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RESIDUES OF HOLOMORPHIC VECTOR FIELDS RELATIVE TO SINGULAR INVARIANT SUBVARIETIES

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1- Introduction

Let $\mathcal{F}$ be a holomorphic foliation with singularities in a smooth complex manifold W, and V an analytic subvariety (not necessarily everywhere smooth), invariant by $\mathcal{F}$ ("invariant", or equivalently "saturated" means: if a point of V belongs to the regular part of $\mathcal{F}$, then the whole leaf through this point is included in V). We shall assume furthermore that the normal bundle to the regular part of V in W has a natural extension ν to the whole V, and even a smooth extension $\tilde{\nu}$ to a germ of neighborhood of V in W, making us able to use connections on $\tilde{\nu}$ and to integrate associated differential forms on compact pieces of V. [For instance, as we shall see, such a natural extension $\tilde{\nu}$ always exists for complex hypersurfaces, for algebraic subsets of $\mathbf{CP}^{p+q}$ defined by q global equations, or for "strongly" locally complete intersections (SLCI: see definition below)].

Denote by p (resp. $p+q$, resp. s) the complex dimension of V (resp. W, resp. of the leaves of $\mathcal{F}$). Then, it is easy to prove that the characteristic classes of ν in dimension $> 2(p-s)$ will "localize" near $\Sigma = [\mathrm{Sing}(\mathcal{F}) \cap V] \cup \mathrm{Sing}(V)$, and give rise to a residue for each connected component Σ_α of Σ: in fact, once we know $\tilde{\nu}$ to exist, the definition and the proof of the existence of this residue work exactly in the same way as in the case where V is smooth (see theorem 3, p.227, in [L]), and we shall omit the theory for $s > 1$. We will concentrate ourselves to the computation of the residue at an isolated point of $[\mathrm{Sing}(\mathcal{F}) \cap V] \cup \mathrm{Sing}(V)$, for Chern numbers, when $s = 1$: we get then formulas generalizing the ones in [LN_1] and [Su] and also, in the spirit of Baum-Bott ([BB_1],[BB_2]), the Grothendieck residues already known when V is smooth ([L]) (see the theorem 1 below, and its third particular case with theorem 2).

This residue has first been defined by C.Camacho and P.Sad ([CS]) when $p = q = s = 1$, V smooth and Σ_α an isolated point. When the invariant curve V may have singularities, the theory has then been generalized by A.Lins Neto [LN_1] for $W = \mathbf{CP}^2$, by M.Soares [So] when the surface W is a complete intersection in $\mathbf{CP}^n$, and in [Su] for arbitrary complex surfaces. It has also been studied in higher dimensions when V is smooth, first in the case $s = p$, $q = 1$ by B.Gmira [G], J.P.Brasselet (unpublished) and A.Lins Neto [LN_2], and then in [L] for the general case with more precise formulas when $s = 1$.

All these results extend by taking, instead of $\tilde{\nu}$, any C^∞ vector bundle on a germ of neighborhood of V in W, the restriction of which to the regular part of V being holomorphic and equipped with an action of a holomorphic vector field X_0 tangent to this regular part (see theorem 1' below). In particular, if we take $T(W)$, with the action $[X_0, .]$ on $T(W)|_V$, we get a formula for computing the index defined in the theorem 8 of [L]. (We were wrong when claiming that the index there defined was the same as the index of [LN_1] for $p = q = s = 1$: there was a mistake in the proof of part (iv) of this theorem, the 3 first parts remaining correct).

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2- Background on locally complete intersections (LCI and SLCI)

Let W be a complex manifold of complex dimension $n = p + q$, and V an analytic irreducible subvariety of pure complex dimension p. We shall call "reduced locally defining function" for V every holomorphic map $f : U \to \mathbf{C}^q$ defined on an open set U of W, such that:
(i) $V \cap U = f^{-1}(0)$,
(ii) the q components of f generate the ideal $I(V \cap U)$ of holomorphic functions which vanish on $V \cap U$; (for instance, if $q = 1$, this condition implies that f may not have factors which are powers).

If $U \supset V$, we say that f is a "reduced defining function", insisting sometimes "globally defined" near V.

The subvariety V is said to be a "locally complete intersection" (briefly: LCI) if the following condition holds: there exists a family $(f_h : U_h \to \mathbf{C}^q)_h$ of reduced locally defining functions for V, such that $\bigcup_h U_h \supset V$. Such a family will be called a "system of reduced equations" for V. Recall the following proposition, well known to the specialists:

Proposition 1

(i) Let $f_1 : U \to \mathbf{C}^q$ and $f_2 : U \to \mathbf{C}^q$ be two reduced locally defining functions for V defined on the same open set U. Then, there exists an holomorphic map $\tilde{g} : U \to gl(q, \mathbf{C})$ taking values in the set $gl(q, \mathbf{C})$ of $q \times q$ matrices with complex coefficients, satisfying $f_1 = < \tilde{g}, f_2 >$, such that the restriction g of $\tilde{g}$ to $V \cap U$ is uniquely defined and takes values in the group $GL(q, \mathbf{C})$ of invertible matrices.

(ii) If V is LCI, and if $(f_h : U_h \to \mathbf{C}^q)_h$ denotes a system of reduced equations for V, let $\tilde{g}_{hk} : U_h \cap U_k \to gl(q, \mathbf{C})$ such that $f_h = < \tilde{g}_{hk}, f_k >$ on $U_h \cap U_k$, and denote by g_{hk} the restriction of $\tilde{g}_{hk}$ to $V \cap U_h \cap U_k$. The family (g_{hk}) is then a system of transition functions for a holomorphic q vector bundle $\nu \to V$. This vector bundle is well defined (it does not depend on the choice of the given system of reduced equations for V).

(iii) The bundle ν is an extension to V of the (holomorphic) bundle normal to $V - \mathrm{Sing}(V)$ in W: more precisely, there exists a natural bundle map $\pi : T_{\mathbf{C}}(W)|_V \to \nu$ which, over the regular part of V, has rank q and the complex tangent bundle to this regular part for kernel (we may therefore identify the restriction of ν to this regular part with the usual normal bundle).

Proof:

Let f_1 and f_2 be such as in (i). Since the components $f_{1,\lambda}$ $(1 \leq \lambda \leq q)$ of f_1 and $f_{2,\lambda}$ of f_2 generate the ideal $I(V \cap U)$, there exists $q \times q$ matrices $\tilde{g}$ and $\tilde{h}$ with holomorphic coefficients such that $f_1 = < \tilde{g}, f_2 >$ and $f_2 = < \tilde{h}, f_1 >$. Furthermore, since f_1 and f_2 vanish on $U \cap V$, we get also on $U \cap V$: $df_1 = < g, df_2 >$ and $df_2 = < h, df_1 >$ (where g and h denote the restrictions of $\tilde{g}$ and $\tilde{h}$ to $U \cap V$). Since $df_1 = < g \circ h, df_1 >$ on $V \cap U$, $g \circ h = Id$ on the regular part of $V \cap U$. By continuity, since this regular part is everywhere dense in $V \cap U$, one still has $g \circ h = Id$ on the whole $V \cap U$: g takes values in $GL(q, \mathbf{C})$. The uniqueness of g is obvious since $g = h^{-1}$. This proves part (i) of the proposition.

From the uniqueness of g in part (i), we deduce immediately that the (g_{hk}) given in part (ii) satisfy the cocycle condition, and are therefore a system of transition functions for a holomorphic vector bundle $\nu \to V$. Let (g'_{hk}) denotes the system of transition functions arising from another system (f'_h) of reduced equations for V (with the same open covering (U_h) for the moment): after part (i), there exists a family $(\tilde{g}_h)$ such $f_h = < \tilde{g}_h, f'_h >$. Denoting (g_h) the induced family on V, the uniqueness in part

(i) implies that the 2 cocycles (g_{hk}) and (g'_{hk}) differ by the coboundary of (g_h): they define therefore isomorphic bundles. If we change the covering (U_h), we can use a common refinement to both coverings, for coming back to the case of a same covering.

Notice that the sections σ of ν may be identified with the families $(\sigma_h : U_h \to \mathbf{C}^q)_h$ of maps such that $\sigma_h =< g_{hk}, \sigma_k >$ on $V \cap U_h \cap U_k$. On the other hand we get also there: $df_h =< g_{hk}, df_k >$. Therefore the family of $(df_h : T_{\mathbf{C}}(W)|_{V \cap U_h} \to \mathbf{C}^q)$ defines a bundle map $\pi : T_{\mathbf{C}}(W)|_V \to \nu$. Furthermore, the kernel of df_h on the regular part of $U_h \cap V$ is exactly the tangent space to this regular part. This achieves the proof of part (iii).

By continuity and reducing the open sets U_h to smaller ones if necessary, we may assume that the functions $\tilde{g}_{hk}$ take themselves values in $GL(q, \mathbf{C})$. However it is not clear that the cocycle condition remains true off V. This justifies the following definition: a LCI subvariety V of W will be said a"strongly" locally complete intersection (shortly SLCI), if there exists a smooth C^∞ vector bundle $\tilde{\nu} \to U$, defined over some neighborhood U of V in W, the restriction of which to V being ν.

Assuming V to be SLCI, and given an extension $\tilde{\nu} \to U$ of ν, we shall call "C^∞" any section of ν which is the restriction of a C^∞ section of $\tilde{\nu}$. Local sections over U_h are given by maps $U_h \to \mathbf{C}^q$, and in particular the q constant functions corresponding to the canonical base of $\mathbf{C}^q$ make a local trivialization of $\tilde{\nu}$ over U_h (or of ν over $V \cap U_h$) called the "trivialization associated" to f_h.

Remarks:

1) Notice that the singular foliations $df_h = 0$ on U_h and $df_k = 0$ on U_k do not coincide in general on $U_h \cap U_k$.

2) We can define a virtual tangent bundle τ to V in the $\tilde{K}U$ theory by

$$[\tau] = \left[T_{\mathbf{C}}(W)|_V\right] - [\nu].$$

3) We do not know if LCI implies automatically SLCI. However, there are many examples of SLCI.

