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On the solution complexes of confluent hypergeometric $\mathcal{D}$-modules

お茶の水女子大学人間文化研究科　石塚寿美子
(Sumiko Ishizuka, Ochanomizu University)
お茶の水女子大学理学部数学科　真島　秀行
(Hideyuki Majima, Ochanomizu University)

1 A point of view for Binet-Stirling formula

The function $\Gamma(z)$ is a meromorphic function in the complex plan, which has the integral representation

$$\Gamma(z)=\int_0^\infty \exp(-\xi)\xi^{z-1}d\xi,$$

and the infinite product representation

$$\frac{1}{\Gamma(z)}=z\exp(\gamma z)\prod_{n=1}^{\infty}(1+\frac{z}{n})\exp(-\frac{z}{n}).$$

It satisfies the functional equalities

$$\Gamma(z+1)=z\Gamma(z),\quad \Gamma(1)=1,$$

and

$$\Gamma(z)\Gamma(1-z)=\frac{\pi}{\sin\pi z}.$$

We also have so-called Binet (1820)-Stirling formula,

$$\begin{aligned}\log\Gamma(z) &= (z-\frac{1}{2})\log z - z+\frac{1}{2}\log 2\pi+\frac{1}{12z}-\frac{1}{360z^3}+\frac{1}{1260z^5}\\ &+ \cdots+\frac{(-1)^{N-1}B_N}{2N(2N-1)z^{2N-1}}+E_N(z),\end{aligned}$$

$$|E_N(z)|\le\frac{K_zB_{N+1}}{(2N+2)(2N+1)|z|^{2N+1}},$$

$$(|z| \to \infty, \quad |\arg z| \leq \frac{1}{2}\pi - \epsilon, \quad K_z \leq \csc 2\epsilon \quad (0 < \epsilon < \frac{1}{4}\pi))$$

where B_N $(N = 0, 1, 2, ...)$ are Bernoulli numbers and therefore we have

$$\lim_{|z|\to\infty} |z^N E_N(z)| = 0.$$

Poincaré obtained the concept of asymptotic expansion from the formula. By his terminology,

$$J(z) = \log\Gamma(z) - (z - \frac{1}{2})\log z + z - \frac{1}{2}\log 2\pi$$

is asymptotically developable to the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}B_n}{2n(2n-1)z^{2n-1}},$$

which does not converge, because of

$$\lim_{n\to\infty} \frac{B_n 2n(2n-1)}{B_{n+1}2(n+1)(2n+1)} = 0.$$

The Binet-Stirling formula is derived form Binet's integral formulae

$$J(z) = \int_0^\infty e^{-zt}\left(\frac{t}{2} - 1 + \frac{t}{e^t - 1}\right)\frac{dt}{t^2} \quad (\Re z > 0) \quad \text{(Binet's 1st integral formula)},$$

$$J(z) = -\int_0^\infty e^{-zt}\left(\frac{t}{2} + 1 - \frac{t}{1 - e^{-t}}\right)\frac{dt}{t^2} \quad (\Re z > 0) \quad \text{(Binet's 1st integral formula)},$$

$$J(z) = 2\int_0^\infty \frac{\arctan\frac{t}{z}dt}{e^{2\pi t} - 1} \quad (\Re z > 0) \quad \text{(Binet's 2nd integral formula)},$$

$$J(z) = \frac{1}{\pi}\int_0^\infty \frac{z}{t^2 + z^2}\log\frac{1}{1 - e^{-2\pi t}}dt \quad (\Re z > 0),$$

and we have these formulae by using

$$\frac{d^2}{dz^2}\log\Gamma(z) =$$

$$\sum_{n=0}^{\infty}\frac{1}{(z+n)^2} = \frac{1}{z^2} + \sum_{n=0}^{\infty}\int_0^\infty te^{-t(z+n)}dt = \frac{1}{z^2} + \int_0^\infty e^{-zt}\frac{t}{e^t - 1}dt,$$

$$\sum_{n=0}^{\infty}\frac{1}{(z+n)^2} = \frac{1}{z} + \frac{1}{2z^2} + \int_0^\infty \frac{4tz}{(z^2+t^2)^2}\frac{dt}{e^{2\pi t} - 1}.$$

