integral is the unique solution to the Airy equation

$$I'' - \lambda^{-2/3} q I = 0.$$

By the relation $Ai_\lambda(q) = \lambda^{1/3}Ai(\frac{q}{\lambda^{2/3}})$ the asymptotics as $\lambda \to 0$ is concentrated to the local behavior of the solution at ∞, where the formal solution has Stokes gap between positive and negative half lines. This causes to the difference of the above asymptotic expantion on the positive and negative sides of the q-line. The asymptotics of the integral

$$I_\lambda(q,r) = \int_{-\infty}^{\infty} \exp \frac{i(p^4 + qp^2 + rp)}{\lambda}\, dp$$

was studied by Pearcey [25]. The wave front associated to this is modeled on the spherical optics and the integral approximates the intensity nearby the cusp of the caustics. We call the family $\{D_d\}$ the web since the wave fronts form configuration of foliations at generic points. In this talk some differential geometric properties of the webs are investigated and the various problems from the web geometrical view point are proposed.

First we apply the singularity theory to our PDE and prove the existence of the (uni)versal model for almost all webs. From a democratic view point in the singularity theory of functions we study the generalized oscillatory integrals

$$I_\lambda(q) = \int_{V_q} \exp \frac{i\ s(p,q)}{\lambda}\, dp, \qquad q \in \mathbb{R}^n,$$

where V_q is a variety parametrized by $q \in \mathbb{R}^n$ and $s(*,q)$ is a function on V_q. We show that all first order PDE are realized as the webs of the generalized oscillatory integrals. In the complex analytic case a result due to Goryunov [14] tells the complement of the singular locus (= caustics) of the generalized integrals are Eilenberg-Maclane $K(\pi,1)$-space: the fundamental groups π are finite index subgroups of the symmetric groups $S(m)$. The order m is given by Milnor number of $s(*,q)$ and the index is the local intersection number of m solutions. Secondly we apply the web geometry to define the affine connection on the configuration space $\mathbb{R}^n$ and show that in some cases the structure of the webs are determined by their curvature forms. In analyzing the moduli of the normal forms of the webs a certain residue of dynamics acting on the phase space (d-line) $\mathbb{R}$ turns out to give a formal invariant,

The webs are defined in another way by first order partial differential equations with the complete integral s (PDE). A *first order partial differential equation* (PDE) on $\mathbb{R}^n$ is a subvariety V of the projective cotangent space $PT^*\mathbb{R}^n$ (or $PT^*\mathbb{C}_n$) furnished with the canonical contact structure [6]. Our subject is the local topological structure of the PDE at the singular points of the projection of the variety to the base space $\mathbb{R}^n$. So $PT^*\mathbb{R}^n$ may be replaced by the 1-jet bundle $J^1(\mathbb{R}^{n-1},\mathbb{R})$ with the contact form $\omega = p\, dx\ -\ dy$, where x, y are respectively the coordinates of $\mathbb{R}^{n-1}, \mathbb{R}$ and p is the coordinate of $\mathbb{R}^{n-1*}$, the fibre of the projection $ev : J^1(\mathbb{R}^{n-1},\mathbb{R}) \to \mathbb{R}^{n-1} \times \mathbb{R} = \mathbb{R}^n$. Assume $V = \mathbb{R}^n$, $ev \circ \mathcal{I}(0) = 0 \in \mathbb{R}^{n-1} \times \mathbb{R}$ and $\mathcal{I} : \mathbb{R}^n \to J^1(\mathbb{R}^{n-1},\mathbb{R})$ is an immersion. The direct image of the pull back $\mathcal{I}^*\omega$ under the projection $ev \circ \mathcal{I}$ defines a multi valued 1-form (implicit differential equation) on the *configuration space* : the base space $\mathbb{R}^{n-1} \times \mathbb{R}$. A *complete integral* is a germ of non singular smooth function $s : \mathbb{R}^n, 0 \to \mathbb{R}, 0$ such that $ds \wedge \mathcal{I}^*\omega$ vanishes identically on a neighbourhood of 0 (from now on we assume the existence of the complete integrals). Then the images $D_d = ev \circ \mathcal{I}(s^{-1}(d)), d \in \mathbb{R}$, constitute the integral submanifolds (possibly singular) of the equation on a neighbourhood of $0 \in \mathbb{R}^n$. We call D_d a *solution* of the equation and call the family $\mathcal{W}_\mathcal{I} = \{D_d, d \in \mathbb{R}\}$ the *solution web* of the equation $\mathcal{I}$. Web geometry [3,9,13] applies to the solution web and defines the various affine connections on the confuguration space $\mathbb{R}^n$. In some cases, this gives a one-to-one correspondence of the moduli space (function moduli) of PDE and the curvature forms of their affine connections.