4) Let $\mathcal{O}_W$ be the sheaf of holomorphic functions on W, and $\mathcal{I}$ the sheaf of ideals defining the subvariety V in W. Thus $\mathcal{O}_V = \mathcal{O}_W/\mathcal{I}$ is the sheaf of holomorphic functions on V. If V is LCI, then the sheaf $\mathcal{I}/\mathcal{I}^2$ is locally free and the sheaf of germs of holomorphic functions of the bundle $\nu \to V$ above is identified with the dual sheaf

$\mathcal{H}om_{\mathcal{O}_V}(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_V)$. Furthermore, the bundle map $\pi : T_{\mathbf{C}}(W)|_V \to \nu$ corresponds, on the sheaf level, to the morphism dual to the one $\mathcal{I}/\mathcal{I}^2 \to \Omega_W \otimes_{\mathcal{O}_W} \mathcal{O}_V$ induced by $f \to df \otimes 1$, where $\Omega_W = \mathcal{O}_W(T^*_{\mathbf{C}}(W))$ denotes the cotangent sheaf of W.

Example 1: Any hypersurface V of W (pure complex codimension 1) is SLCI. In fact, if we set $\tilde{g}_{hk} = \frac{f_h}{f_k}$ where (f_h) denotes a family of local defining functions without factors which are powers, then the system $(\tilde{g}_{hk})$ satisfies the cocycle condition and it defines an holomorphic extension $\tilde{\nu}$ of ν defined on the union of the domains U_h of f_h.

Example 2: Any algebraic set V in $W = \mathbf{CP}^n$ which is globally a complete intersection is SLCI. In fact, denote by $(X_0, X_1, ..., X_n)$ homogeneous coordinates in $\mathbf{CP}^n$, and $F_1, F_2, ..., F_q$ homogeneous polynomials in the variables $(X_0, X_1, ..., X_n)$ of respective degree $d_1, d_2, ..., d_q$ such that V has pure complex codimension q, and is defined by the q equations $F_\lambda = 0$ $(1 \leq \lambda \leq q)$. In the affine open subset U_i of $\mathbf{CP}^n$ defined by $X_i \neq 0$, $V \cap U_i$ has for equation with respect to the affine coordinates $(\frac{X_j}{X_i})_{j,j \neq i}$: $\frac{1}{(X_i)^{d_\lambda}} F_\lambda = 0, (1 \leq \lambda \leq q)$. Therefore, on $U_i \cap U_j$ the change of equations $\tilde{g}_{ij}$ is equal to the diagonal $q \times q$ matrix $\left(\frac{X_j}{X_i}\right)^{d_1}, ..., \left(\frac{X_j}{X_i}\right)^{d_q}$. [In fact, in this case, it is not necessary to assume that the components $\frac{1}{(X_i)^{d_\lambda}} F_\lambda$ $(1 \leq \lambda \leq q)$ generate the ideal $I(V \cap U_i)$]! Denoting by $\check{L} \to \mathbf{CP}^n$ the hyperplane bundle (dual of the tautological bundle), $\tilde{\nu}$ is defined on the whole $\mathbf{CP}^n$ by the formula

$$\tilde{\nu} = \oplus_{\lambda=1}^q (\check{L})^{\otimes d_\lambda}.$$

Hence: $1 + c_1(\tilde{\nu}) + \cdots + c_q(\tilde{\nu}) = \Pi_{\lambda=1}^q (1 + d_\lambda c)$, with $c = c_1(\check{L})$.

3- Statement of the theorems 1 and 1'

Assume from now on that V is invariant by a holomorphic vector field with singularities X_0 on U. Let θ_{X_0} the $\mathbf{C}$-linear operator defined for any section $\pi(Y)$ over the regular part of V by: $\theta_{X_0}(\pi(Y)) = \pi([X_0, \tilde{Y}]|_V)$, $\tilde{Y}$ denoting some local extension of Y near V.

In case V is LCI, let $f_h = 0$ be a local reduced equation of V: each component $(df_h(X_0))_\lambda$ $(1 \leq \lambda \leq q)$ of the derivative $df_h(X_0)$ has to vanish on $V \cap U_h$, and must be therefore a linear combination with holomorphic coefficients of the components $(f_h)_\lambda$ of f_h: there exists a $q \times q$ matrix $\tilde{C}_h$ with holomorphic coefficients such that: $df_h(X_0) = < \tilde{C}_h, f_h >$. Denote by $C_h = ((C^\mu_{h,\lambda}))$ the restriction of $\tilde{C}_h$ to $V \cap U_h$.

Lemma 1

(i) $\theta_{X_0}(\pi(Y))$ *depends only on* $\pi(Y)$, *not on* Y *nor on* $\tilde{Y}$.

(ii) $\theta_{X_0}(u\sigma) = u\theta_{X_0}(\sigma) + (X_0.u)\sigma$, *for any function* u *on* V *which is the restriction of a* C^∞ *function* $\tilde{u} : U \to \mathbf{C}$.

(iii) If V *is LCI, and* $f_h = 0$ *a local reduced equation, we have, denoting* $(\sigma_1, \ldots, \sigma_q)$ *the trivialization associated to* f_h*:*

$$\theta_{X_0}(\sigma_\lambda) = -\sum_\mu C^\mu_{h,\lambda}\sigma_\mu.$$

(In particular, over the regular part of $V_h = V \cap U_h$, C_h *depends only on* f_h, *not on the choice of* $\tilde{C}_h$*).*

Parts (i) and (ii) of the lemma are proved in lemma 2-1 p.220 of [L]. For proving part (iii), take a partition $\{i_1, \ldots, i_p\} \cup \{j_1, \ldots, j_q\}$ of $\{1, \ldots, n\}$ such that $\frac{D(f_{h,1}, \ldots, f_{h,q})}{D(z_{j_1}, \ldots, z_{j_q})} \neq 0$ near some point of the regular part of V_h: then, near this point, $(z_{i_1}, \ldots, z_{i_p}, f_{h,1}, \ldots, f_{h,q})$ is a new system of local coordinates denoted by $(x_1, \ldots, x_p, y_1, \ldots, y_q)$, the local trivialization of ν associated to f_h becoming $\pi(\frac{\partial}{\partial y_\lambda})$, $(1 \leq \lambda \leq q)$. Hence if X_0 writes locally $\sum_{i=1}^p P_i \frac{\partial}{\partial x_i} + \sum_{\mu=1}^q Q_\mu \frac{\partial}{\partial y_\mu}$, then $X_0.f_{h,\mu} = X_0.y_\mu = Q_\mu = \sum_{\lambda=1}^q y_\lambda \tilde{C}'^\mu_{h,\lambda}$: hence, $C'^\mu_{h,\lambda} = \frac{\partial Q_\mu}{\partial y_\lambda}|_{y=0}$. On the other hand, $\pi[X_0, \frac{\partial}{\partial y_\lambda}] = -\sum_{\mu=1}^q \left(\frac{\partial Q_\mu}{\partial y_\lambda}|_{y=0}\right) \pi(\frac{\partial}{\partial y_\mu})$: this proves part (iii) of the lemma.

Denote by Σ (resp. $(\Sigma_\alpha)_\alpha$) the singular set $\Sigma = [\mathrm{Sing}(X_0) \cap V] \cup \mathrm{Sing}(V)$ (resp. its connected components). (Recall that a singular point of X_0 is either a point where X_0 is not defined, or a point where it vanishes).

Assume Σ_α to be compact, and denote by U_α an open neighborhood of Σ_α in W, and $U_0 = U - \Sigma$. Let $V_\alpha = V \cap U_\alpha$. We shall assume furthermore that $U_\alpha \cap U_\beta = \emptyset$, for $\alpha \neq \beta$. (In particular, $V_\alpha - \Sigma_\alpha$ is in the regular part of V).

Denote by $\tilde{\mathcal{T}}_\alpha$ a compact real manifold with boundary, of real dimension $2n$, included in U_α, such that Σ_α be inside the interior of $\tilde{\mathcal{T}}_\alpha$, and the boundary $\partial\tilde{\mathcal{T}}_\alpha$ of which being transverse to $V - \Sigma$. Put: $\mathcal{T}_\alpha = \tilde{\mathcal{T}}_\alpha \cap V$, $\partial\mathcal{T}_\alpha = \partial\tilde{\mathcal{T}}_\alpha \cap (V - \Sigma)$.

Assume:
(i) U_α is included in the domain of a local holomorphic chart $(z_1, \ldots, z_n)$ of W,
(ii) U_α is one of the U_h's above, the index α being one of the indices h. (Write f_α and C_α the corresponding terms).

Let:

$$X_0|_{U_\alpha} = \sum_{i=1}^{n} A_i(z_1, \ldots, z_n)\frac{\partial}{\partial z_i}.$$

Denote by $\mathcal{V}_i$ $(1 \le i \le n)$ the open set of points m in $\partial\mathcal{T}_\alpha$ such that $A_i(m) \neq 0$. These open sets $\mathcal{V}_i$ constitute an open covering $\mathcal{V}$ of $\partial\mathcal{T}_\alpha$. Let $\mathcal{U}$ be any subcovering of $\mathcal{V}$. (Such a $\mathcal{U}$ always exists: take for instance $\mathcal{V}$ itself; see also the particular cases 2 and 3 below). We will denote by (R_i), $(1 \le i \le n)$ any system of "honey-cells" adapted to this covering $\mathcal{U}$ (see the definition in [L], section 1, under the name of "système d'alvéoles"). For instance, if the real hypersurfaces $|A_i| = |A_j|$ $(i \neq j)$ in U_α are in general position, we may take for R_i the cell defined by: $|A_i| \ge |A_j|$ for all $j, j \neq i, \mathcal{V}_j \in \mathcal{U}$.

Denote by $\mathcal{M}$ the set of multiindices $u = (u_1, u_2, \ldots, u_p)$ such that $1 \le u_1 < u_2 < \ldots < u_p \le n$, and by $\mathcal{M}(\mathcal{U})$ the subset of those such that $\mathcal{V}_{u_j} \in \mathcal{U}$ and $\cap_{j=1}^{p}\mathcal{V}_{u_j}$ be not empty (that is the set of p simplices in the "nerve" of $\mathcal{U}$). For any $u \in \mathcal{M}(\mathcal{U})$, define $R_u = R_{u_1 u_2 \ldots u_p} = \cap_{j=1}^{p} R_{u_j}$, oriented as in section 1 of [L].

Let $\varphi \in (\mathbf{Z}[c_1, \ldots, c_q])^{2p}$ be a Chern polynomial having integral coefficients with respect to the Chern classes, and defining a characteristic class of dimension $2p$.