Fom the Binet's second integral formula, we have

$$E_n(z) = \frac{2(-1)^n}{z^{2n-1}}\int_0^\infty \left\{\int_0^t \frac{u^{2n}du}{u^2 + z^2}\right\}\frac{dt}{e^{2\pi t} - 1},$$

from which we have the estimate(for example, see Whittaker-Watson [13]) and by using

$$2\pi = \left(\limsup \sqrt[2n]{\frac{|B_n|}{(2n)!}}\right)^{-1},$$

we have also the estimate with Gevrey order 1=2-1

$$\left|\frac{(-1)^{N-1}B_N}{2N(2N-1)}\right| \leq K((2N-2)!)^1 \left(\frac{1}{2\pi}\right)^{2N-2},$$

$$|E_N(z)| \leq K(2N)!^1 \left(\frac{1}{2\pi}\right)^{2N} |z|^{-2N},$$

$$(|z| \to \infty, \quad |\arg z| \leq \frac{1}{2}\pi - \epsilon, \quad (0 < \epsilon < \frac{1}{4}\pi)).$$

According to the Binet's first integral formula, we know the following remarkable thing:the difference equation

$$J(z+1) - J(z) = -1 - \left(z + \frac{1}{2}\right)\log\left(1 + \frac{1}{z}\right)$$

has a formal power-series solution

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}B_n}{2n(2n-1)z^{2n-1}}$$

of which the Borel transform is equal to

$$\left(\frac{t}{2} - 1 + \frac{t}{e^t - 1}\right)\frac{1}{t^2} = -\left(\frac{t}{2} + 1 - \frac{t}{1 - e^{-t}}\right)\frac{1}{t^2}$$

and as the Laplace transform, we have

$$J(z) = \log\Gamma(z) - \left\{\left(z - \frac{1}{2}\right)\log z - z + \frac{1}{2}\log 2\pi\right\}.$$

Then, we can derive the Binet-Stirling formula by using Watson's Lemma: $q(t)$ has N-th derivative and

$$|q^{(k)}(t)| \leq Me^{\sigma t} \quad (k = 0, \quad 1, \cdots, N),$$

then

$$\int_0^{\infty} e^{-zt}q(t)dt = \sum_{k=0}^{N-1} \frac{q^{(k)}(0)}{z^{k+1}} + \frac{M}{|z|^N(\Re z - \sigma)}$$

$$(|z| \to \infty, \quad |\arg z| \leq \frac{1}{2}\pi - \epsilon, \quad (0 < \epsilon < \frac{1}{4}\pi)).$$

2 Poincaré's asymptotic expansion and asymptotic expansion with Gevrey order

A function $f(z)$ defined on S is asymptotically developpable to a formal series $\hat{f}(z) = \sum_{k=0}^{\infty} a_k z^{-k}$ as $|z| \to \infty$ in the sense of Poincaré , if, for any positive integer N and for any open subsector S' , we have

$$|f(z) - \sum_{k=0}^{N-1} a_k z^{-k}| \leq constant|z|^{-N},$$

where the series is said to be asymptotic series. A function defined in a sector S at the infinity has an asymptotic expansion with Gevrey order $\sigma = s - 1$ as $|z| \to \infty$, if it is asymptotically developpable and the asymptotic sereis $\hat{f}(z)$ satisfies the following conditions:

$$|a_k| \leq C(k!)^s A^k \quad (k = 0, \quad 1, \quad 2, \cdots),$$

and for any integer N and for any subsector S', there exists K and B ,

$$|f(z) - \sum_{k=0}^{N-1} a_k z^{-k}| \leq K(N!)^{\sigma} B^N |z|^{-N}.$$

3 Index theorems of ordinary differential operator and its irregularity

Consider a linear ordinary differential operator with coefficients in holomorphic functions at the origin in the complex plan

$$Pu = (\sum_{i=0}^{m} a_i(x)(d/dx)^i)u.$$

Let $\mathcal{O}$,$\hat{\mathcal{O}}$, $\mathcal{K}$, $\hat{\mathcal{K}}$ and $\mathcal{E}$ be the ring of convergent power-series, the ring of formal power-series, the ring of convergent Laurent series with finite negative order terms, the ring of formal, the ring of formal Laurent series with finite negative term and the ring of convergent Laurent series, respectively.