Two webs $\mathcal{W}_\mathcal{I} = \{D_d\}, \mathcal{W}_\mathcal{J} = \{D'_d\}$ of PDE's $\mathcal{I}, \mathcal{J}$ are C^∞ *equivalent* if there exists a germ of diffeomorphisms $\psi : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ and $k : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$ such that $\psi(D_d) = D'_{k(d)}$ for d in a neighbourhood of $0 \in \mathbb{R}$. In other words $\mathcal{W}_\mathcal{I}, \mathcal{W}_\mathcal{J}$ are C^∞ equivalent if there exists a germ of a contact diffeomorphism $d\psi$ of the Legendre fibration $J^1(\mathbb{R}^{n-1}, \mathbb{R}) \to \mathbb{R}^n \times \mathbb{R}$ sending the image of $\mathcal{I}$ to that of $\mathcal{J}$: $\mathcal{I}, \mathcal{J}$ are *strictly* C^∞ *equivalent* if k is the identity. We discuss some classification problem of PDE in terms of the geometry of the solution webs. We say that the solution web $\mathcal{W}_\mathcal{I}$ is a *non singular m-web at* $q \in \mathbb{R}^n$ if the number of d for which D_d passes through q is m, those solutions $D_{d_i}, i = 1, ..., m$, are smooth and meet in general position at q and D_d forms a non-singular foliation at q as d varies nearby d_i for each $i = 1, ..., m$. We call the maximum of such d for q nearby $0 \in \mathbb{R}^n$ the *web number*. The web number is the topological multiplicity of $ev \circ \mathcal{I}$ for generic $\mathcal{I}$. The *singular locus* $\mathrm{Sing}(\mathcal{W}_\mathcal{I})$ of $\mathcal{W}_\mathcal{I}$ is the set of those q where $\mathcal{W}_\mathcal{I}$ is not a nonsingular m-web, m being the web number. By an easy calculation we see that the C^∞ equivalence classes of m-webs, $n + 1 \leq m$, form subsets of infinite codimension in the jet space of m-tuples of level functions defined at $q \in \mathbb{R}^n$. So C^∞-classification of the solution webs fails in the ordinary sense. In fact Arnol'd [2], Carneiro [7], Dufour [12] and Hayakawa-Ishikawa-Izumiya-Yamaguchi [16] showed that C^∞ classes of some PDE have moduli of infinite dimension called the *function moduli*, which are parametrized by the space of smooth functions defined on the configuration space $\mathbb{R}^n$ at 0.

A "*versal* PDE" $\mathcal{I}'$ (versal unfolding) of a PDE $\mathcal{I}$ defined on $\mathbb{R}^{n+r}$ has the following properties.

(1) The solution web $\mathcal{W}_\mathcal{I}$ is induced from $\mathcal{W}_{\mathcal{I}'}$ by the natural imbedding $\mathbb{R}^n \to \mathbb{R}^{n+r}$ "transverse" to the solutions of $\mathcal{I}'$,

(2) For a deformation $\mathcal{I}_t$ of $\mathcal{I}$, there exists a family of imbeddings $i_t : \mathbb{R}^n \to \mathbb{R}^{n+r}$ transverse to the solutions of $\mathcal{I}'$ by which the solution web $\mathcal{W}_{\mathcal{I}_t}$ is induced from $\mathcal{W}_{\mathcal{I}'}$,

(3) $\mathcal{W}_{\mathcal{I}'}$ has the web number $\leq n + r$ and any deformation is trivial

(The imbedding of $\mathbb{R}^n$ is not transversal to the $\mathcal{W}_{\mathcal{I}'}$ in general in the ordinary sense.) The property (2) suggests that $\mathcal{W}_{\mathcal{I}'}$ possesses a certain universality. The classification problem falls into the following two problems.

(A) Classification of versal PDE's $\mathcal{W}_{\mathcal{I}'}$.

(B) Geometry of sections of $\mathcal{W}_{\mathcal{I}'}$.

Problem A is reduced to the singularity theory of functions $s(*, q)$ on varieties V_q (see c.f. [4,14,15, 21,22]). Problem B is closely related to the web geometry of the solution webs, which is mentioned in the later part this note.