Theorem 1

Assume V to be SLCI. Define:

$$I_\alpha(\mathcal{F}, V, \varphi, \nu) = (-1)^{[\frac{p}{2}]} \sum_{u \in \mathcal{M}(\mathcal{U})} \int_{R_u} \frac{\varphi(-C_\alpha) dz_{u_1} \wedge dz_{u_2} \wedge \ldots \wedge dz_{u_p}}{\prod_{j=1}^{p} A_{u_j}}.$$

(i) $I_\alpha(\mathcal{F}, V, \varphi, \nu)$ does not depend on the various choices of $(z_1, \ldots, z_n), \mathcal{U}, \tilde{\mathcal{T}}_\alpha$, $f_\alpha, \tilde{C}_\alpha, R_i$, and depends only on the foliation $\mathcal{F}$ defined by X_0, but not on X_0 itself.

(ii) Assume furthermore V to be compact: $\sum_\alpha I_\alpha(\mathcal{F}, V, \varphi, \nu)$ is then an integer.

(iii) This integer depends only on V and φ , but not on $\mathcal{F}$: it is equal to the evaluation $< \varphi(\nu), V >$ of $\varphi(\nu)$ on the fundamental class $[V]$ of V.

Remark:

The index above depends obviously only on $\mathcal{F}$ and not on X_0: if we take uX_0 instead of X_0 (u denoting some holomorphic non vanishing function on U), each A_i is multiplied by $u|_V$, the matrix C_α also, and the term under integration does not change.

In fact, we could write the theorem for a foliation $\mathcal{F}$ with singularities, defined only locally by an holomorphic vector field but non necessarily globally.

Particular cases:

1) For $p = q = 1$, $I_\alpha(\mathcal{F}, V, c_1, \nu)$ coincides with the index defined in [LN_1] by A.Lins Neto, if V_α is a (locally) irreducible curve. For a possibly reducible V_α, it coincides with the one in [Su] (notice that the sum of the indices of Lins Neto over the irreducible components is different from the above index: see [Su] (1.3) Remarks 1° and (1.4) Proposition). In fact, in this case, the 1-forms $\frac{dz_1}{A_1}$ and $\frac{dz_2}{A_2}$ coincide over $\mathcal{V}_1 \cap \mathcal{V}_2$ and glue therefore together, defining a 1-form η_α on $\partial\mathcal{T}_\alpha$, while $X_0.f_\alpha$ may be written $g_\alpha f_\alpha$ for some holomorphic function g_α. The formula of theorem 1 becomes now:

$$I_\alpha(\mathcal{F}, V, c_1, \nu) = \frac{-1}{2i\pi}\left[\int_{R_1}(-g_\alpha)\eta_\alpha + \int_{R_2}(-g_\alpha)\eta_\alpha\right] = \frac{1}{2i\pi}\int_{\partial\mathcal{T}_\alpha} g_\alpha\eta_\alpha.$$

On the other hand, when f is irreducible, if $k\omega = \bar{h}.df + f\bar{\alpha}$ according to the notations of [LN] p.198 (up to the bars for avoiding confusions with our notations), his index is then equal to $\frac{-1}{2i\pi}\int_{\partial\mathcal{T}_\alpha}\frac{\bar{\alpha}}{\bar{h}}$. But $\frac{-\bar{\alpha}}{\bar{h}}$ and $g_\alpha\eta_\alpha$ are equal on $\partial\mathcal{T}_\alpha$, because they both take the same value g_α when applied to the restriction of X_0, Q.E.D. See (1.1) Lemma and (1.2) in [Su], when f is possibly reducible. This coincidence is also obvious from the theorem 2 and the remark below. Thus the above theorem 1 may be seen as a generalization of the theorems A and C of [LN_1] and the theorem (2.1) of [Su]. In particular, since the sum of our indices is the self-intersection number of the curve V, the integer $3dg(S) - \chi(S) + \sum_B \mu(B)$, lying in the theorem A of [LN_1], is equal to $dg(S)^2$, if the curve S is locally irreducible at each of its singular points. In general, the integer is different from $dg(S)^2$ (see the theorems (2.1) and (2.5) in [Su], in fact, $dg(S)^2$ is equal to $3dg(S) - \chi(S) + \sum_p c(S,p)$ by the adjunction formula).

More generally, for $p = 1$ and any q, there exists a 1-form η_α on $\partial\mathcal{T}_\alpha$, the restriction of which to each $\mathcal{V}_i$ being equal to $\frac{dz_i}{A_i}$. Then, still defining g_α by the same formula $X_0.f_\alpha = g_\alpha f_\alpha$, the formula of theorem 1 becomes:

$$I_\alpha(\mathcal{F}, V, c_1, \nu) = \frac{1}{2i\pi}\int_{\partial\mathcal{T}_\alpha} g_\alpha\eta_\alpha.$$

2) When Σ_α is in the regular part of V, we may take for local chart:

$$(z_1, \ldots, z_n) = (x_1, \ldots, x_p, y_1, \ldots, y_q)$$

such that $f_\lambda = y_\lambda$ for any $\lambda = 1, \ldots, q$. Then $A_{p+\lambda}$ vanishes on V_α, in such a way that all open sets $\mathcal{V}_{p+\lambda}$ are empty, and that we may take $\mathcal{U} = \mathcal{V}_1, ..., \mathcal{V}_p$: Then, $u = \{1, \ldots, p\}$ is the unique element of $\mathcal{M}(\mathcal{U})$. On the other hand, c_λ^μ and $\frac{\partial A_{p+\mu}}{\partial y_\lambda}$ are equal on V_α. We recover therefore the formula of theorem 1 in [L], writing $I_\alpha(\mathcal{F}, V, \varphi, \nu)$ as a Grothendieck residue. Note that there are some sign errors in [L]. In the third line of p.237, the factor $(-1)^{[\frac{p}{2}]}$ should be omitted, in Théorème 1 of p.217, the integral giving the residue should be multiplied by $(-1)^{p+[\frac{p}{2}]} = (-1)^{[\frac{p+1}{2}]}$ instead of $(-1)^p$ and in Théorème 1' of p.233, the integral should be multiplied by $(-1)^{[\frac{p}{2}]}$.

3) Assume that Σ_α is a point m_α isolated in V, and that X_0 is meromorphic near m_α (thus X_0 has a zero, a pole or both at m_α). Then, we have the following

Theorem 2

There exists a local holomorphic chart $(z_1, ..., z_n)$ near m_α in W, such that $\mathcal{V}_1, \mathcal{V}_2, ..., \mathcal{V}_p$ cover ∂T_α ($p = dim_{\mathbf{C}} V$).

For this covering $\mathcal{U}$, $\mathcal{M}(\mathcal{U})$ has a unique element $u_0 = \{1, ..., p\}$. Writing R instead of R_{u_0}, the formula of theorem 1 becomes now:

$$I_\alpha(\mathcal{F}, V, \varphi, \nu) = (-1)^{[\frac{p}{2}]} \int_R \frac{\varphi(-C_\alpha) dz_1 \wedge dz_2 \wedge \ldots \wedge dz_p}{\prod_{i=1}^p A_i}.$$

Proof:

Let us write $X_0 = \sum_{i=1}^n A_i \frac{\partial}{\partial z_i}$, $A_i = \frac{P_i}{Q_i}$ with P_i and Q_i holomorphic near m_α. We think of P_i and Q_i as being in the ring $\mathcal{O}_n$ of germs of holomorphic functions at the origin O in $\mathbf{C}^n$ and assume that they are relatively prime for each i. Let Q be the least common multiple of the Q_i's. Then QX_0 is a holomorphic vector field leaving V invariant.

Lemma 2

The holomorphic vector field QX_0 has an isolated zero at m_α on V.

In fact suppose QX_0 had a non-isolated zero at m_α on V and let V' be a positive dimensional irreducible subvariety of V containing m_α and contained in the zero set of QX_0. For each i, we write $Q = Q_i Q_i'$, where $Q_1', \ldots, Q_n'$ have no common factors. Since $QX_0 = \sum_{i=1}^n P_i Q_i' \frac{\partial}{\partial z_i}$, the functions $P_i Q_i'$ are all in the defining ideal $I(V')$ of V'. Hence, since $I(V')$ is prime and X_0 is non-zero away from m_α, there exists i_0 such that $Q_{i_0}' \in I(V')$. Thus there is a prime factor P of Q_{i_0}' such that $P \in I(V')$.

Now, since $Q_iQ'_i = Q = Q_{i_0}Q'_{i_0}$, P is a factor of $Q_iQ'_i$ for any i. On the other hand, since the pole of X_0 is the union of the zero sets of the Q_i's, we have $Q_i \notin I(V')$, by the assumption that the pole of X_0 is at most isolated on V. Therefore, P must be a factor of Q'_i for all i. This contradicts the fact that the Q'_i's have no common factors. This proves the lemma.

In the above situation, since the zero set of $P_iQ'_i$ is not smaller than that of P_i, it suffices to prove the proposition for vector fields holomorphic near m_α. Note that the index of X_0 at m_α is equal to that of QX_0. Note also that if X_0 has an isolated pole on V, then V is in fact 1-dimensional, since the pole of X_0 has codimension 1 in the ambiant space and in V.

In what follows, for an ideal I in the ring $\mathcal{O}_n$, we denote by $\operatorname{ht} I$ its height and by $V(I)$ the (germ of) the analytic set defined by I. Thus $\operatorname{ht} I = \operatorname{codim} V(I)$. Also, for germs $a_1, \dots, a_r$ in $\mathcal{O}_n$, we denote by $(a_1, \dots, a_r)$ the ideal generated by them.

Lemma 3

Let $A_1, \dots, A_n, f_1, \dots, f_q$ be germs in $\mathcal{O}_n$, $n = p + q$, with $\operatorname{ht}(f_1, \dots, f_q) = q$ *and* $\operatorname{ht}(A_1, \dots, A_n, f_1, \dots, f_q) = n$. *Then there exist germs $A'_1, \dots, A'_p$ in $\mathcal{O}_n$ such that*

(i) $A'_1, \dots, A'_p$ are linear combinations of $A_1, \dots, A_n$ with **C** *coefficients,*

(ii) $\operatorname{ht}(A'_1, \dots, A'_p, f_1, \dots, f_q) = n$.