Denote by F one of $\mathcal{O}$, $\hat{\mathcal{O}}$, $\mathcal{K}$, $\hat{\mathcal{K}}$ and $\mathcal{E}$. We consider P as an operator from F to itself. Then, $\mathrm{Ker}(P; F)$ and $\mathrm{Coker}(P; F)$ are finite dimensional, and has a index $\chi(P; F) = \dim_C \mathrm{Ker}(P; F) - \dim_C \mathrm{Coker}(P; F)$, which can be calculated as follow:

$\chi(P; \mathcal{O}) = m - v(a_m)$, $\chi(P; \hat{\mathcal{O}}) = \sup\{i - v(a_i) : i = 1, ..., m\}$,
$\chi(P; \mathcal{K}) = m - v(a_m) - \sup\{i - v(a_i) : i = 1, ..., m\}$, $\chi(P; \hat{\mathcal{K}}) = 0$,
$\chi(P; \mathcal{E}) = 0$.

At the origin, the folloings are the same and the quantity is said to be the irregularity of P at the origin, denoted by $\mathrm{Irr}(P)_0$:

$\chi(P;\hat{\mathcal{O}})-\chi(P;\mathcal{O})$, $\chi(P;\hat{\mathcal{O}}/\mathcal{O})$,
$\chi(P;\hat{\mathcal{K}})-\chi(\mathcal{K})$, $-\chi(P;\mathcal{K})$, $\chi(P;\hat{\mathcal{K}}/\mathcal{K})$,
$\chi(P;\mathcal{E})-\chi(P;\mathcal{K})$, $\chi(P;\mathcal{E}/\mathcal{K})$,
$\chi(P;\mathcal{E}/\mathcal{O})-\chi(P;\mathcal{K}/\mathcal{O})$,
$\dim_C \mathrm{Ker}(P;\hat{\mathcal{O}}/\mathcal{O})$,
$\dim_C \mathrm{Ker}(P;\hat{\mathcal{K}}/\mathcal{K})$,
$\dim_C \mathrm{Ker}(P;\mathcal{E}/\mathcal{K})$,
$\dim_C \mathrm{Ker}(P;(\mathcal{E}/\mathcal{O})/(\mathcal{K}/\mathcal{O}))$.

In spiring the characterization of regular singularity by Fuchs using the coefficients and by Deligne as the validity of comparison theorem, Malrange [9] got another characterization:

The opetator P is regular singular at the origin if and only if

$$\sup\{i-v(a_i) : i+1,...,m\}-\{m-v(a_m)\}=0,$$

which is equivalent to
(zero irregularity)

$$\mathrm{Irr}(P)_0=0,$$

(validity of comparison theorem between $\mathcal{O}$ and $\hat{\mathcal{O}}$)

$$\mathrm{Ker}(P;\hat{\mathcal{O}})\simeq \mathrm{Ker}(P;\mathcal{O}),\quad \mathrm{Coker}(P;\hat{\mathcal{O}})\simeq \mathrm{Coker}(P;\mathcal{O}),$$

(validity of comparison theorem between $\mathcal{K}$ and $\hat{\mathcal{K}}$)

$$\mathrm{Ker}(P;\hat{\mathcal{K}})\simeq \mathrm{Ker}(P;\mathcal{K}),\quad \mathrm{Coker}(P;\hat{\mathcal{K}})\simeq \mathrm{Coker}(P;\mathcal{K}),$$

