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A problem on the singulalities of a real algebraic vector field

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I would like to thank the organizers for giving me time for this problem session. I have a problem on the singularities of a real algebraic vector field. I am not at all a specialist of this field. My problem might be familiar or easy for specialists.

1. A vector fields.

Let $M(n)$ be the algebra of all $n \times n$ complex matrices, χ a monic complex polynomial of degree n, $M(\chi)$ the subset of all $X \in M(n)$ such that the characteristic polynomial of X is given by χ. $M(\chi)$ is a complex algebraic subvariety of $M(n)$. Moreover, let $N(n)$ be the set of all $n \times n$ normal matrices, $N(\chi) := N(n) \cap M(\chi)$.

Consider a real algebraic vector field V on $M(n)$ defined by

$$V(X) := [[X^*, X], X] \qquad \text{at} \quad X \in M(n),$$

where X^* is the Hermitian adjoint of X. We provide $M(n)$ with the Hermitian inner product and the Hermitian norm defined by

$$(X, Y) := \mathrm{Trace}(XY^*), \qquad \|X\| := \sqrt{(X, X)}.$$

The vector field V arises as the gradient flow of the functional φ : $M(n) \to \mathbf{R}$ defined by

$$\varphi(X) := \frac{1}{4}\|[X^*, X]\|^2.$$

LEMMA 1.1. *The fixed point set of V is $N(n)$.*

The vector field V preserves each conjugacy class of $M(n)$, where a conjugacy class means a $GL(n)$-orbit of the group action

$$M(n) \times GL(n) \to M(n), \qquad (X, g) \mapsto g^{-1}Xg.$$

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In particular, for any χ, V preserves $M(\chi)$, and hence one can consider the restriction V_χ of V into $M(\chi)$:

$$V_\chi = V|_{M(\chi)}.$$

The vector field V arises as the gradient flow of another variational problem. To state it, let C be any conjugacy class and consider the functional $\psi : C \to \mathbf{R}$ defined by

$$\psi(X) := \frac{1}{2}\|X\|^2.$$

C is a locally closed complex submanifold of $M(n)$ and its tangent space at $X \in C$ is given by

$$T_X C = \text{Image of} \quad \text{Ad}(X) : M(n) \to M(n), \quad Y \mapsto [X,Y].$$

If $T(X)$ is the orthogonal complement of Ker Ad(X), then we have an isomorphism Ad$(X) : T(X) \to T_X C$. We provide $T_X C$ with a Hermitian inner product so as to make Ad$(X) : T(X) \to T_X C$ an isometry. Thus we have obtained a Hermitian metric on C. The gradient flow of the functional $\psi : C \to \mathbf{R}$ with respect to this Hermitian metric gives the vector field $V_C := V|_C$ on C.

2. Stratification.

$M(\chi)$ consists of a finite number of $GL(n)$-orbits. Let $\mathcal{O}(\chi)$ be the set of all orbits in $M(\chi)$. $\mathcal{O}(\chi)$ gives a stratification of $M(\chi)$ by locally closed complex submanifolds. We introduce a partial order $<$ in $\mathcal{O}(\chi)$: For $C_1, C_2 \in \mathcal{O}(\chi)$, we put $C_1 < C_2$ if and only if $C_1 \subset \overline{C_2}$. Let $E(\chi)$ be the set of all $e = (e_1, e_2, \ldots, e_n)$ such that

(1) e_i is a monic polynomial, $(i = 1, 2, \ldots, n)$,
(2) e_i divides e_{i+1}, $(i = 1, 2, \ldots, n-1)$, and
(3) $e_1 e_2 \cdots e_n = \chi$.

For any $C \in \mathcal{O}(\chi)$, we denote by $e_i(C)$ be the i-th elementary divisor of C and put $e(C) := (e_1(C), e_2(C), \ldots, e_n(C))$.