Since $\operatorname{ht}(f_1, \dots, f_q) = q$, it suffices to show the following for $r = 1, \dots, p$:

(*) if $A'_1, \dots, A'_{r-1}$ are linear combinations of $A_1, \dots, A_n$ (with **C** coefficients) with $\operatorname{ht}(A'_1, \dots, A'_{r-1}, f_1, \dots, f_q) = r - 1 + q$, then there exists A'_r which is a linear combination of $A_1, \dots, A_n$ (with **C** coefficients) with $\operatorname{ht}(A'_1, \dots, A'_r, f_1, \dots, f_q) = r + q$.

To show this, let $V(A'_1, \dots, A'_{r-1}, f_1, \dots, f_q) = V_1 \cup \cdots \cup V_s$ be the irreducible decomposition of $V(A'_1, \dots, A'_{r-1}, f_1, \dots, f_q)$. Since $\operatorname{ht}(A_1, \dots, A_n, f_1, \dots, f_q) = n$, for any point x in $V(A'_1, \dots, A'_{r-1}, f_1, \dots, f_q)$ near O but different from O, there exists A_i with $A_i(x) \neq 0$. Hence we see that there exists A'_r which is a linear combination of $A_1, \dots, A_n$ with $V_k \not\subset V(A'_r)$ for $k = 1, \dots, s$. We have

$$V(A'_1, \dots, A'_r, f_1, \dots, f_q) = (V_1 \cap V(A'_r)) \cup \cdots \cup (V_s \cap V(A'_r)).$$

Since each V_k is irreducible and $V_k \not\subset V(A'_r)$, we have $\dim(V_k \cap V(A'_r)) < \dim V_k$. Therefore, we get $\operatorname{ht}(A'_1, \dots, A'_r, f_1, \dots, f_q) = r + q$, hence the lemma.

Note that the condition $\mathrm{ht}\,(f_1,\ldots,f_q)=q$ means that the variety V defined by $f_1=\cdots=f_q=0$ is a complete intersection and the condition $\mathrm{ht}\,(A_1,\ldots,A_n,f_1,\ldots,f_q)=n$ means that the singularity of the holomorphic vector field $X=\sum_{i=1}^n A_i\frac{\partial}{\partial z_i}$ is isolated in V.

In the above situation, if we choose a suitable coordinate system $(z_1,\ldots,z_n)$ in $\mathbf{C}^n$, then we may suppose that $\mathrm{ht}\,(A_1,\ldots,A_p,f_1,\ldots,f_q)=n$. The theorem 2 follows.

Remark:

Let V_α be defined by $f_\lambda=0$, $\lambda=1,\ldots,q$. Suppose that V_α is invariant by a holomorphic vector field X_0 and that Σ_α is an isolated point m_α in V_α. Then as is shown above, there exists a holomorphic chart $(z_1,\ldots,z_n)$ near m_α such that, when we write $X_0=\sum_{i=1}^n A_i\frac{\partial}{\partial z_i}$, $\mathrm{ht}\,(A_1,\ldots,A_p,f_1,\ldots,f_q)=n$, i.e., $A_1,\ldots,A_p,f_1,\ldots,f_q$ form a regular sequence. We may set

$$\tilde{T}_\alpha=\{\,z=(z_1,\ldots,z_n)\mid |A_i(z)|\le\varepsilon,\ |f_\lambda(z)|\le\varepsilon,\ i=1,\ldots,p,\ \lambda=1,\ldots,q\,\}.$$

Thus we have $T_\alpha=\{\,z\mid |A_i(z)|\le\varepsilon,\ f_\lambda(z)=0\,\}$ and we may also set

$$R_i=\{\,z\in\partial T_\alpha\mid |A_i(z)|\ge|A_j(z)|\text{ for }j\ne i\,\}.$$

Then we have

$$R=R_{12\cdots p}=\{\,z\mid |A_i(z)|=\varepsilon,\ f_\lambda(z)=0,\ i=1,\ldots,p,\ \lambda=1,\ldots,q\,\},$$

which is a smooth closed submanifold of real dimensiom p in ∂T_α, the link of the singularity V_α. If we set $\theta_i=\arg A_i(z)$, R is oriented so that the form $(-1)^{[\frac{p}{2}]}d\theta_1\wedge\ldots\wedge d\theta_p$ is positive. Thus if we set $R'=(-1)^{[\frac{p}{2}]}R$ so that $d\theta_1\wedge\ldots\wedge d\theta_p$ is positive on R', we get

$$I_\alpha(\mathcal{F},V,\varphi,\nu)=\int_{R'}\frac{\varphi(-C_\alpha)dz_1\wedge dz_2\wedge\ldots\wedge dz_p}{\prod_{i=1}^p A_i}.$$

More generally, let $E\to V$ be a continuous complex vector bundle of rank $r\ge1$, the restriction of which to the regular part of V being holomorphic, and such that there exists a C^∞ extension $\tilde{E}\to U$ of E to some neighborhood U of V in W. We shall assume also that there exists a $\mathbf{C}$ action of X_0 over $E|_{V-\Sigma}$ in the sense of Bott ([B$_2$]): a $\mathbf{C}$- linear operator θ_{X_0} from the space of C^∞ sections of $E|_{V-\Sigma}$ into itself is given, such that:

$\theta_{X_0}(\sigma)$ is holomorphic whenever σ is holomorphic,
$\theta_{X_0}(u\sigma) = (X_0.u)\sigma + u\theta_{X_0}(\sigma)$ for any C^∞ function u and any section σ.

Let $\varphi \in (\mathbf{Z}[c_1, \ldots, c_r])^{2p}$. We have the following generalization of theorem 1:

Let $(\sigma_1, \ldots, \sigma_r)$ be a trivialization of $E|_{U_\alpha}$ (assumed to be trivial), and M_α be the $r \times r$ matrix with holomorphic coefficients $(M_\alpha)_a^b : V_\alpha - \Sigma_\alpha \to \mathbf{C}$ such that $\theta_{X_0}(\sigma_a) = \sum_b (M_\alpha)_a^b \sigma_b$.

Theorem 1'

Define:

$$I_\alpha(\theta_{X_0}, V, \varphi, E) = (-1)^{[\frac{p}{2}]} \sum_{u \in \mathcal{M}(\mathcal{U})} \int_{R_u} \frac{\varphi(M_\alpha) dz_{u_1} \wedge dz_{u_2} \wedge \ldots \wedge dz_{u_p}}{\prod_{j=1}^{p} A_{u_j}}.$$

(i) $I_\alpha(\theta_{X_0}, V, \varphi, E)$ does not depend on the various choices of $(z_1, \ldots, z_n), \mathcal{U}, \tilde{\mathcal{T}}_\alpha$, $(\sigma_1, \ldots, \sigma_r), R_i$,

(ii) Assume V to be compact : $\sum_\alpha I_\alpha(\theta_{X_0}, V, \varphi, E)$ is then an integer.

(iii) This integer depends only on V, φ and E, but not on X_0 and θ_{X_0}. It is in fact equal to the evaluation $< \varphi(E), V >$ of $\varphi(E)$ on the fundamental class $[V]$ of V.

Remarks:

1) For theorem 1', V does not need to be SLCI not even LCI; this assumption was only useful for being sure that ν and $\tilde{\nu}$ exist in the example 1 below. This is still true, even for theorem 1, if we have some other reason to know that ν and $\tilde{\nu}$ exist.

2) If V is smooth, we recover the theorem 1' of [L], some particular cases of which being also in Baum-Bott [when $E = T_{\mathbf{C}}(W)|_V$ ([BB$_1$])], and in Bott ([B$_2$]) [when X_0 is non degenerate along Σ_α].

3) Let V_α be defined by $f_\lambda = 0$, $\lambda = 1, \ldots, q$ and invariant by a holomorphic vector field X_0. Suppose that Σ_α is an isolated point m_α in V_α, X_0 still being holomorphic near m_α. Then, as in the previous remark, there exists a holomorphic chart $(z_1, \ldots, z_n)$ near m_α such that $A_1, \ldots, A_p, f_1, \ldots, f_q$ form a regular sequence. In this case, we have

$$I_\alpha(\theta_{X_0}, V, \varphi, E) = \int_{R'} \frac{\varphi(M_\alpha) dz_1 \wedge dz_2 \wedge \ldots \wedge dz_p}{\prod_{i=1}^{p} A_i},$$

where

$$R' = \{ z \mid |A_i(z)| = \varepsilon,\ f_\lambda(z) = 0,\ i = 1, \ldots, p,\ \lambda = 1, \ldots, q \},$$

which is oriented so that the form $d\theta_1 \wedge \ldots \wedge d\theta_p$ is positive, $\theta_i = \arg A_i(z)$.

Example 1

Assume V to be SLCI. Take $E = \nu$, and θ_{X_0} defined such as in section 2 above, with $M_\alpha = -C_\alpha$. Then we get the theorem 1 above from the theorem 1'. We shall write in this case $I_\alpha(\mathcal{F}, V, \varphi, \nu)$ instead of $I_\alpha(\theta_{X_0}, V, \varphi, \nu)$.

Example 2

Take $E = T_{\mathbf{C}}(W)|_V$, and define $\theta_{X_0}(Y) = [X_0, \tilde{Y}]|_V$, depending only on the vector field Y tangent to W along V, and not on its extension $\tilde{Y}$ to some neighbourhood of V. Then, $M_\alpha = -\frac{D(A_1,\ldots,A_n)}{D(z_1,\ldots,z_n)}$. The index is now this one defined in section 8 of [L], theorem 1' giving a formula for computing it. In this case, we shall write $I_\alpha(X_0, V, \varphi, T_{\mathbf{C}}(W))$ instead of $I_\alpha(\theta_{X_0}, V, \varphi, T_{\mathbf{C}}(W)|_V)$. [Notice that if we replace here X_0 by uX_0 as in theorem 1, the index is now changing!]

3- Proof of theorem 1'

Let ω be a connexion on $\tilde{E}|_{U_0}$, defined by a derivation law ∇ satisfying:

$$\begin{cases} \nabla_{X_0}\tilde{\sigma}|_{V-\Sigma} = \theta_{X_0}\sigma \text{ for every section } \sigma \text{ of } E, \\ \nabla_Z\, \sigma = 0 \text{ for every section } Z \in T^{0,1}(V-\Sigma) \\ \quad \text{and every holomorphic section } \sigma \text{ of } E \end{cases}$$

(We shall say that such an ω is *special* relatively to θ_{X_0}.)

Let us give also an arbitrary connection ω_α on $\tilde{E}|_{U_\alpha}$.