(validity of comparison theorem between $\mathcal{K}$ and $\mathcal{E}$, Deligne [1])

$$\mathrm{Ker}(P;\mathcal{E})\simeq \mathrm{Ker}(P;\mathcal{K}),\quad \mathrm{Coker}(P;\mathcal{E})\simeq \mathrm{Coker}(P;\mathcal{K}).$$

Let $\mathcal{D}_0$ be the sheaf of germs of linear ordinary differential operators with holomorphic coefficients, and put $\mathcal{M}_0=\mathcal{D}_0/\mathcal{D}_0P$. Then, $\mathcal{M}_0$ has a projective resolution

$$0\leftarrow \mathcal{M}_0\leftarrow \mathcal{D}_0 \xleftarrow{P} \mathcal{D}_0\leftarrow 0,$$

from which, by operating the functor $\mathcal{H}om_{\mathcal{D}_0}(\cdot,\mathcal{F}_0)$, we have the solution complex with values in $\mathcal{F}$ at the origin,

$$\mathcal{S}ol(\mathcal{M}_0,\mathcal{F}_0) : \mathcal{F}_0 \xrightarrow{P} \mathcal{F}_0\rightarrow 0.$$

We have the isomorphism:

$$\mathrm{Ext}^0(\mathcal{M}_0, \mathcal{F}_0) \simeq \mathrm{Ker}(\mathcal{F}_0; P), \quad \mathrm{Ext}^1(\mathcal{M}_0, \mathcal{F}_0) \simeq \mathrm{Coker}(\mathcal{F}_0; P).$$

Therefore, the index as $\mathcal{D}$-module at the origin,

$$\chi(\mathcal{M}; \mathcal{F})_0 = \dim_C \mathrm{Ext}^0(\mathcal{M}_0, \mathcal{F}_0) - \dim_C \mathrm{Ext}^1(\mathcal{M}_0, \mathcal{F}_0),$$

is equal to the index $\chi(P; F)$, and the irregularity as $\mathcal{D}$-module at the origin,

$$\mathrm{Irr}(\mathcal{M})_0 = \chi(\mathcal{M}_0; \hat{\mathcal{O}}) - \chi(\mathcal{M}_0; \mathcal{O}),$$

is equal to the irregularity $\mathrm{Irr}(P)_0$ and

$$\mathrm{Irr}(\mathcal{M})_0 = \chi(\mathcal{M}_0; \hat{\mathcal{K}}) - \chi(\mathcal{M}_0; \mathcal{K}),$$

$$\mathrm{Irr}(\mathcal{M})_0 = \chi(\mathcal{M}_0; \mathcal{E}) - \chi(\mathcal{M}_0; \mathcal{K}),$$

$$\mathrm{Irr}(\mathcal{M})_0 = \chi(\mathcal{M}_0; \mathcal{E}/\mathcal{O}) - \chi(\mathcal{M}_0; \mathcal{K}/\mathcal{O}).$$

Ramis [10], [11] obtained index theorems with Gevrey order.

4 Indices of holonomic $\mathcal{D}$-modules and their irregularities

Let $\mathcal{D}$ be the sheaf of germs of linear partial differential operetors with coefficients of holomorphic functions on a manifold M and let $\mathcal{M}$ be a holonomic $\mathcal{D}$-module. The module $\mathcal{M}$ has a projective resolution

$$0 \leftarrow \mathcal{M} \leftarrow \mathcal{D}^{m_0} \xleftarrow{P_0} \mathcal{D}^{m_1} \xleftarrow{P_1} \mathcal{D}^{m_2} \xleftarrow{P_2} \cdots \xleftarrow{P_{2n-1}} \mathcal{D}^{m_{2n}} \leftarrow 0$$

from which, by operating the functor $\mathcal{H}om_{\mathcal{D}}(\cdot, \mathcal{F})$, we have the solution complex with values in $\mathcal{F}$,

$$\mathcal{S}ol(\mathcal{M}, \mathcal{F}) : \mathcal{F}^{m_0} \xrightarrow{P_0^t} \mathcal{F}^{m_1} \xrightarrow{P_1^t} \cdots \xrightarrow{P_{2n-1}^t} \mathcal{F}^{m_{2n}} \rightarrow 0.$$