Hayakawa-Ishikawa-Izumiya-Yamaguchi [16] explained a link of the first order PDE's with complete integral and the singularity theory of the so-called generating functions of legendrian submanifolds using, and classified generic PDE with complete integral on the plane as follows. First recall a well known result due to Hörmander and Zakalyukin [17,24]. Let $h:\mathbb{R}^{n+k},0\to\mathbb{R}^{n+1},0$ a germ of smooth map of corank 1: we may assume $h(z,x)=(x,h_x(z)), z\in\mathbb{R}^k, x\in\mathbb{R}^n$. Assume that $(\partial h_x/\partial z_1,\dots,\partial h_x/\partial z_k)|\mathbb{R}^n\times 0$ is non singular. Then $\Sigma(h)$ is a smooth submanifold of dimension n, on which h restricts to a finite-to-one and generically immersive mapping to the discriminant set $D(h)$ assuming a certain generic condition. The Legendre submanifold associated to h is the image of the map

$$(x,\ h_x(z),\ \partial h_x/\partial x_1(z),\dots,\partial h_x/\partial x_n(z))$$

of $\Sigma(h)$ into $J^1(\mathbb{R}^n,\mathbb{R})$, which is nothing but the Nash blow up of the discriminant set $D(h)$. The family of functions h_x is called the *generating function.* It is known that all germs of Legendre submanifolds of $J^1(\mathbb{R}^n,\mathbb{R})$ are obtained in this way (see e.g. [17,24]).

Next we recall an idea from the paper [16]. Let $\mathcal{I}=(\mathcal{I}_x,\mathcal{I}_y,\mathcal{I}_p):\mathbb{R}^n\to J^1(\mathbb{R}^{n-1},\mathbb{R})=\mathbb{R}^{n-1}\times\mathbb{R}\times\mathbb{R}^{n-1*}$ be a germ of imbedding at $0\in\mathbb{R}^n$ transverse to the contact elements and assume it admits a complete integral s. Define the germ of Legendre imbedding

$$\tilde{\mathcal{I}}=(\mathcal{I}_x,s,\mathcal{I}_y,\mathcal{I}_p,\alpha):\mathbb{R}^n\to J^1(\mathbb{R}^n,\mathbb{R})=\mathbb{R}^{n-1}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}^{n*}$$

with a function α in $\mathbb{R}^n$ satisfying $\tilde{\mathcal{I}}^*_{\tilde\omega}=\mathcal{I}^*\omega+\alpha\ ds=0$, where $\omega,\tilde\omega$ are respectively the canonical contact forms of the jet spaces $J^1(\mathbb{R}^{n-1},\mathbb{R}), J^1(\mathbb{R}^n,\mathbb{R})$. Then by the above construction of Legendre submanifolds, the image of the $\tilde{\mathcal{I}}$ is identified with the image of a map

$$(x_1,\dots,x_n,h_x,\partial h_x/\partial x_1,\dots,\partial h_x/\partial x_n):\Sigma(x,h_x)\to J^1(\mathbb{R}^n,\mathbb{R}).$$

We identify the divergent diagram

$$\mathbb{R}\xleftarrow{\ s\ }\mathbb{R}^n\xrightarrow{ev\circ\mathcal{I}=(\mathcal{I}_x,\mathcal{I}_y)}\mathbb{R}^{n-1}\times\mathbb{R}$$

with the restriction of the divergent diagram

$$(*)\qquad \mathbb{R}\xleftarrow{\ x_n\ }\Sigma(x,h_x)\xrightarrow{f=(x_1,\dots,x_{n-1},h_x)}\mathbb{R}^{n-1}\times\mathbb{R}.$$

Then the integral curve $s^{-1}(d)\subset\Sigma(x,h_x)$ is the critical point set of the restriction of $f=(x_1,\dots,x_{n-1},h_x):\mathbb{R}^{n-1}\times t\times\mathbb{R}^k\to\mathbb{R}^{n-1}\times\mathbb{R}$. The solution D_d is the discriminant (critical value) set of the restriction, which is the intersection of the discriminant $D(x,h_x)$ with $\mathbb{R}^{n-1}\times d\times\mathbb{R}$.

In this way we are led to the study of the divergent diagrams (**) with corank $(f,x_n)=1$,

$$(**)\qquad \mathbb{R}\xleftarrow{\ s=x_n\ }\mathbb{R}^{n+k}\xrightarrow{\ f\ }\mathbb{R}^{n-1}\times\mathbb{R}.$$

Define the *solution* by $D_d=D(f,s)\cap\mathbb{R}^n\times d$: the discriminant set of the restriction $f_d:s^{-1}(d)\to\mathbb{R}^n$. We denote the family of the solutions $D_d, d\in\mathbb{R}$ by $\mathcal{W}_{f,s}$ and call the *solution web* of (f,s). On the one hand the diagram (**) can be regarded as a family of the restrictions of s to the fibres of f,

$$s_q:\ f^{-1}(q)\longrightarrow\mathbb{R}$$

with the parameter space $\mathbb{R}^n$. By definition of the solution web, for $q \in \mathbb{R}^n - D(f)$, $q \in D_d$ if and only if s_q has the critical value d.

