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$SL(2;\mathbf{R})$-Actions on Surfaces

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The aim of this talk was to give a fairly elementary method to classify the differential structures of non-transitive $SL(2;\mathbf{R})$-actions on 2-dimensional manifolds, as well as to give an account on how Lie groups act on manifolds, especially on very low dimensional ones, beginning with a Lie's theorem of more than one hundred years ago. Most of the arguments here are on Lie algebra level.

1 Lie Groups Acting on 1-Manifolds

Standard examples of the actions of finite dimensional Lie groups on 1-manifolds are coming from the projective action of $PSL(2;\mathbf{R})$ on RP^1 by taking its covering or the restrictions to its subgroups.

Theorem (Lie, [4], [5], [3]) *Essentially those are all.*

The author would not like to clarify what word *essentially* could imply. However, one can consider that it is true on an orbit, in the local Lie algebra sense.

The proof roughly goes as follows. Take an element $X\in\mathbf{g}$ and a point and its neighbourhood with local coordinate on the manifold so that X looks just like $\partial/\partial x$. Any element $Y\in\mathbf{g}$ can be expressed as $f(x)\partial/\partial x$ on that neighbourhood. Then it is quite easy to see that $[X,Y]=f'(x)\partial/\partial x$. Thus, $(ad(X))^n(Y)=f^{(n)}(x)\partial/\partial x$.

Our assumption that the Lie algebra $\mathbf{G}$ is finite dimensional implies that the function f must satisfy some linear ODE's with constant coefficients. As we know all of their solutions, *i.e.,* they are in the form $\sum$ **polynomial** $\times$ **exponential** in the complexified sense, we can easily classify them to generate finite dimensional Lie algebras. Actually, we have only three possibilities below when $\mathbf{g}$ is 3-dimensional, and in the lower dimensional case are realized as their subalgebras ;

Type 1. $\mathbf{g}=\{[\mathit{polynimials of degree} \le 2]\times\partial/\partial x\}$

Type 2. $\mathbf{g}=\{\langle\exp(ax),\exp(-ax),1\rangle\times\partial/\partial x\}$

Type 3. $\mathbf{g}=\{\langle\sin(ax),\cos(-ax),1\rangle\times\partial/\partial x\}$

Sophus Lie obtained a similar result on the complex plane $\mathbf{C}$.

2 Lie Groups Acting on 2-Manifolds

In the 2-dimensional case (and also in higher dimensional case) such classification problems become those of PDE's.

Analytic 2-dimensional orbits are listed and classified in [6] and in [2] in terms of Lie algebra of vector fields. There we can see easily that for any integer $n > 1$ there exists a nilpotent Lie group of dimension n which acts effectively and transitively on $\mathbf{R}^2$. Therefore we can not bind the dimension of Lie groups. However;

Theorem (Epstein-Thurston, [1]) *If* $\mathbf{g}$ *is a solvable Lie algeba acting transitively on m-dimensional manifold, then* $\mathbf{d}$*(=the derived length)* $\leq m + 1$ *holds. In the case that* $\mathbf{g}$ *is nilpotent, we have* $\mathbf{d} \leq m$.

As to simple Lie groups, thanks to the list we know every homogeneous 2-spaces.

3 Non-transitive Actions on 2-Manifolds

The next problem is to classify non-transtive actions of (semi-)simple Lie algebras on surfaces. Modifying Thurston's argument of the generalized Reeb stability [10] into the Lie algebra version, Plante proved the following.

Theorem (Plante, [7]) *If such an action has a fixed point, the Lie algebra is isomorphic to* $sl(2;\mathbf{R})$.

The essential idea is to modify the generalized Reeb stability of Thurston [10] into the Lie algebra version, *i.e.*, if the tangential representation of the Lie algebra $\mathbf{g}$ on T_xM at a fixed point $x \in M$ vanishes, Thurston's argument implies that $\mathbf{g}$ has a non-trivial abelian homomorphism. Therefor the simplicity implies that $\mathbf{g}$ is isomorphic to a simple subalgebra of $gl(2;\mathbf{R})$. Combining this with Lie's theorm, it turns out that among simple Lie algebras only $sl(2;\mathbf{R})$ can act on surfaces non-transitively. Furthermore, such actions around fixed points are diffeomorphic to the standard linear action.

Remark Durling this meeting, Etienne Ghys gave a comment to the author that simple Lie group actions around a fixed point can be linearized smoothly and this is a classical result.

Therefore the actions of $\mathbf{g}=sl(2;\mathbf{R})$ on surfaces with a 1-dimensional orbit $\mathcal{L}$ and two 2-dimensional orbits $\mathcal{O}_+$ and $\mathcal{O}_-$ on each side of $\mathcal{L}$ are of interest. Up to coverings, 2-dimensional orbits are classified into three types G/H, G/P, and G/K, according to typical three elements $H=\begin{pmatrix}1,0\\0,1\end{pmatrix}$, $P=\begin{pmatrix}0,1\\0,0\end{pmatrix}$, and $K=\begin{pmatrix}0,-1\\1,0\end{pmatrix}$. Here G/X denotes the quotient of $G = PSL(2;\mathbf{R})$ or its covering by the 1-parameter subgroup generated by X. The problem is **which combinations** of types of 2-dimensional orbits can occur and **how** they are glued together along 1-dimensional orbit $\mathcal{L}$.