For a point p, the index of holonomic $\mathcal{D}$-module $\mathcal{M}$with values in $\mathcal{F}$ is defined by

$$\chi(\mathcal{M}; \mathcal{F})_p = \sum_{i=0}^{2n} \dim_C (-1)^i \mathcal{E}xt^i(\mathcal{M}, \mathcal{F})_p.$$

For the point p, the irregularity of holonomic $\mathcal{D}$-module $\mathcal{M}$is defined by

$$\mathrm{Irr}(\mathcal{M})_p = \chi(\mathcal{M}; \mathcal{O}_{M\hat{|}H}))_p - \chi(\mathcal{M}; \mathcal{O}_{M|H})_p,$$

where $\mathcal{O}$ is the sheaf of germs of holomorphic functions on M , H is the set of singular points of $\mathcal{M}$, $\mathcal{O}_{M|H}$ is the zero-extension of the restriction of $\mathcal{O}$ on H and $\mathcal{O}_{M\hat{|}H}$ is the Zariski completion of $\mathcal{O}$ along H.

5 Holonomic $\mathcal{D}$-module defined by confluent hypergeometric partial differential equations Φ_2

In the sequel, we consider the solution complexes of holonomic $\mathcal{D}$-module defined by confluent hypergeometric partial differential equations Φ_2 and give the calculation of the cohomology groups.

We put $M = P^1_C \times P^1_C$ and $H = \{(\infty, y); y \in P^1_C\} \cup \{(x, \infty); x \in P^1_C\}$.
For a domain U included in $\{(\infty, y); y \in P^1_C\}$, we define

$$\mathcal{O}_{\widehat{M|H},s,A}(\infty, U) = \left\{\sum_{j\geq 0} f_j(y)x^{-j}; \exists C > 0, \forall n, s.t. \sup_{y\in U} |f_n(y)| < CA^n\{(n-1)!\}^{s-1}\right\},$$

and for a domain V included in $\{(x, \infty); x \in P^1_C\}$, we define

$$\mathcal{O}_{\widehat{M|H},s,A}(V, \infty) = \left\{\sum_{j\geq 0} f_j(x)y^{-j}; \exists C' > 0, \forall n, s.t. \sup_{x\in V} |f_n(x)| < C'A^n\{(n-1)!\}^{s-1}\right\}.$$

For a point $p \in H\backslash(\infty, \infty)$, if $p \in \{(\infty, y); y \in P^1_C\}$ then we put

$$(\mathcal{O}_{\widehat{M|H},s,A})_p = \operatorname*{Ind\,lim}_{p\in U\subset H} \mathcal{O}_{\widehat{M|H},s,A}(\infty, U),$$

and if $p \in \{(x, \infty); x \in P^1_C\}$, then we put

$$(\mathcal{O}_{\widehat{M|H},s,A})_p = \operatorname*{Ind\,lim}_{p\in V\subset H} \mathcal{O}_{\widehat{M|H},s,A}(V, \infty).$$

We define as follow:

$$\begin{aligned}
(\mathcal{O}_{\widehat{M|H},s})_p &= \operatorname*{Ind\,lim}_{A>0}(\mathcal{O}_{\widehat{M|H},s,A})_p\,, \\
(\mathcal{O}_{\widehat{M|H},(s)})_p &= \operatorname*{Proj\,lim}_{A>0}(\mathcal{O}_{\widehat{M|H},s,A})_p\,, \\
(\mathcal{O}_{\widehat{M|H},s,A-})_p &= \operatorname*{Ind\,lim}_{0<B<A}(\mathcal{O}_{\widehat{M|H},s,B})_p\,, \\
(\mathcal{O}_{\widehat{M|H},(s,A+)})_p &= \operatorname*{Proj\,lim}_{B>A}(\mathcal{O}_{\widehat{M|H},s,B})_p\,.
\end{aligned}$$