Let $\epsilon(n)$ denote the local ring of the function germs on $\mathbb{R}^n$ at 0 and $m(n)$ the maximal ideal which consists of the function germs vanishing at 0. Similarly to the contact equivalence relation for map germs, we say divergent diagrams $(f,s),(g,t)$ are ***algebraically S-equivalent*** if there exist an $\mathbb{R}$-algebra isomorphism $\phi^* : Q(g) \to Q(f)$ and a germ of diffeomorphism $\chi : \mathbb{R},0 \to \mathbb{R},0$ such that $\phi^*(t) = \chi \circ s$, and we say ***strictly algebraically S-equivalent*** if χ is the identity, where $Q(f) = \epsilon(n+k)/f * m(n)$. Roughly stating $(f,s),(g,t)$ are algebraically S-equivalent if the restrictions s_o, t_o are equivalent.

Two diagrams $(f,s),(g,t)$ are *equivalent* if there exist germs of diffeomorphisms $\phi, \bar{\psi}, \psi$ such that the following diagram commutes

$$\begin{array}{ccccc} \mathbb{R}^1 & \xleftarrow{\;s\;} & \mathbb{R}^{n+k} & \xrightarrow{\;f\;} & \mathbb{R}^n \\ \chi\big\downarrow & & \bar{\psi}\big\downarrow & & \psi\big\downarrow \\ \mathbb{R}^1 & \xleftarrow[\;t\;]{} & \mathbb{R}^{n+k} & \xrightarrow[\;g\;]{} & \mathbb{R}^n. \end{array}$$

We denote this diagram by $(\chi, \bar{\psi}, \psi) : (f,s) \to (g,t)$ and call the triple an *equivalence.* By definition, $\psi(\mathcal{W}_{f,s}) = \mathcal{W}_{g,t}$ holds: $\phi(D_d) = D'_{\chi(d)}$ for $d \in \mathbb{R}$, where D'_d denotes the solution of (g,t). We say $(f,s),(g,t)$ are *strictly equivalent* if χ is the identity. An *unfolding of a diagram (f,s) of dimension s* is a pair of a diagram (F,S) and a triple of imbeddings $i = (i_1, i_2, i_3)$ such that i_3 is transverse to $F = \pi \circ (F,S)$ and (f,s) is given by the following commutative diagram of fiber product

$$\begin{array}{ccccc} \mathbb{R}^1 & \xleftarrow{\;S\;} & \mathbb{R}^{n+r+k} & \xrightarrow{\;F\;} & \mathbb{R}^{n+r} \\ \big\| & & i_2\big\uparrow & & i_3\big\uparrow \\ \mathbb{R}^1 & \xleftarrow[\;s\;]{} & \mathbb{R}^{n+k} & \xrightarrow[\;f\;]{} & \mathbb{R}^n. \end{array}$$

We denote $i : (f,s) \to (F,S)$ and call i a *morphism.*

In the manner of Thom-Mather theory we say diagrams $(f,s),(g,t)$ are (strictly) S-*equivalent* if they admit unfoldings, which are (strictly) equivalent.

Let $\theta(n)$ denote the $\epsilon(n)$-module of germs of smooth vector fields on $\mathbb{R}^n$ at 0. For a map germ $f : \mathbb{R}^{n+k}, 0 \to \mathbb{R}^n, 0$, let $\theta(f)$ denote the $\epsilon(n)$-module of germs of sections of the pull back $f^*T\mathbb{R}^n$. We say (f,s) is *(infinitesimally)* stable if the morphism

$$T(f,s) : \theta(n+k) \,\oplus\, \theta(n) \to \theta(s) \oplus \theta(f)$$

defined by

$$T(f,s)(\chi,\xi) = (ts(\chi),\; tf(\chi) - \omega f(\xi))$$

is surjective, where tf, ts denote the diferentials of f, s and ωf the pull back by df. The following is an easy consequence of the deformation theory of singularities of functions.

Theorem 1. *Let $\{(f_i, s_i), i = 1, \ldots, r\}$ be a generator over $\mathbb{R}$ of the module*

$$\frac{\theta(s) \oplus \theta(f)}{Im\ T(f,s) + f^*m(n)(\theta(s) \oplus \theta(f))}.$$

Then (f,s) admits the strictly stable unfolding (F,S) : $\mathbb{R}^{n+s+k}, 0 \longrightarrow \mathbb{R}^{n+r+1}, 0 \longrightarrow \mathbb{R}^{n+r}, 0$ defined by

$$F(z,u) = (f(z) + \sum_{i=1}^{r} u_i f_i,\ u), \quad S(z,u) = s(z) + \sum_{i=1}^{r} u_i s_i\ ,\ z \in \mathbb{R}^{n+k},\ u \in \mathbb{R}^r\ .$$

Stable diagrams possess the following properties.