Lemma 2.1. *There is a one-to-one correspondence:*

$$\mathcal{O}(\chi) \to E(\chi), \quad C \mapsto e(C).$$

For any $C_1, C_2 \in O$, we have $C_1 < C_2$ if and only if

$$\prod_{j=1}^{i} e_j(C_2) \quad \textit{divides} \quad \prod_{j=1}^{i} e_j(C_1), \quad (i = 1, 2, \ldots, n).$$

Remark 2.2: There are a unique maximal orbit $C_{max}(\chi)$ and a unique minimal orbit $C_{min}(\chi)$ in $\mathcal{O}(\chi)$ with respect to the partial order $<$.

Lemma 2.3. *Let $C \in \mathcal{O}(\chi)$.*
(i) The following three assertions are equivalent:

(1) $C = C_{min}(\chi)$.
(2) *C is closed in $M(\chi)$.*
(3) *C is semisimple.*

(ii) $C = C_{max}(\chi)$ if and only if C is open in $M(\chi)$.
(iii) $C_{min}(\chi) = C_{max}(\chi)$ if and only if χ has distinct n roots,
(iv) $X \in M(n)$ is smooth in $M(n)$ if and only if $X \in C_{max}(\chi)$, and
(v) $N(\chi) = N(n) \cap C_{min}(\chi)$.

Lemma 2.3 implies that, if χ has a multiple root, then $N(\chi)$ lies in the singularities of $M(\chi)$. If χ has distinct n roots, then $M(\chi)$ is smooth everywhere.

Consider the vector field V_χ on $M(\chi)$ This is a real algebraic *stratified vector field* on $M(n)$. In this symposium, Prof. Brasselet talked about complex analytic stratified vector fields.

Lemma 2.4. *The fixed point set of V_χ is $N(\chi)$. Moreover, the ω-limit set of V_χ is $N(\chi)$.*

3. Semisimple trajectries.

Consider the trajectry $\{X(t)\}_{t\geq 0}$ of V_χ starting from $X_0 \in M(\chi)$. $X(t)$ exists for all $t \geq 0$. If $X_0 \in C_{min}(\chi)$, then $X(t)$ is called a *semisimple* trajectry and, if $X_0 \notin C_{min}(\chi)$, then $X(t)$ is called a *non-semisimple* trajectry, respectively.

Notation 3.1: Let $\{z_1, z_2, \ldots, z_k\}$ be the set of mutually distinct roots of χ. We put

$$a(\chi) := \begin{cases} 0 & (k = 1), \\ \min_{i\neq j} |z_i - z_j|^2, & (k > 1). \end{cases}$$

Remark 3.2: (i) If $a(\chi) = 0$, then $C_{min}(\chi)$ consists of a single point which is a scalar matrix. So the trajectry $X(t)$ is a single point. Everything is trivial in this case.
(ii) If $a(\chi) > 0$, then $N(\chi)$ is a compact real analytic manifold of positive dimension. $N(\chi)$ is a $U(n)$-orbit.

Theorem 3.3. *There exists a continuous function $K : C_{min}(\chi) \to \mathbf{R}$ such that the following condition holds: For any $X_0 \in C_{min}(\chi)$ there exists a normal matrix $X_\infty \in N(\chi)$ such that the trajectry $X(t)$ starting from X_0 satisfies*

$$\|X(t) - X_\infty\| \leq K(X_0)\|[X_0^*, X_0]\|e^{-2a(\chi)} \qquad (t \geq 0).$$

Remark 3.4: (i) The function K can be given more explicitly (see [**Iw**]). (ii) Theorem 3.3 implies that each semisimple trajectry in $M(\chi)$ converges exponentially to a normal matrix in $N(\chi)$ as $t \to \infty$.

4. Non-semisimple trajectries.

What can we say about the non-semisimple trajectries ? We have at least the following:

Theorem 4.1. *For any non-semisimple trajectry $X(t)$,*

$$t\|[X^*(t), X(t)]\|^2 \to 0 \quad \text{as} \quad t \to \infty,$$

but

$$\int_0^\infty t\|[X^*(t), X(t)]\|^2 dt = \infty.$$

Now we propose the following:

Problem 4.2. *Does any non-semisimple trajectry converge as $t \to \infty$? If a non-semisimple trajectry does not converge, how does it behave?*

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