The system of confluent hypergeometric partial differential equations Φ_2 [2] is as follows:

$$\Phi_2 : \begin{cases} x\dfrac{\partial^2 u}{\partial x^2} + y\dfrac{\partial^2 u}{\partial x \partial y} + (c-x)\dfrac{\partial u}{\partial x} - bu = 0 & \text{(denoted by} L_1u = 0) \\ y\dfrac{\partial^2 u}{\partial y^2} + x\dfrac{\partial^2 u}{\partial x \partial y} + (c-y)\dfrac{\partial u}{\partial y} - b_p u = 0 & \text{(denoted by} L_2u = 0) \end{cases}$$

where b, b_p, c are not non-negative integers.
We consider the $\mathcal{D}$-module $\mathcal{M}_2$ defined by Φ_2, namely we put

$$\mathcal{M}_2 = \mathcal{D}/(\mathcal{D}L_1 + \mathcal{D}L_2).$$

We have a projective resolution

$$0 \longleftarrow \mathcal{M}_2 \longleftarrow \mathcal{D} \longleftarrow \mathcal{D}^3 \longleftarrow \mathcal{D}^2 \longleftarrow 0$$

and we have the solution complex $Sol(\mathcal{M}_2, \mathcal{F})$ with values in $\mathcal{F}$

$$\mathcal{F} \xrightarrow{\nabla_0} \mathcal{F}^3 \xrightarrow{\nabla_1} \mathcal{F}^2 \longrightarrow 0,$$

where

$$\nabla_0 = \begin{pmatrix} L_1 \\ L_2 \\ L_3 \end{pmatrix},$$

$$\nabla_1 = \begin{pmatrix} -L_2 & L_1 & 0 \\ \dfrac{\partial}{\partial y} & -\dfrac{\partial}{\partial x} & 1 \end{pmatrix}.$$

by using Takayama's Kan [12] and we have the following

Theorem 1 . *Let* $M = P^1_C \times P^1_C$, $H = \{(\infty, y); y \in P^1_C\} \cup \{(x, \infty); x \in P^1_C\}$, $p \in H \backslash (\infty, \infty)$ *be as above. The dimensions of chohomology groups of the solution complexes for the* $\mathcal{D}$*-module defined by* Φ_2 *are as folow:*

(1) *If* $1 \leq s < 2$,

for $\mathcal{F} = \mathcal{O}_{\widehat{M|H},(s)}, \mathcal{O}_{\widehat{M|H},s,A-}, \mathcal{O}_{\widehat{M|H},(s,A+)}, \mathcal{O}_{\widehat{M|H},s}$,

$$\dim_C \mathrm{Ext}^j((\mathcal{M}_2)_p, \mathcal{F}_p) = \begin{cases} 0, & (j = 0, 2) \\ 1, & (j = 1) \end{cases}$$

(2) *If* $s > 2$,

for $\mathcal{F} = \mathcal{O}_{\widehat{M|H},(s)}, \mathcal{O}_{\widehat{M|H},s,A-}, \mathcal{O}_{\widehat{M|H},(s,A+)}, \mathcal{O}_{\widehat{M|H},s}$,

$$\dim_C \mathrm{Ext}^j((\mathcal{M}_2)_p, \mathcal{F}_p) = 0, \quad (j = 0, 1, 2).$$

(3) *In the case of* $s = 2$,

(i) *if* $A > 1$,

for $\mathcal{F} = \mathcal{O}_{\widehat{M|H},2,A-}, \mathcal{O}_{\widehat{M|H},(2,A+)}$,

$$\dim_C \mathrm{Ext}^j((\mathcal{M}_2)_p, \mathcal{F}_p) = 0, \quad (j = 0, 1, 2).$$