(1) s is non singular,

(2) f is stable as a map germ, i.e. for any deformation g of f a germ of g at an (z', u') nearby the origin is equivalent to the germ f,

(3) $(f,s) : \mathbb{R}^{n+k}, 0 \to \mathbb{R}^{n+1}, 0$ is stable as a map germ,

(4) the rectriction $f_0 : s^{-1}(0), 0 \to \mathbb{R}^n, 0$ is stable as a map germ. In particular a solution web of a strictly stable-divergent diagram is a one parameter family of discriminant sets of stable map germs f_d. The restriction s_q has at most n critical values and the following conditions are equivalent.

(i) The web $\mathcal{W}_{f,s}$ is a non singular m-web at a q,

(ii) The first projection $\pi : D(f,s) \to \mathbb{R}^n$ is a non singular m-sheet covering over q,

(iii) $s_q : f^{-1}(q) \to \mathbb{R}$ is a Morse function with m distinct critical values.

The next theorem is fundamental in the singularity theory of composite map germs

Theorem 2. *Let $(f,s), (g,t) : \mathbb{R}^{n+k}, 0 \to \mathbb{R}^{n+1}, 0 \to \mathbb{R}^n, 0$ be strictly stable divergent diagrams. Then*

(1) $(f,s), (g,t)$ are strictly equivalent if and only if strictly algebraically S-equivalent.

(2) $(f,s), (g,t)$ are equivalent if and only if algebraically S-equivalent.

This reduces Problem (A) to the classification of germs of functions on varieties. In the complex analytic case the tuples of the critical values of s_q defines a mapping of the configuration space $\mathbb{C}^n$ onto the quotient space $\mathbb{C}^m/S(m)$ of $\mathbb{C}^m$ by the permutation group $S(m)$, m being the web number.

Define the $\mathbb{R}^+$-equivalence relation of functions germs of varieties to be generted by the S-equivalence and the relation $s_o \sim s_o + c, c \in \mathbb{R}$. A result due to Gryunov [14] is interpreted as follows.

Theorem 3. *Let $(f,s) : \mathbb{C}^{n+k}, 0 \to \mathbb{C}^{n+1}, 0 \to \mathbb{C}^n$ be a strictly stable divergent diagram. Then the singular locus $\mathrm{Sing}(\mathcal{W}_{f,s})$ of the solution web of the PDE associated to the diagram is a germ of a hypersurface. If the diagram is simple, i.e. s_0 is simple in the sense of the singularity of functions, the complement of $\mathrm{Sing}(\mathcal{W}_{f,s})$ is a $K(\pi, 1)$ space. Here π is a finite index subgroup of the braid group $S(m)$, m being the web number, and the index of π is the intersection number of $D_{d_1} \cdots D_{d_m}$ for generic distinct $d_1, \ldots, d_m$ close to 0.*

This suggests a relation to the ADE problem. Now we explain the relation of the function moduli and versal PDE. Let $i : (f,s) \to (F,S), (F,S) : \mathbb{R}^{n+r+k} \to \mathbb{R}^{n+r+1} \to \mathbb{R}^{n+r}$ a morphism into a strictly stable unfolding of dimension s given in Theorem 1. Given a deformation $(f_v, s_v), v \in \mathbb{R}^r$ of $(f,s) = (f_o, s_o)$, define the unfolding (F', S') of (F,S) with the parameter v by $F' = (F + (f_v - f\ ,\ o)\ ,\ v), S' = S + (s_v - s)$. Then by Theorem 1, (F', S') is stable and by Theorem 2 strictly equivalent to the trivial unfolding of (F,S) of dimension r. The composition of the trivialization of (F', S') and the projection of the trivial family to (F,S) restricts to the subfamily (f_v, s_v) to give a morphism into (F,S). By this idea we obtain

Theorem 4 (Function moduli). *Let $(f_v, s_v), v \in \mathbb{R}^{r'}$ be a smooth family of divergent diagrams. Then (f_v, s_v) is strictly equivalent to a germ of (f, s'_v) at an $z_v \in \mathbb{R}^{n+k}$ nearby 0, where s'_v is of the form*

$$s'_v(z) = S \circ i_{1v}(z) = s_0(z) + \sum_{j=1}^{r'} \alpha_{v,j}(z) \cdot s_j(z)$$

$$= s_0(z) + \sum_{j=1}^{r'} \beta_{v,j}(f(z)) \cdot s_j(z).$$

The second term in the theorem is called *function moduli.*

Theorem 5 (Versality theorem). *Let (g,t) be strictly algebraically S-equivalent to an (f,s) and assume f, g are stable and f is minimal: f is not equivalent to a trivial unfolding of an f'. Then (g,t) is strictly equivalent to a diagram $(f, s'), s' \in M$, where M is the $\epsilon(n)$-module generated by $s_i, i = 1, \dots, r$.*

Example (Verslity and function moduli).