Examples 1) The projective linear action of $PSL(2;\mathbf{R})$ as a subgroup of $PSL(2;\mathbf{C})$ on CP^1 has an orbit decompostion [*the upper half space*, RP^1, *the lower half space*] as a model of $[\mathcal{O}_+,\mathcal{L},\mathcal{O}_-] = [G/K\,|\,G/K]$.
2) If we take the projective sphere of the linear representation $\mathbf{g}_{ad}$, we find a model of $[G/K\,|\,G/H]$.
3) Like in 2) taking the projective spheres of linear representations, we obtain models of $[G/H \mid G/H]$ and $[G/P \mid G/P]$. Remark that they are analytic.
4) If we take the projective sphere of the linear representation $\mathbf{g}_{ad}\oplus\mathbf{R}$, we find a C^1-model of $[G/K \mid G/P]$ and also that of $[G/H \mid G/P]$ by finding C^1-invariant submanifolds.

By solving PDE Schneider [8] and Stowe [9] classified analytic structures around $\mathcal{L}$. According to them, for the models $[G/H\,|\,G/H]$, $[G/K\,|\,G/K]$, and $[G/H\,|\,G/K]$ we have countably infinitely many solutions. $[G/P\,|\,G/P]$ has also infinitely many solutions, however, neither $[G/H\,|\,G/P]$ nor $[G/K\,|\,G/P]$ has analytic solutions.

Conjectures 1) The dimension of any Lie group which can act on a **closed** n-manifold does not exceed $n^2 + 2n$.
2) Only the projective linear actions on the projective spheres and the conformal actions on the standard spheres are the possible **transitive** actions of **non-compact** Lie groups on **compact** manifolds.

4 Characteristic Functions

Now we give an elementary method to classify actions in the previous section without solving PDE's. This method is valid for more general higher dimensional cases, but efficient only for non-transitive actions.

For a while let $\mathbf{g}$ denote any simple Lie algebra with its Killing form $B : \mathbf{g} \to \mathbf{g}^*$, and let M denote a Riemannian manifold with its metric $R : TM \to T^*M$ on which $\mathbf{g}$ acts. For each point $x \in M$, we have a linear endomophism $E_x = B^{-1} \circ A_x{}^* \circ R \circ A_x : \mathbf{g} \to \mathbf{g}$, where A_x is the evaluation $\mathbf{g} \to \mathcal{X}(M) \to T_xM$ and $A_x{}^*$ is its dual. For any invariant polynomial ϕ on $gl(\mathbf{g})$ we obtain a function $\phi(x) = \phi(E_x)$ on M.

Theorem 1) *On any d-dimensional orbit, σ_d does not change its sign, where σ_d is the d-th elemetary symmetric polynomial on eigenvalues.*
2) *Especially, for any $(d+1)$-dimensional (semi-)simple Lie group G and its closed 1-parameter subgroup <X> generated by $X \in \mathbf{g}$ and for any Riemannian metric on G/X, we have $\sigma_d > 0$, $\equiv 0$, or < 0 on G/X according to $B(X, X) > 0$, $= 0$, or < 0.*
3) *For $sl(2; \mathbf{R})$-actions in the previous section, we have $\sigma_2 > 0$ on G/K, $\equiv 0$ on G/P, and < 0 on G/H.*

Taking an analytic Riemannian metric around $\mathcal{L}$, quite easily we obtain the followings.

Corollary 1) *We have analytic solutions neither for $[G/H \mid G/P]$ nor for $[G/K \mid G/P]$.*
2) *The equivalence classes of germs of σ_2 around $\mathcal{L}$ under local analytic transformations classify the analytic structures of all analytic slutions of $[G/H \mid G/H]$, $[G/K \mid G/K]$, and $[G/H \mid G/K]$.*

The similar statements hold for higher dimensional cases, *e.g.*, for the Lie groups $SO(p, q)$ in the situation of Theorem 2).

Closely looking into all analytic solutions of $[G/H \mid G/H]$, $[G/K \mid G/K]$, and $[G/H \mid G/K]$, it turns out that they all belong to a single family, as explained below. Under analytic conjugacy, Example 2) has the following expression for the basis $H = \begin{pmatrix} 1,0 \\ 0,1 \end{pmatrix}$, $L = \begin{pmatrix} 0,1 \\ 1,0 \end{pmatrix}$, and $K = \begin{pmatrix} 0,-1 \\ 1,0 \end{pmatrix}$ around $\mathcal{L}$.

$$
\begin{aligned}
K &= \partial/\partial x \\
H &= (1+y) \sin x \partial/\partial x + (2y + y^2) \cos x \partial/\partial y \\
L &= (1+y) \cos x \partial/\partial x - (2y + y^2) \sin x \partial/\partial y
\end{aligned}
$$

Pulling back this model by the maps $\Phi_{n,\pm}(\xi,\eta) = (x{=}\xi,\ y{=}\pm\eta^n)$, we obtain all analytic solutions of $[G/K\,|\,G/K]$ $\{$*resp.* $[G/H\,|\,G/H]\}$ taking $(n,\ \pm) =$ (*even*, $+$) $\{$*resp.* (*even*, $-$)$\}$, and those of $[G/H\,|\,G/K]$ taking (*odd*, $\pm$). (If we take $y(\eta)$ to be non-analytic smooth function, we obtain smooth solutions.) We can recover completely this function Φ from σ_1 and σ_2. Thus Corollary 2) is strengthened a little.

Essentially these arguments also work for any reductive Lie algebras. If M admits a symplectic structure $\Omega : TM \to T^*M$ we can replace the Riemannian metric with it. In the 2-dimensional case, we obtain the same results.

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