(ii) *if* $0 < A < 1$,

for $\mathcal{F} = \mathcal{O}_{\widehat{M|H},2,A-}, \mathcal{O}_{\widehat{M|H},(2,A+)}$,

$$\dim_C \mathrm{Ext}^j((\mathcal{M}_2)_p, \mathcal{F}_p) = \begin{cases} 0, & (j = 0, 2) \\ 1, & (j = 1) \end{cases}$$

(iii) *if* $A = 1$,

$$\dim_C \mathrm{Ext}^j((\mathcal{M}_2)_p, (\mathcal{O}_{\widehat{M|H},2,1-})_p) = \begin{cases} 0, & (j = 0, 2) \\ 1, & (j = 1) \end{cases} .$$

$$\dim_C \mathrm{Ext}^j((\mathcal{M}_2)_p, (\mathcal{O}_{\widehat{M|H},(2,1+)})_p) = 0, \quad (j = 0, 1, 2).$$

(iv) $\dim_C \mathrm{Ext}^j((\mathcal{M}_2)_p, (\mathcal{O}_{\widehat{M|H},(2)})_p) = \begin{cases} 0, & (j = 0, 2) \\ 1, & (j = 1) \end{cases}$.

$\dim_C \mathrm{Ext}^j((\mathcal{M}_2)_p, (\mathcal{O}_{\widehat{M|H},2})_p) = 0, \quad (j = 0, 1, 2).$

(4) $\dim_C \mathrm{Ext}^j((\mathcal{M}_2)_p, (\mathcal{O}_{\widehat{M|H}})_p) = 0, \quad (j = 0, 1, 2).$

Corollary 1 . *The indexes of $\mathcal{D}$-module defined by Φ_2 are as follow:*
(1) *If* $1 \le s < 2$,

for $\mathcal{F} = \mathcal{O}_{\widehat{M|H},(s)}, \mathcal{O}_{\widehat{M|H},s,A-}, \mathcal{O}_{\widehat{M|H},(s,A+)}, \mathcal{O}_{\widehat{M|H},s}$,

$$\mathcal{X}((\mathcal{M}_2)_p, \mathcal{F}_p) = -1.$$

(2) *If* $s > 2$,

for $\mathcal{F} = \mathcal{O}_{\widehat{M|H},(s)}, \mathcal{O}_{\widehat{M|H},s,A-}, \mathcal{O}_{\widehat{M|H},(s,A+)}, \mathcal{O}_{\widehat{M|H},s}$,

$$\mathcal{X}((\mathcal{M}_2)_p, \mathcal{F}_p) = 0.$$

(3) *In the case of* $s = 2$

(i) *if* $A > 1$,

for $\mathcal{F} = \mathcal{O}_{\widehat{M|H},2,A-}, \mathcal{O}_{\widehat{M|H},(2,A+)}$,

$$\mathcal{X}((\mathcal{M}_2)_p, \mathcal{F}_p) = 0.$$

(ii) *if* $0 < A < 1$,

for $\mathcal{F} = \mathcal{O}_{\widehat{M|H},2,A-}, \mathcal{O}_{\widehat{M|H},(2,A+)}$,

$$\mathcal{X}((\mathcal{M}_2)_p, \mathcal{F}_p) = -1.$$

(iii) *if* $A = 1$,

$$\mathcal{X}((\mathcal{M}_2)_p, (\mathcal{O}_{\widehat{M|H},2,1-})_p) = -1.$$

$$\mathcal{X}((\mathcal{M}_2)_p, (\mathcal{O}_{\widehat{M|H},(2,1+)})_p) = 0.$$

(iv) $\mathcal{X}((\mathcal{M}_2)_p, (\mathcal{O}_{\widehat{M|H},(2)})_p) = -1.$

$\mathcal{X}((\mathcal{M}_2)_p, (\mathcal{O}_{\widehat{M|H},2})_p) = 0.$

(4) $\mathcal{X}((\mathcal{M}_2)_p, (\mathcal{O}_{\widehat{M|H}})_p) = 0.$

Corollary 2 . *The irregularity* $\mathrm{Irr}((\mathcal{M}_2)_p) = 1$.

We have the results for Φ_3 similar to those as above.

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