Consider the following (non versal) differential equation in the x_1y-plane

$$y = x_1 y' + (y')^3. \tag{1}$$

This defines a nonsingular variety $V = \{y = x_1 p + p^3\} \subset J^1(\mathbb{R}, \mathbb{R})$, on which the projection ev to the base space restricts to the Whitney cusp mapping. The variety V is not transverse to the contact elements at the singular locus. This equation admits the family of algebraic solutions $y = x_1 x_2 + x_2^3$, with the parameter $x_2 \in \mathbb{R}$. The variety V is the image of the imbedding $\mathcal{I}(x_1, x_2) = (x_1, x_1x_2 + x_2^3, x_2)$ and the complete integral is given by $s = x_2$. This admits the Legendre imbedding into $J^1(\mathbb{R}^2, \mathbb{R})$ defined by $\tilde{\mathcal{I}}(x_1, x_2) = (x_1, x_2, x_1x_2 + x_2^3, x_2, x_1 + 3x_2^2)$, which is given by the generating function $h_x(z) = z^2 + x_1x_2 + x_2^3$.

Now we will construct the versal unfolding of (1). The divergent diagram

$$\mathbb{R} \xleftarrow{s = x_2} \mathbb{R}^3 \xrightarrow{f = (x_1, h_x)} \mathbb{R}^2$$

is not strictly stable i.e. a deformation is not strictly equivalent to the trivial unfolding. By Theorem 5.1, this admits the stable unfolding with one parameter

$$\mathbb{R} \xleftarrow{S = x_2} \mathbb{R}^4 \xrightarrow{F = (x_1, u, h_{x,u})} \mathbb{R}^3,$$

where $h_{x,u}(z) = z^2 + x_1(x_2 - u) + (x_2 - u)^3$. The singular locus $\Sigma(F,S)$ of the map $(F,S) : \mathbb{R}^4 \to \mathbb{R}^3 \times \mathbb{R}$ is the x_1x_2u-space defined by $z = 0$, on which the above divergent diagram restricts to the integral diagram

$$\mathbb{R} \xleftarrow{S=x_2} \Sigma(F,S) = \mathbb{R}^3 \xrightarrow{F} \mathbb{R}^3,$$

and the restriction of F is given by $F(x_1, x_2, u) = (x_1, u, x_1(x_2 - u) + (x_2 - u)^3)$. The level surfaces of the complete integral $s = x_2$ in $\mathbb{R}^3$ pjoject by F to the solutions in the x_1yu-space $\mathbb{R}^3$, which satisfy the following versal PDE

$$(2) \qquad \begin{Bmatrix} y = x_1 y_{x_1} + (y_{x_1})^3, \\ y_u = -x_1 - 3(y_{x_1})^2 \end{Bmatrix}$$

By definition $F^{-1}(x_1, u, y) \subset \mathbb{R} \times u \times \mathbb{R} = \mathbb{R}^2$ is the parallel translation of $f^{-1}(x_1, y) \subset \mathbb{R}^2$ by $(u, 0) \in \mathbb{R}^2$. Identifying $F^{-1}(x_1, u, y)$ with $f^{-1}(x_1, y)$ naturally, the restrictions $s_{x_1,y}, S_{x_1,u,y}$ of the complete integral x_2 satisfy $s_{x_1,y} + u = S_{x_1,u,y}$. Let i be the imbedding of x_1y-space into x_1uy-space defined by $i(x_1, y) = (x_1, \phi(x_1, y), y)$ and j the imbedding of x_1x_2z-space into x_1x_2uz-space defined by $j(x_1, x_2, z) = (x_1, x_2, \ \phi(x_1, h_x(z)), z)$. These imbeddings define the divergent diagram (g, t) by $F \circ j = i \circ g$ and $t = S \circ j$. Clearly $f = g$. By definition $S_{x_1,\phi(x_1,y),y} = s_{x_1,y} + \phi(x_1, y) : f^{-1}(x_1, y) \to \mathbb{R}$. This shows that t restricts to $s + \phi(f)$ on $\Sigma(g, t)$. The second term $\phi(f)$ is the *function moduli.* The solutions of the equation (1) are the transverse intersections of the solutions of this equation (2) with the x_1y-plane naturally imbedded in x_1yu-space, and any deformation of (1) is obtained by a suitable deformation of the natural imbedding.

