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# On Regular Algebras

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**Abstract**

The notation of a (non-commutative) regular, graded algebra is introduced in [AS]. The results of that paper, combined with those in [ATV1], gives a complete description of the regular graded ring of (global) dimension three. Further M.Artin [A] defined Quantum Proj for non-commutative graded algebras and studied projective geometry of quautum proj.

In this paper, we shall explain those results.

## 1 Regular algebras

Let $k$ be an algebraically closed field of characteristic zero. A graded algebra $A$ will mean a (connected) $\mathbf{N}$ - graded algebra, generated in degree one; thus $A = \bigoplus_{i\geq 0} A_i$, where $A_0 = k$ is central, $\dim_k A_i < \infty$ for all $i$, and $A$ is generated as an algebra by $A_1$. M.Artin and W.Schelter defined the regular graded algebra as follows.

**Definition 1** *A graded algebra $A$ is regular of dimension $d$ provided that*
*(1) $A$ has global dimension $d$; that is every graded (left) $A$ -modules has projective dimension $\leq d$*
*(2) $A$ has polynomial growth; that is there exists $\rho \in \mathbf{R}$ such that $\dim A_n \leq n^\rho$ for all $n$.*
*(3) $A$ is Gorenstein; that is $Ext^q_A(k, A) = \delta_{d,q} k$*

These conditions put strong restriction on $A$. For example, if $A$ is commutative, and regular, then $A$ must be a polynomial ring. If d = 1, the only such $A$ is the polynomial ring $k[x]$. If $d = 2$, then $A$ is of the form $k\langle x, y\rangle$ (free algebra of rank two) with a single quadratic relation, which is either $yx - xy = x^2$, or $yx = \lambda xy$ for some $0 \neq \lambda \in k$. In particular, the quantum plane gives a regular algebra. If $d = 3$, then things begin to get interesting. there are 13 class of regular algebras (for detailed see [AS],[ATV1]), these algebras are of the forms $k\langle x, y\rangle$ with two cubic relations, or $k\langle x, y, z\rangle$ with three quadratic relations. However two such classes are of particular interest.

Fix $(a,b,c) \in \mathbf{P}^2$, and let $A = \mathbf{C}\langle x,y,z\rangle$ with defining relations

$$ax^2 + byz + czy = 0$$

$$ay^2 + bzx + cxz = 0$$

$$az^2 + bxy + cyx = 0$$

This algebra is very closely related to the subvariety of $\mathbf{P}^2$, $E$ say, defied by the equation $(a^3+b^3+c^3)xyz - abc(x^3+y^3+z^3) = 0$. Usually $E$ is an elliptic curve. If $(a,b,c) = (0,1,-1)$, then $E = \mathbf{P}^2$ and $A$ is the polynomial ring. Suppose that $(a,b,c)$ is such that $E$ is an elliptic curve. Then $A$ is regular algebra, and noetherian domain. In general, let $A$ be a graded algebra of the form

$$A = k\langle x_1,\cdots,x_r\rangle/(f_1,\cdots,f_s)$$

where $f_i$ are homogeneous elements. Then multilinearization of $\{f_1,\cdots,f_s\}$ defines a scheme $E$ in $(\mathbf{P}^{r-1})^{s-1}$. Further projective scheme $E$ define the homogeneous coordinate ring $B$. This is isomorphic to $\bigoplus_{n\geq 0}\Gamma(E,\varphi)$, where $\varphi$ is the invertible sheaf $vartheta(1)$. Let $\sigma$ be an automorphism of $E$ and denote the pullback $\sigma^*\varphi$ by $\varphi^\sigma$, then we set

$$B_n = \Gamma(E, \varphi\otimes\varphi^\sigma\otimes\cdots\otimes\varphi^{\sigma^{n-1}})$$

for all $n \geq 0$ and $B = \bigoplus_{n\geq 0} B_n$. Multiplication of section is defined by the rule that if $a \in B_m$ and $b \in B_n$, then

$$a\cdot b = a\otimes b^{\sigma^m}$$

If $E = Spec(R)$ and $\sigma$ is an automorphism of $E$, then $B = R[t,t^{-1};\sigma]$, where $ta = a^\sigma t$. If $A$ is a regular algebra, then the next theorem is proved in [ATV1].