Let $\varphi \in (\mathbf{Z}[c_1, \ldots, c_r])^{2p}$ be a Chern polynomial having integral coefficients with respect to the Chern classes $c_1, \ldots, c_r]$, and defining a characteristic class of dimension $2p$. We use the notations Δ_ω for the Chern-Weil homomorphism defined by a connection ω, and $\Delta_{\omega_0\omega_1\cdots\omega_r}(\varphi)$ the Bott's operator for iterated differences ([B$_1$]), such that: $d \circ \Delta_{\omega_0\omega_1\cdots\omega_r} = \sum_{j=0}^{r}(-1)^j\Delta_{\omega_0\cdots\hat{\omega}_j\cdots\omega_r}$.
(In particular: $d \circ \Delta_{\omega\omega'} = \Delta_{\omega'} - \Delta_\omega$).

Proposition 2

Let: $J_\alpha(\theta_{X_0}, V, \varphi, E) = \int_{\tilde{T}_\alpha} \Delta_{\omega_\alpha}(\varphi) + \int_{\partial\tilde{T}_\alpha} \Delta_{\omega_\alpha\omega}(\varphi)$.

(i) $J_\alpha(\mathcal{F}, V, \varphi, E)$ *does not depend on the choices of* $\tilde{T}_\alpha$, ω , ω_α.

(ii) Assume V *to be compact :* $\sum_\alpha J_\alpha(\theta_{X_0}, V, \varphi, E)$ *is then an integer.*

(iii) This integer depends only on V and φ , but not on $\mathcal{F}$. (It is in fact nothing else but the evaluation $< \varphi(E), V >$ of $\varphi(E)$ on the fundamental class $[V]$ of V).

[Notice that, in Proposition 2, we do not assume neither that U_α is included in the domain of a local chart, nor that $E|_{U_\alpha}$ is trivial].

The proof is exactly the same as the proof of the 3 first parts in theorem 8 of [L], just writing $\nabla_{X_0}\sigma = \theta_{X_0}\sigma$, instead of $\nabla_{X_0}Y = [X_0, Y]$.

The theorem 1' (hence the theorem 1) will follow immediately from Proposition 2 above, and from

Proposition 3

When U_α is included in the domain of a local chart, and when $\tilde{E}|_{U_\alpha}$ is trivial, then

$$I_\alpha(\theta_{X_0}, V, \varphi, E) = J_\alpha(\theta_{X_0}, V, \varphi, E).$$

In the formula of proposition 2, we may choose ω_α equal to the trivial connection ω_0 whose connection form with respect to the trivialization $(\sigma_1, \ldots, \sigma_r)$ of $\tilde{E}|_{U_\alpha}$ is the matrix 0. Hence,

$$J_\alpha(\theta_{X_0}, V, \varphi, E) = \int_{\partial T_\alpha} \Delta_{\omega_0 \omega}(\varphi).$$

Remarks:

1) Notice that the integration of the same expression over only one of the connected components of $\partial T_\alpha \cap V$ would give the partial index corresponding to the corresponding "sheet" or "branch" through Σ_α.

2) If V is not LCI, we still can define $I_\alpha(\mathcal{F}, V, \varphi, \nu)$ and $J_\alpha(\mathcal{F}, V, \varphi, \nu)$ under the condition that the bundle $\nu|_{V_\alpha - \Sigma_\alpha}$ is trivializable, and conclusion of proposition 3 will still remain true. But this index will now depend on the choice of the homotopy class of the trivialization. Furthermore, if this is possible at any point of Σ, the sum of these indices has now no reason neither to be an integer nor to be independant on $\mathcal{F}$.

There are 3 steps in the proof of proposition 3:

1) We first study the properties of the holomorphic connections ω_i on $E|_{V_i}$, the connection form of which with respect to the given trivialization being $\frac{dz_i}{A_i} M_\alpha$.

2) Then, we prove that $\Delta_{\omega_0 \omega}(\varphi)$, which is a cocycle on ∂T_α, is cohomologous, when imbedded in the total Čech-de Rham complex $CDR^*(\mathcal{U})$, to the element μ in

$CDR^{2p-1}(\mathcal{U})$ defined by:

$$\begin{cases} \mu_u = \Delta_{\omega_0\ \omega_{u_1}\ \omega_{u_2}\ldots\omega_{u_p}}(\varphi) \text{ for } u \in \mathcal{M}(\mathcal{U}), \\ \mu_I = 0 \text{ for any simplex } I \text{ of dimension } \neq p-1 \text{ in the nerve of } \mathcal{U}. \end{cases}$$

3) Finally, we prove that

$$\mu_u = \frac{\varphi(M_\alpha)\ dz_{u_1} \wedge dz_{u_2} \wedge \ldots \wedge dz_{u_p}}{\prod_{j=1}^{p} A_{u_j}}.$$

Using integration on $CDR^*(\mathcal{U})$ as recalled in lemma 6 below, this will achieve the proof of proposition 3.

First step:

Let Ω be an open set in $V_\alpha - \Sigma_\alpha$, Y a holomorphic non vanishing vector field tangent to Ω, and Γ a holomorphic map from Ω into the space of $r \times r$ matrices with complex coefficients. A connection $\bar{\omega}$ on $E|_\Omega$ will be said "adapted" to (Y, Γ) if its connection form relatively to the trivialization $(\sigma_1, \ldots, \sigma_r)$ of $E|_\Omega$, still denoted $\bar{\omega}$, satisfies:

$$\begin{cases} \bar{\omega}(Y) = \Gamma, \\ \bar{\omega}(Z) = 0 \text{ for every section } Z \text{ of } T^{0,1}(V_\alpha - \Sigma_\alpha). \end{cases}$$

Hence the restriction to Ω of a "special" connection, such as defined for proposition 2, is adapted to (X_0, M_α), while the restriction to Ω of the trivial connection ω_0 is adapted to any $(Y, matrix\ 0)$ for Y holomorphic tangent to Ω. From the usual vanishing theorem (Bott ([B$_1$]), Kamber-Tondeur ([KT]), we deduce the

Lemma 4

Let dim $\varphi = 2p$.

$$\begin{cases} \textit{If } \bar{\omega} \textit{ is adapted to some } (Y, \Gamma),\ \Delta_{\bar{\omega}}(\varphi) = 0, \\ \textit{If } \bar{\omega}_1, \ldots, \bar{\omega}_k \textit{ are adapted to the same } (Y, \Gamma),\ \Delta_{\bar{\omega}_1, \ldots, \bar{\omega}_k}(\varphi) = 0. \end{cases}$$

For any q multiindex $I = (1 \leq i_1, i_2, \ldots, i_q \leq n)$ (the i_j's being all distinct), define

$$D_I = \det \frac{D(f_1, \ldots, f_q)}{D(z_{i_1}, \ldots, z_{i_q})}.$$

For any $u \in \mathcal{M}$, define the q multiindex $\bar{u} = (\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_q)$ so that $1 \leq \bar{u}_1 < \bar{u}_2 < \ldots < \bar{u}_q \leq n$, and $\{1, 2, \ldots, n\} = \{u_1, u_2, \ldots, u_p\} \cup \{\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_q\}$, and by $\Omega_{\bar{u}}$ the open set of points in V_α where $D_{\bar{u}} \neq 0$: $\Omega_{\bar{u}}$ is a union of open sets where the restrictions of the functions $z_{u_1}, \dots, z_{u_p}$ constitute a system of local coordinates. For any $q+1$ multiindex $I = (1 \leq i_0, i_1, \dots, i_q \leq n)$, Y_I will denote the holomorphic vector field:

$$Y_I = \sum_{k=0}^{q} (-1)^k D_{I-i_k} \frac{\partial}{\partial z_{i_k}}.$$

Lemma 5

(i) Y_I is tangent to V.

(ii) For $m \in \mathcal{V}_i$ $(1 \leq i \leq n)$, there exists $u \in \mathcal{M}$ containing i such that $D_{\bar{u}} \neq 0$ at the point m.

(iii) For any i $(1 \leq i \leq n)$, the connection $\omega_i = \frac{dz_i}{A_i} M_\alpha$ on $E|_{\mathcal{V}_i}$ satisfies the following condition: for any $u \in \mathcal{M}$ containing i, the restriction of ω_i to $\Omega_{\bar{u}}$ is simultaneously adapted to (X_0, M_α) and to any $(Y_{u_j+\bar{u}}, matrix\ 0)$ such that $u_j \neq i$.

Let in fact I be some $q+1$ multi index such that $D_{I-i_k} \neq 0$ at some point m in V for some $i_k \in I$: it means that the restrictions $\tilde{z}_i$ to V of the functions z_i constitute, for i belonging to $\{1, 2, \dots, n\} - \{I - i_k\}$ (in particular for $i = i_k$), a system of local coordinates on V near m. But then, the restriction of Y_I to the domain of such a local chart is equal to $(-1)^k D_{I-i_k} \frac{\partial}{\partial \tilde{z}_{i_k}}$ and is therefore tangent to V, hence part (i) of the lemma.

The condition for X_0 to be tangent to V may be written:

$$\sum_{j=1}^{n} A_j (f_\lambda)'_{z_j} = 0 \text{ on } V_\alpha \text{ for all } \lambda = 1, \dots, q.$$

Hence, if $m \in \mathcal{V}_i$, the q dimensional vector $\left((f_\lambda)'_{z_i}\right)_{\lambda=1,\dots,q}$ is, on V_α, a linear combination of the others $\left((f_\lambda)'_{z_j}\right)_{\lambda=1,\dots,q}$. $(j \neq i)$: D_J must be zero at m for any q multiindex J containing i. But, since $\mathcal{V}_i$ is in the regular part of V, one at least of the D_J must be $\neq 0$: the only possibility is therefore that $i \notin J$ for such an J, hence part (ii) of the lemma.

On $\Omega_{\bar{u}}$, $X_0 = \sum_{j=1}^{p} A_{u_j} \frac{\partial}{\partial \tilde{z}_{u_j}} = \frac{1}{D_{\bar{u}}} \sum_{j=1}^{p} A_{u_j} Y_{u_j+\bar{u}}$ and, on $\mathcal{V}_i \cap \Omega_{\bar{u}}$, the p holomorphic vector fields X_0 and $\left(Y_{u_j+\bar{u}}\right)_{u_j \neq i}$ are linearly independant. The part (iii) of the lemma becomes now obvious to check, since $\mathcal{V}_i$ is covered by the $\Omega_{\bar{u}}$ such that $i \in u$.