Generic PDE's with complete integrals are classified in [16] as follows.

Theorem 6 [16]. *The diagrams* $(*)$ $(f, s) : \mathbb{R} \leftarrow \Sigma(f, s) \to \mathbb{R}^2$ *associated to generic differential equation in* xy*-plane are equivalent to one of the following five forms as divergent diagrams.*
(0) $f(z, x) = (z, x), \quad s(z, x) = z,$
(1) $f(z, x) = (z^2, x), \quad s(z, x) = z + x,$
(2) $f(z, x) = (z^2, x), \quad s(z, x) = z^3 + x,$
(3) $f(z, x) = (z^3 + xz, x), \quad s(z, x) = z + \phi(f),$
(4) $f(z, x) = (z^3 + xz, x), \quad s(z, x) = \frac{3}{4}z^4 + \frac{1}{2}xz^2 + \phi(f),$
(5) $f(z, x) = (z^3 + xz^2, x), \quad s(z, x) = z^2 + \phi(f),$
where ϕ *is an arbitrary smooth function defined on a neighbourhood of the origin in the* xy*-plane.*

In the above list the normal forms has at most "one" function moduli. This is explined as follows. Let $\mathcal{I}$ be a PDE on $\mathbb{R}^n$, $\mathcal{I}'$ its (mini) versal PDE on $\mathbb{R}^{n+s}$ with smallest s, and let $(f, s), (F, S)$ be their associated divergent diagrams. Define the partition $\mathcal{S}$ of the configuration space $\mathbb{R}^{n+s}$ by the $\mathbb{R}^+$-equivalence class of the singular points of the restrictions $S_{(q,u)}, (q, u) \in \mathbb{R}^{n+s}$. If $s_o = S_o$ is $\mathbb{R}^+$-simple, then by the definition of simplicity, $\mathcal{S}$ is locally finite stratification by submanifolds and generic perturbations of the natural imbedding of $\mathbb{R}^n$ into $\mathbb{R}^{n+s}$ are transversal to $\mathcal{S}$. This transversality means that the extended family $s_{q,c} = s_q + c, (q \in \mathbb{R}^n, c \in \mathbb{R})$ is a versal family and the unfolding (f', s'), $(f' = (f, c), s' = s + c)$, of dimension 1 is stable by the criterion in Theorem 1. By the property of stable diagrams, this has the web number at most $n+1$ hence (f, s) has the web number at most $n+1$ and by the argument of versality of stable diagrams (f, s) has at most one function moduli of type $\phi(f)$. Therefore it would be important to estimate the codimension c of the union of non $\mathbb{R}^+$-simple functions of varieties. Theorem 6 assirts that $c \geq 3$.

Theorem 7. *For $n < c$, generic PDE's have the web number less than or equal to $n+1$ and at most one function moduli of type $\phi(f)$.*

The function moduli of type $\phi(f)$ has two meanings. The first one is seen by the obvious calculation

$$\begin{aligned} I_\lambda(q) &= \int_{V_q} \exp \frac{i\,(s(p,q) \;+\; \phi \circ f(q))}{\lambda}\, dp \\ &= \exp i\frac{\phi \circ f(q)}{\lambda} \times \int_{V_q} \exp \frac{i\; s(p,q)}{\lambda}\, dp. \end{aligned}$$

The function moduli changes only the phase of the oscillatory integral. In particula the zero of the integral does not change. On the other hand the web structure of the wave fronts D_d changes topologically [11,12] and futhermore the contour of the phase function d (multi valued) on $\mathbb{R}^n$ form a quasi periodic (Penrose) tiling [30], which changes vigorously. The structure of those fine structure seems not yet well understood. Here we propose the approach from the web geometry.

$n+1$-webs of codimension 1 on $\mathbb{R}^n$ is one of the well-deveropped parts of the web geometry historically studied by Lie Darboux, Blashke, Chern, ... [3,9,13]. To give a brief introduction we assume $n = 2$. Assume that a germ of 3-web $\mathcal{W} = (\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3)$ of $\mathbb{R}^2$ is given by nonsingular 1-forms $\omega_1, \omega_2, \omega_3$ in general position. Since ω_i are linearly dependent we may assume $\omega_1 + \omega_2 + \omega_3 = 0$ by multiplying units to those 1-forms. It is easy to see that there exists the unique 1-form θ such that $d\omega_i = \theta \wedge \omega_i$ for $i = 1, 2, 3$. The affine connection on $\mathbb{R}^2$ associated to $\mathcal{W}$ is defined by the connection form $\begin{pmatrix} \theta & 0 \\ 0 & \theta \end{pmatrix}$. The derivative $K(\mathcal{W}) = d\theta$ is independent of the choice of ω_i and called the ***web curvature form.*** If the curvature form vanishes identically, $\mathcal{W}$ is diffeomorphic to the (hexagonal) web defined by parallel lines with three distinct directions. (The affine connections and web curvatures are defined also for $n+1$-webs of $\mathbb{R}^n$. See the book [3,13] for the details.)