**Theorem 1** *If $A$ is a regular algebra of dimension 3, then* $\dim E = 1,2$. *If* $\dim E = 1$, *then $A/gA \cong B^\sigma$, where $g$ is an element of $A$ such that $gA = Ag$. If* $\dim E = 2$, *then $A \cong B$.*

Next suppose that $d = 4$. Not all the regular algebras are known for $d = 4$, however there is one class that has been studied to some extent. This is a family of algebras defined by E.Sklyanin [Sk1],[Sk2]. Let $(\alpha,\beta,\gamma) \in \mathbf{P}^3$ lie on the surface $\alpha+\beta+\gamma+\alpha\beta\gamma = 0$. Let $A = \mathbf{C}\langle a,x,y,z\rangle$ with defining relations

$$ax - xa = \alpha(yz+zy) \quad xy - yx = az + za$$

$$ay - ya = \beta(xz+zx) \quad yz - zy = ax + xa$$

$$az - za = \gamma(xy + yx) \quad zx - xz = ay + ya$$

If $\{\alpha, \beta, \gamma\} \cap \{0, +1, -1\} = \emptyset$, then $A$ is a regular algebra of dimension 4, and has the same Hilbert series as the polynomial ring. Further if $(\alpha, \beta, \gamma) = (0, \delta, -\delta)$ $(\delta \neq 0, -1)$, then $A$ is a quotient of $U_q(sl(2))$ (quantum group of $sl(2)$).

# 2 Quantum Proj

Let $A$ be a finitely generated commutative graded $k$ - algebra which is generated in degree 1. Let $X = Proj(A)$, and denote by $C$ the quotient category $(gr - A)/\tau$, where $(gr - A)$ is the category of finite graded $A$ - modules and $\tau$ is its full subcategory of modules of finite length. Serre's theorem (cf. [Se]) asserts that there is a natural equivalence of categories

$$\tau \longrightarrow (mod - \vartheta)$$

between the quotient category $\vartheta$ and the category $(mod - \vartheta)$ of coherent sheaves on $Proj(A)$. The shift $M(\mu)$ of module $M$, defined by $M(\mu)_n = M_{n+\mu}$, correspond to the tensor product by the polarizing invertible sheaf:

$$M \rightsquigarrow M(1) = M \otimes \vartheta(1)$$

This shift operation defines an autoequivalence of $C$. The class of $A$ - modules which corresponds to a coherent sheaf $M$ on $X$ is represented by the module

$$\Gamma(M) := \bigotimes_{n=0}^{\infty} \Gamma(X, M(n))$$

In particular, $\Gamma(\vartheta) = \otimes_n \Gamma(X, \varphi^{\otimes n})$ agree with in a sufficient high degree, where $\varphi$ is a invertible sheaf. Thus $Proj(A)$ can recovered from category $C$.

M.Artin (cf.[A],[ATV1],[AV]) has used this correspondence to define quantum Proj.

**Definition 2** *Let $A$ be a non-commutative graded algebra, generated in degree 1. Then $Proj(A)$ is the triple $(C, \vartheta, s)$, where $C = (gr - A)/\tau$, $\vartheta$ is the object of $C$ which is represented by the right module $A$, and $s$ is the operation $M \rightsquigarrow M(1)$ on $C$ induced by the shift of degree on an $A$ - modules.*

Suppose that $R = \mathbf{C}[x_0, \cdots, x_n]/J$ is a graded quotient ring of the commutative polynomial ring endowed with its ususal graded structure. Let $V(J) \subset \mathbf{P}^n$ be the projective variety cut out by $J$. To each point $p \in V(J)$ we may associate the graded $R$ - module $M(p) = R/I(p) \cong \mathbf{C}[X]$, where $I(p)$ is the ideal generated by the homogeneous polynomials vanishing at $p$. Since $\mathbf{C}[X]$ is a domain, every proper quatient of $M(p)$ is finite dimensional, whence $M(p)$ is an irreducible object in $Proj(R)$. This motivates the following definition.

**Definition 3** *([A], [ATV2]) A point module is a graded cyclic A - module M with Hilbert series* $(1-t)^{-1}$.
*A line module is a graded cyclic A - module M with Hilbert series* $(1-t)^{-2}$
*A plane module is a graded cyclic A - module M with Hilbert series* $(1-t)^{-3}$

By using these modules, projective geometry over graded regular algebras of dimension 3 (quantum plane) is expanded (cf. [A]). In the case of dimension 4, projective geometry of regular algebra which obtained by homogenization of $sl(2)$ ([LBS]).

# 3 Remark and Problem

(1) In the definition of regular algebras, can the Gorenstein condition be changed to domain ? This is true in the case that $gl.dimA \leq 2$ (cf [K1]) and it is known that regular algebras of dimension $\leq 4$ are Noetherian domain (cf. [SS]).
(2) In the non-graded case, is it possible to define a quantum algebraic geomerty ? One direction has suggested by Manin ([M1],[M2]).

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