Second step:

For any k simplex $I = (i_0 \cdots i_k)$ in the nerve of $\mathcal{U}$, write:
$\Delta_{\omega_0\ \omega\ \omega_I}(\varphi) = \Delta_{\omega_0\ \omega\ \omega_{i_0}\cdots\omega_{i_k}}(\varphi)$, $\Delta_{\omega\ \omega_I}(\varphi) = \Delta_{\omega\ \omega_{i_0}\cdots\omega_{i_k}}(\varphi)$,
and $\Delta_{\omega_0\ \omega_I}(\varphi) = \Delta_{\omega_0\ \omega_{i_0}\cdots\omega_{i_k}}(\varphi)$.

Define $\gamma \in CDR^{2p-1}(\mathcal{U})$ as the family $(\gamma_I)_I$ given by:
$\gamma_I = (-1)^{[\frac{k+1}{2}]}\Delta_{\omega_0\ \omega\ \omega_I}(\varphi)$, where k denotes the dimension $|I|$ of I.

Then, the total differential $D\gamma$ of γ in $CDR^*(\mathcal{U})$ is given by:

$$\begin{aligned}\left(D\gamma\right)_I &= (-1)^{[\frac{k+1}{2}]+k}\left(\Delta_{\omega\ \omega_I}(\varphi) - \Delta_{\omega_0\ \omega_I}(\varphi) + \textstyle\sum_{\alpha=0}^{k}(-1)^{\alpha}\Delta_{\omega_0\ \omega\ \omega_{I-i_\alpha}}(\varphi)\right)\\ &\quad + \textstyle\sum_{\alpha=0}^{k}(-1)^{[\frac{k}{2}]+\alpha+1}\Delta_{\omega_0\ \omega\ \omega_{I-i_\alpha}}(\varphi)\\ &= (-1)^{[\frac{k+1}{2}]+k}\left(\Delta_{\omega\ \omega_I}(\varphi) - \Delta_{\omega_0\ \omega_I}\right)(\varphi), \text{ for } |I| > 0,\end{aligned}$$

and $\left(D\gamma\right)_i = \Delta_{\omega\ \omega_i}(\varphi) - \Delta_{\omega_0\ \omega_i}(\varphi) + \Delta_{\omega_0\ \omega}(\varphi)$ for $|I| = 0$.

But all terms $\Delta_{\omega\ \omega_I}(\varphi)$ vanish because the connections $\omega, \omega_{i_0}, \cdots, \omega_{i_k}$ are all adapted to the same (X_0, M_α), all terms $\Delta_{\omega_0\ \omega_I}(\varphi)$ vanish for $|I| < p-1$ because the connections $\omega_0, \omega_{i_0}, \cdots, \omega_{i_k}$ are all adapted to a same $(Y, matrix\ 0)$, and all terms of $\left(D\gamma\right)_I$ vanish for $|I| \geq p$ because $\Delta_{\tilde\omega_0\cdots\tilde\omega_r}(\varphi)$ is always 0 for any family of $r+1$ connections when $r > p$. Therefore, it remains only:
$\left(D\gamma\right)_i = \Delta_{\omega_0\ \omega}(\varphi)$ for $I = \{i\}$ of dimension 0,
$\left(D\gamma\right)_u = -\mu_u$ for $u \in \mathcal{M}(\mathcal{U})$ of dimension $p-1$,
all others $\left(D\gamma\right)_I$'s being 0. This proves: $D\gamma = \iota\left(\Delta_{\omega_0\ \omega}(\varphi)\right) - \mu$,
where ι denotes the natural imbedding of the de Rham complex $\Omega^*_{DR}(\partial\mathcal{T}_\alpha)$ into $CDR^*(\mathcal{U})$.

Third step:

The set $\mathcal{V}_u$ equal to $\cap_{j=1}^{p}\mathcal{V}_{u_j}$ is included into $\Omega_{\bar u}$. In fact, as already seen at lemma 5, if m belongs to $\mathcal{V}_i$, D_I must be zero when $i \in I$: so if $m \in \mathcal{V}_u$, u is the only possible element v in $\mathcal{M}(\mathcal{U})$ such that $D_v \neq 0$.

For computing $\Delta_{\omega_0\ \omega_{u_1}\cdots\omega_{u_p}}$, we introduce (Bott [B$_1$]) the connection $\tilde\omega$ on $(\tilde E|_{\mathcal{V}_u}) \times \Delta^p \to \mathcal{V}_u \times \Delta^p$, ($\Delta^p$ denoting the p-simplex $0 \leq \sum_{i=1}^{p} t_i \leq 1$, $0 \leq t_i \leq 1$, in $\mathbf{R}^p$), defined by $\tilde\omega = \sum_{i=1}^{p} t_i\omega_i + \left[1 - (\sum_{i=1}^{p} t_i)\right]\omega_0 = \left(\sum_{j=1}^{p}\frac{t_j}{A_{u_j}}dz_{u_j}\right)M_\alpha$.
The curvature $\tilde\Omega$ of this connection is then equal to

$$\tilde\Omega = \Big(\sum_{j=1}^{p} dt_j \wedge \frac{1}{A_{u_j}}dz_{u_j}\Big)M_\alpha + \quad (\text{terms without any} \quad dt_k)\,.$$

Therefore, for every polynomial φ in $Chern^{2p}[c_1 \dots c_n]$,

$$\Delta_{\tilde{\omega}}(\varphi) = p!(-1)^{[\frac{p}{2}]} dt_1 \wedge dt_2 \wedge \cdots \wedge dt_p \wedge \frac{\varphi(M_\alpha) dz_{u_1} \wedge \cdots \wedge dz_{u_p}}{\prod_{j=1}^{p} A_{u_j}}$$
$$+ (\text{terms of degree} < p \quad \text{in} \quad dt_j)$$

By integration over Δ^p, and using the equality $\int_{\Delta^p} dt_1 \wedge \cdots \wedge dt_p = \frac{1}{p!}$, we get ([B1] p.64):

$$\Delta_{\omega_0 \omega_1 \cdots \omega_p}(\varphi) = \frac{\varphi(M_\alpha)\ dz_{u_1} \wedge dz_{u_2} \wedge \ldots \wedge dz_{u_p}}{\prod_{j=1}^{p} A_{u_j}}.$$

This achieves the proof of proposition 3, hence of theorems 1' and 1, once using:

Lemma 6

There exists a linear map $L : CDR^{2p-1}(\mathcal{U}) \to \mathbf{C}$ with the following properties:
i) L vanishes on the total coboundaries $D\Big(CDR^{2p-2}(\mathcal{U})\Big)$,
ii) L extends simultaneously the integration $\int_{\partial \mathcal{T}_\alpha} : \Omega_{DR}^{2p-1}(\partial \mathcal{T}_\alpha) \to \mathbf{C}$,
and the map: $(-1)^{[\frac{p}{2}]} \sum_{u \in \mathcal{M}(\mathcal{U})} \int_{R_u} : C^{p-1}(\mathcal{U}, \Omega_{DR}^p) \to \mathbf{C}$.

Proof: See section 6 of [L].

4- Examples

Let W be the 3-dimensional complex projective space $\mathbf{CP}^3$, of points $[X, Y, Z, T]$ with homogeneous coordinates X, Y, Z, T. Take for V the cone V_l of equation

$$X^l + Y^l + Z^l = 0 \quad (l \text{ being any integer } \geq 1),$$

which has a single isolated singular point $O = [0, 0, 0, 1]$. Denote by U_T, U_Z and U_Y the affine spaces $T \neq 0, Z \neq 0$ and $Y \neq 0$ with respective coordinates $(x = \frac{X}{T}, y = \frac{Y}{T}, z = \frac{Z}{T}), (x' = \frac{X}{Z}, y' = \frac{Y}{Z}, t' = \frac{T}{Z})$ and $(x" = \frac{X"}{Y}, z" = \frac{Z}{Y}, t" = \frac{T}{Y})$. The 3 open sets U_T, U_Z, U_Y cover V_l since the point $[1, 0, 0, 0]$ does not belong to V_l. The corresponding equations of V_l may be written respectively: $f_T = 0$, $f_Z = 0$, $f_Y = 0$, with:
$f_T(x, y, z) = x^l + y^l + z^l$,
$f_Z(x', y', t') = x'^l + y'^l + 1$, and $f_Y(x", z", t") = x"^l + z"^l + 1$.
The bundle $\tilde{\nu}$ is defined by the cocycle

$$(g_{TZ} = z^l = \frac{1}{t'^l},\quad g_{TY} = y^l = \frac{1}{t"^l},\quad g_{ZY} = y'^l = \frac{1}{z"^l}).$$

In general, for a hypersurface V_l of degree l in $\mathbf{CP}^n$ ($\dim_{\mathbf{C}} V_l = p = n-1$), we have (see Example 2 in section 2)

$$< (c_1)^p(\nu), V_l >= l^{n-1} \int_{V_l} c^{n-1} = l^n.$$

Also, from $T_{\mathbf{C}}(\mathbf{CP}^n) \oplus 1 = (n+1)\check{L}$, we have:

$$1 + c_1(T_{\mathbf{C}}) + c_2(T_{\mathbf{C}}) + \cdots = (1+c)^{n+1},$$

hence:

$$c_1(T_{\mathbf{C}}(\mathbf{CP}^n)) = (n+1)c, \quad c_2(T_{\mathbf{C}}(\mathbf{CP}^n)) = \frac{(n+1)n}{2}c^2, \ldots .$$

In particular, for $p = 2$, $q = 1$, we get:

$$< (c_1)^2(T_{\mathbf{C}}(\mathbf{CP}^3)), V_l >= (3+1)^2 \int_{V_l} c^2 = 16l,$$

$$< c_2(T_{\mathbf{C}}(\mathbf{CP}^3)), V_l >= \frac{4\cdot 3}{2} \int_{V_l} c^2 = 6l.$$

Example 1:

Take for X_0 the extension H to the whole $\mathbf{CP}^3$ of the vector field of infinitesimal homotheties $x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}$ in U_T. (In U_Z and U_Y, H is equal respectively to $-t'\frac{\partial}{\partial t'}$ and $-t"\frac{\partial}{\partial t"}$). This vector field has for singular set the union of $\{O\}$ and of the hyperplane $T=0$, and Σ has 2 connected components: Σ_1 is the isolated point $\{O\}$, and Σ_2 the curve ($X^l + Y^l + Z^l = 0$, $T=0$). Notice however that Σ_2 does not contain any singularity for the foliation $\mathcal{F}$ generated by H, so that we can already assert:

$$I_2(\mathcal{F}, V_l, (c_1)^2, \nu) = 0.$$

1) Computation of $I_1(\mathcal{F}, V_l, (c_1)^2, \nu)$ and $I_1(H, V_l, \varphi, T_{\mathbf{C}}(W))$ ($\varphi = (c_1)^2$ or c_2):

For $E = \nu$, $H.f_T = lf_T$ and $M_0 = -C_0$ is the 1×1 constant matrix $(-l)$. For $E = T_{\mathbf{C}}(W)|_V$, $M_0 = -\frac{D(x,y,z)}{D(x,y,z)}$ is equal to the opposite of the 3×3 identity matrix, in such a way that for $E=\nu$, $(c_1)^2(M_0)$ is a constant equal to $\frac{-l^2}{4\pi^2}$,

while for $E = T_{\mathbf{C}}(W)|_V$, $\varphi(M_0)$ is also a constant equal to $\begin{cases} \dfrac{-9}{4\pi^2} & \text{if } \varphi = (c_1)^2, \\ \dfrac{-3}{4\pi^2} & \text{if } \varphi = c_2. \end{cases}$

(Recall that, c_k applied to some matrix is equal to $(\frac{i}{2\pi})^k$ times the k th elementary symmetric function of the eigenvalues).