Assume that a solution web $\mathcal{W}_\mathcal{I}$ of a PDE $\mathcal{I}$ on $\mathbb{R}^2$ has the web number 3 and has function moduli $\phi(f)$. On the nonsingular locus of $\mathcal{W}_\mathcal{I}$ the web curvature form $K(\mathcal{W}_\mathcal{I})$ is defined. Clearly the web curvature form is independent of the right equivalence $s \to \chi \circ s$ with a germ of diffeomorphism χ of $\mathbb{R}$. Therefore we obtain the morphism

$$K : \text{ Function moduli}/ \sim^{\text{right}} \;\to \text{Web curvature.}$$

Proposition 8. *Let (f, s) be the normal forms in Theorem 6 (3) - (5). Then there exists a germ of analytic (respectively formal) diffeomorphism χ in case (3) (resp. in cases (4), (5)) such that, respectively,*

(3) $\chi \circ \phi$ vanishes identically on the discriminant locus $D(f) = \{27u^2 + 4v^3 = 0\}$,

(4) $\chi \circ \phi = \pm v + av^2$ on the double point locus = v-axis, $a \in \mathbb{R}$,

(5) $\chi \circ \phi = \pm v + av^2$ on the cusp point locus = v-axis, $a \in \mathbb{R}$.

The proposition is proved by normalizing the dynamics on the range $\mathbb{R}$ of s, which sends a critical value of the function s_q to the other critical value for those q, respectively, in $D(f)$ and v-axis. The potential function s_q in Case (3) for $q = (0, a)$ is of the form

$$s_q(p) = q^4 + aq^2 + \phi(a),$$

which has the critical values $\phi(a)$ and $\phi(a) + \frac{1}{4}a^2$. Assume that ϕ is nonsingular. Then the dynamics which sends the first to the later is smoothly conjugate to the diffeomorphism

$$a + \frac{1}{4}(\phi^{-1}(a))^2.$$

It is known [28] that germs of diffeomorphism h of $\mathbb{R}, 0$ (as well as $\mathbb{C}$) are formally classified by their residue. Here we introduce the (reduced) *residue* $\mathrm{Res}(f) = \mathrm{res}(f) + \frac{k+1}{2}$ (where f is k-flat), which is defined by

$$f^{\circlearrowleft^k} = f^{(2\pi\sqrt{-1}\ \mathrm{Res}(f))},$$

in other words, writing formally as $f = \exp \chi$ with a formal vector field χ on $\mathbb{R}$,

$$\exp \circlearrowleft^k \chi = \exp\,(2\pi\sqrt{-1}\ \mathrm{res}(\chi))\ \chi.$$

Here $\circlearrowleft^k$ stands for the "analytic" continuation of the iteration f^t of t-times, t being a complex number, along the k-fold anti clockwise cycle from 0 to a nearby point in the time space $\mathbb{C}$. The residue of our dynamics $a + \frac{1}{4}(\phi^{-1}(a))^2$ is a formal invariant under the right equivalence. (It is proved in [20] that the formal conjugacy is realized by C^∞-conjugacy by a result due to Takens.) Introducing in this way the residue seems an adhoc invariant, although, the following Proposition 9 suggests there might be an "intrinsic" definition in terms of the web curvature form as well as the affine connection of the solutions.

Proposition 9. *For the webs of the normal form in (3),*

(1) Function moduli/ $\sim^{\text{right}} = \{\phi \mid \phi \equiv 0$ on $D(f)\}$,

(2) the web curvature form $K(\mathcal{W}_\mathcal{I})$ extends analytically to $\mathbb{R}^2$,

(3) K is a bijection of the reduced function moduli and the set of germs of analytic 2-forms.

The statement in (1) follows from Proposition 8 (3). The statement (2) follows from the theory of invariant forms. The statement (3) is shown by a direct calculation and remains valid for some other cases. This work is in progress involving the Gauss map of the webs determined by the frame of the tangent space by the differential of the phase function s.

Finally I would propose the following questions.

(1) Is K injective?

(2) Study the singularities of $K(\mathcal{W}_\mathcal{I})$.

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