We compute the indices in two ways; first directly by the definition in theorem 1 or 1' and then applying theorem 2.

(i) Take for $\tilde{T}$ the ball Sup $(|x|,|y|,|z|) \leq \varepsilon$ for some positive constant ε. Let R_z be the region in the boundary ∂T defined by $|z| \geq |x|, |z| \geq |y|$, and define R_x and R_y similarly. The index $I_1(\theta_H, V_l, \varphi, E)$ at the origin O is equal in both cases to

$$-\varphi(M_0)\left(\int_{R_{xy}} \frac{dx}{x} \wedge \frac{dy}{y} + \int_{R_{yz}} \frac{dy}{y} \wedge \frac{dz}{z} + \int_{R_{xz}} \frac{dx}{x} \wedge \frac{dz}{z}\right).$$

On R_{xy}, we may write: $x = \varepsilon e^{i\theta}$, $y = \varepsilon e^{i\sigma}$, and $\frac{dx}{x} \wedge \frac{dy}{y} = -d\theta \wedge d\sigma$, which is positive on R_{xy}. [In fact, remember ([L]) the convention about the orientation of R_{xy} by the normal from R_x to R_y: let us write $x = re^{i\theta}$ and $y = se^{i\sigma}$ on T; then $dr \wedge d\theta \wedge ds \wedge d\sigma$ is positive on T with r increasing when approaching $\partial T \cap R_x$, $r = \varepsilon$ and $d\theta \wedge ds \wedge d\sigma$ is positive on R_x with s increasing when approaching the boundary near R_{xy}, in such a way that $-d\theta \wedge d\sigma$ is positive on R_{xy}]. But there, we have $z^l = -(x^l + y^l) = -2\varepsilon^l cos \frac{l(\sigma-\theta)}{2} e^{i\frac{l(\sigma+\theta)}{2}}$, so that R_{xy} is an l-fold covering of the set of (θ, σ) such that $2\varepsilon^l |cos(\sigma - \theta)| \leq \varepsilon^l$ (because $|z| \leq \varepsilon$ on R_{xy}). It is easy to check that the set of (θ, σ) in the square $[0, 2\pi]^2$ where the previous condition holds is made of l strips, the area of each one being $\frac{2\pi}{3} \times 2\pi = \frac{4\pi^2}{3}$. Then, because of the l sheets of the covering, we get: $\int_{R_{xy}} \frac{dx}{x} \wedge \frac{dy}{y} = \frac{4l\pi^2}{3}$. The computation is the same for the two others integrals, so that

$$\int_{R_{xy}} \frac{dx}{x} \wedge \frac{dy}{y} + \int_{R_{yz}} \frac{dy}{y} \wedge \frac{dz}{z} + \int_{R_{xz}} \frac{dx}{x} \wedge \frac{dz}{z} = 4l\pi^2.$$

(ii) We observe that, in this case, x, y and f_T form a regular sequence (see the Remark after Theorem 2 and Remark 3) after Theorem 1'), and we may take for $\tilde{T}$ the ball Sup $(|x|,|y|,|f_T|) \leq \varepsilon$. The index $I_1(\theta_H, V_l, \varphi, E)$ at the origin O is equal to

$$\varphi(M_0)\int_{R'} \frac{dx}{x} \wedge \frac{dy}{y},$$

where R' is the 2-submanifold in the boundary ∂T given by

$$R' = \{(x,y,z) \mid |x| = |y| = \varepsilon,\ x^l + y^l + z^l = 0\}.$$

On R', we may write: $x = \varepsilon e^{i\theta}$, $y = \varepsilon e^{i\sigma}$, and $\frac{dx}{x} \wedge \frac{dy}{y} = -d\theta \wedge d\sigma$, which is negative on R'. But there, we have $z^l = -(x^l + y^l)$, so that R' is an l-fold covering of the set of (θ, σ) in the square $[0, 2\pi]^2$. Thus we get

$$\int_{R'} \frac{dx}{x} \wedge \frac{dy}{y} = -4l\pi^2.$$

In either way we get:

$$I_1(\mathcal{F}, V_l, (c_1)^2, \nu) = l^3, \text{ and}$$

$$I_1(H, V, \varphi, T_{\mathbf{C}}(W)) = \begin{cases} 9l & \text{if } \varphi = (c_1)^2, \\ 3l & \text{if } \varphi = c_2, \end{cases}$$

2) Computation of $I_2(H, V_l, \varphi, T_{\mathbf{C}}(W))$:

Since Σ_2 is a smooth compact holomorphic manifold in the regular part of V_l, we may use the Bott's theorem ([B$_2$] p.314) for computing the index, under the condition that the infinitesimal action of H on the bundle N normal to Σ_2 in V_l be non degenerate. Since V_l is compact, this action will be of constant type along Σ_2, and the same thing is true for the action $\theta_H|_{\Sigma_2}$ of H. So, it is enough to calculate them for instance along $\Sigma_2 \cap U_Z$. Since $\frac{\partial f_Z}{\partial x'} = lx'^{l-1}$, and $\frac{\partial f_Z}{\partial y'} = ly'^{l-1}$, and because both coordinates x' and y' may not vanish simultaneously over $\Sigma_2 \cap U_Z$, we may assume for instance $x' \neq 0$. Near such a point in $\Sigma_2 \cap U_Z$, we may replace the coordinates (x', y', t') by $(u = f_Z(x', y', t'), v = y', w = t')$, so that V_l has now for local equation $u = 0$, while Σ_2 is now locally defined by $u = 0, w = 0$. The bundle N is generated by $\frac{\partial}{\partial w}$, $H = -w\frac{\partial}{\partial w}$, and $[H, \frac{\partial}{\partial w}] = \frac{\partial}{\partial w}$: therefore this action, represented by the constant 1×1 matrix $(+1)$, is effectively non degenerate. On the other hand, ν is generated by $\frac{\partial}{\partial u}$, so that $[H, \frac{\partial}{\partial u}] = 0$, while the third bracket $[H, \frac{\partial}{\partial v}]$ being also 0, the action $\theta_H|_{\Sigma_2}$ on $T_{\mathbf{C}}(W)$ will be represented by the constant matrix

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Denote a, b, c the formal classes such that the k th Chern class of W is equal to the k th elementary symmetric function of a, b, c. After Bott, we have:

$$I_2(H, V_l, \varphi, T_{\mathbf{C}}(W)) = < \frac{\hat{\varphi}\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c+1 \end{pmatrix}}{1 + c_1(N)}, \Sigma_2 >,$$

where $\hat{\varphi}$ denotes $(a+b+c+1)^2$ for $\varphi = (c_1)^2$, and $ab + (a+b)(c+1)$ for $\varphi = c_2$. Hence, we get:

$$I_2(H, V, \varphi, T_{\mathbf{C}}(W)) = \begin{cases} < 2c_1(T_{\mathbf{C}}(W)) - c_1(N), \Sigma_2 >, \text{ for } (c_1)^2, \\ \text{and } < a + b, \Sigma_2 > \text{ for } c_2. \end{cases}$$

Notice that N coincides with the restriction to Σ_2 of the hyperplane bundle $\check{L} \to \mathbf{CP}^2$ after identification of $\mathbf{CP}^2$ with the hyperplane $T = 0$ in $\mathbf{CP}^3$, while $T_{\mathbf{C}}(W)$ is stably equivalent to $4\check{L}$, and $(a+b)|_{\mathbf{CP}^2} = c_1(\mathbf{CP}^2) = 3c_1(\check{L})$. We get therefore $7 < c_1(\check{L}), \Sigma_2 >= 7l$ for $(c_1)^2$, and $3 < c_1(\check{L}), \Sigma_2 >= 3l$ for c_2.

Finally, we recover:

$$< (c_1)^2(\nu), V_l >= l^3 + 0 = l^3,$$

$$< (c_1)^2(T_{\mathbf{C}}(W)), V_l >= 9l + 7l = 16l, \text{ and } < c_2(T_{\mathbf{C}}(W)), V_l >= 3l + 3l = 6l.$$

In particular, for $l = 2$, we get:

$$< (c_1)^2(\nu), V_2 >= 8, \quad \text{and}$$

$$< (c_1)^2(T_{\mathbf{C}}(W)), V_2 >= 32, \ < c_2(T_{\mathbf{C}}(W)), V_2 >= 12.$$

Example 2:

Take $l = 2$. with now for X_0 the extension $\mathcal{R}$ to the whole $\mathbf{CP}^3$ of the vector field of infinetisimal "complex rotations" $y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}$ in U_T.

In U_Z (resp. in U_Y), $\mathcal{R}$ writes $y'\frac{\partial}{\partial x'} - x'\frac{\partial}{\partial y'}$ (resp. $(x"^2+1)\frac{\partial}{\partial x"} + x"z"\frac{\partial}{\partial z"} + x"t"\frac{\partial}{\partial t"}$). Now Σ is made of 3 isolated points: $m_1 = [0,0,0,1]$, $m_2 = [i,1,0,0]$ and $m_3 = [-i,1,0,0]$. Notice that V_2 is regular at m_2 and m_3. We have :
$\mathcal{R}.f_T = 0$, $\mathcal{R}.f_Z = 0$, and $\mathcal{R}.f_Y = 2x"f_Y$, this proves that $\mathcal{R}$ still preserves V, and that $I_1(\mathcal{R}, V, (c_1)^2, \nu) = 0$ since $m_1 \in U_T$.

1) Computation of $I_1(\mathcal{R}, V_2, \varphi, T_{\mathbf{C}}(W))$:

In this case, y, $-x$ and f_T form a regular sequence and we may take for $\tilde{\mathcal{T}}$ the ball
Sup $(|x|, |y|, |f_T|) \leq \varepsilon$ for some positive constant ε. The index $I_1(\theta_{X_0}, V, \varphi, E)$ at the origin O is then equal to

$$\int_{R'} \varphi(M_1)\frac{dx \wedge dy}{-xy},$$

where R' is the 2-submanifold in the boundary ∂T given by

$$R' = \{ (x,y,z) \mid |y| = |-x| = \varepsilon, \ x^2 + y^2 + z^2 = 0 \}.$$

If we write: $x = \varepsilon e^{i\theta}$, $y = \varepsilon e^{i\sigma}$ on R', $d\sigma \wedge d\theta$ is positive on R'. Hence we have $\int_{R'} \frac{dx\wedge dy}{-xy} = -8\pi^2$. When $E = T_{\mathbf{C}}(W)|_V$, M_1 is now the matrix $\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$:

$\varphi(M_1)$ is still a constant, now equal to 0 for $\varphi = (c_1)^2$, and to $\frac{-1}{4\pi^2}$ for $\varphi = c_2$. Then we have,
$I_1(\mathcal{F}, V_2, (c_1)^2, \nu) = I_1(X_0, V_2, (c_1)^2, T_{\mathbf{C}}(W)) = 0$, and $I_0(X_0, V_2, c_2, T_{\mathbf{C}}(W)) = 2$.

2) Computation of indices at points m_2 and m_3:

Observe that $\frac{\partial f_Y}{\partial x"} = 2x" \neq 0$ near these points. Then we may use $(u = f_Y, v = z", w = t")$ instead of $(x", z", t")$ as local coordinates, with $\mathcal{R}$ $= x(2u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v} + w\frac{\partial}{\partial w})$. The tangent space to V is generated by $\frac{\partial}{\partial v}$ and $\frac{\partial}{\partial w}$. Since the restriction $x(v\frac{\partial}{\partial v} + w\frac{\partial}{\partial w})$ is nondegenerate at m_2 and m_3, with eigenvalues $(\varepsilon i, \varepsilon i)$ with $\varepsilon = 1$ (resp. -1) at m_2 (resp. m_3), we may use the Bott's formula. The normal bundle ν is generated by $\frac{\partial}{\partial u}$, and the action of R on ν at points m_2 and m_3 is given by the 1×1 matrix $(-2\varepsilon i)$, and :

$$I_2(\mathcal{F}, V, (c_1)^2, \nu) = I_3(\mathcal{F}, V, (c_1)^2, \nu) = 4.$$

The action of $\mathcal{R}$ on $T_{\mathbf{C}}(W)$ is given by the matrix $-\varepsilon i \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, and

$$I_2(\mathcal{R}, V_2, (c_1)^2, T_{\mathbf{C}}(W)) = I_3(\mathcal{R}, V_2, (c_1)^2, T_{\mathbf{C}}(W)) = 16,$$

$$I_2(\mathcal{R}, V_2, c_2, T_{\mathbf{C}}(W)) = I_3(\mathcal{R}, V_2, c_2, T_{\mathbf{C}}(W)) = 5.$$

We may notice that we still have, as in example 1:

$< (c_1)^2(\nu), V_2 >= 0 + 4 + 4 = 8$,

$< (c_1)^2(T_{\mathbf{C}}(W)), V_2 >= 0 + 16 + 16 = 32$,

and $< c_2(T_{\mathbf{C}}(W)), V_2 >= 2 + 5 + 5 = 12$.

Example 3:

Take still $l = 2$, with now for X_0 the linear combination $X_\omega = aH + b\mathcal{R}$ of examples 1 and 2, where $\omega \in [0, \frac{\pi}{2}[$, $a =\cos\omega$, $b =\sin\omega$, $(a \neq 0)$. In U_T, $X_\omega = a\left[x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\right] + b\left[y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right]$ has only for singular point the origin m_1. In U_Z, $X_\omega = b(y'\frac{\partial}{\partial x'} - x'\frac{\partial}{\partial y'}) - at'\frac{\partial}{\partial t'}$, has no singular point on V_2. In U_Y, $X_\omega = b(x"^2 + 1)\frac{\partial}{\partial x"} + bx"z"\frac{\partial}{\partial z"}) + t"(bx" - a)\frac{\partial}{\partial t"}$ has the same singular points m_2 and m_3 as in example 2.

1) Computation of indexes at point m_1:

Since, $X_\omega . f_T = 2af_T$, the 1×1 matrix C_1 is constant equal to $((-2a))$, so that $(c_1)^2(C_1) = \frac{-a^2}{\pi^2}$.

Write: $A = ax + by$, $B = -bx + ay$ and $C = az$.

We have $\frac{D(A,B,C)}{D(x,y,z)} = \begin{pmatrix} a & b & 0 \\ -b & a & 0 \\ 0 & 0 & a \end{pmatrix}$, and $\varphi(-\frac{D(A,B,C)}{D(x,y,z)})$ is still a constant equal to

$$\begin{cases} \dfrac{-9a^2}{4\pi^2} & \text{if } \varphi = (c_1)^2, \\ \dfrac{-(3a^2+b^2)}{4\pi^2} & \text{if } \varphi = c_2. \end{cases}$$

In this case, A, B and f_T form a regular sequence and we may take for $\tilde{\mathcal{T}}$ the ball $\text{Sup}\,(|A|, |B|, |f_T|) \le \varepsilon$ for some positive constant ε. The index $I_1(\mathcal{F}, V_2, \varphi, E)$ at the origin O is equal to

$$\varphi(M_1) \int_{R'} \frac{dx \wedge dy}{AB},$$

where R' is the 2-submanifold in the boundary $\partial\mathcal{T}$ given by

$$R' = \{ (x,y,z) \mid |A| = |B| = \varepsilon,\ x^2 + y^2 + z^2 = 0 \}.$$

Since $dx \wedge dy = dA \wedge dB$, the integral is computed as in example 1 to get: $\int_{R'} \frac{dx \wedge dy}{AB} = -8\pi^2$. Thus we have

$$I_1(\mathcal{F}, V_2, \varphi, E) = \begin{cases} 8a^2 \text{ for } E = \nu \text{ and } \varphi = (c_1)^2, \\ 18a^2 \text{ for } E = T_{\mathbf{C}}W \text{ and } \varphi = (c_1)^2, \\ 2(3a^2 + b^2) \text{ for } E = T_{\mathbf{C}}W \text{ and } \varphi = c_2. \end{cases}$$

2) Computation of indices at points m_2 and m_3:

We already observed that $\frac{\partial f_Y}{\partial x"} = 2x" \neq 0$ near these points, so that we may use $(u = f_Y, v = z", w = t")$ instead of $(x", z", t")$ as local coordinates, with $X_\omega = bx"(2u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v}) + (bx" - a)w\frac{\partial}{\partial w}$. The tangent space to V_2 is generated by $\frac{\partial}{\partial v}$ and $\frac{\partial}{\partial w}$. The restriction $bx"v\frac{\partial}{\partial v} + (bx" - a)w\frac{\partial}{\partial w}$ of X_ω to V_2 has for eigenvalues $(b\varepsilon i, b\varepsilon i - a)$ with $\varepsilon = 1$ (resp. -1) at m_2 (resp. m_3). It is therefore nondegenerate at these points, and we may use the Bott's formula.

The normal bundle ν is generated by $\frac{\partial}{\partial u}$, the action of X_ω on ν at points m_2 and m_3 is given by the 1×1 matrix $((-2b\varepsilon i))$, and :

$I_2(\mathcal{F}, V, (c_1)^2, \nu) = -\frac{4b^2}{ib(ib-a)} = 4b(b - ai)$, while

$I_3(\mathcal{F}, V, (c_1)^2, \nu) = 4b(b + ai)$. We recover:

$$< (c_1)^2(\nu).V_2 >= 8a^2 + 4b(b - ai) + 4b(b + ai) = 8.$$

The action of X_ω on $T_{\mathbf{C}}(W)$ has for eigenvalues: $(-2b\varepsilon i, -b\varepsilon i, -(b\varepsilon i - a))$.
$I_2(X_\omega, V_2, (c_1)^2, T_{\mathbf{C}}(W)) = \frac{(4ib-a)^2}{ib(ib-a)} = (16b^2 + 7a^2) - i\frac{a(8b^2-a^2)}{b}$, while
$I_3(X_\omega, V_2, (c_1)^2, T_{\mathbf{C}}(W)) = (16b^2 + 7a^2) + i\frac{a(8b^2-a^2)}{b}$. We recover:

$$< (c_1)^2(T_{\mathbf{C}}(W)), V_2 >= 18a^2 + 2(16b^2 + 7a^2) = 32.$$

$I_2(X_\omega, V_2, c_2, T_{\mathbf{C}}(W)) = \frac{2(bi)^2 + 2bi(bi-a) + bi(bi-a)}{ib(ib-a)} = 5b^2 + 3a^2 - 2iab$, while
$I_3(X_\omega, V_2, c_2, T_{\mathbf{C}}(W)) = 5b^2 + 3a^2 + 2iab$. We recover:

$$< c_2(T_{\mathbf{C}}(W)), V_2 >= 2(3a^2 + b^2) + 2(5b^2 + 3a^2) = 12.$$

We may notice, in accordance with the theory, that the indices themselves are not necessarily integers and depend on a, b, contrary to their sum. Notice also that we recover the values of example 1 ($l = 2$) for $\omega = 0$, and of example 2 for $\omega = \frac{\pi}{2}$. However the calculation for this last case had to be done separately, because we assumed explicitely $C \neq 0$ near m_0 in the calculation of example 3.

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