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THE ORBIT STRUCTURE OF PSEUDOGROUP ON RIEMANN SURFACES VS. DYNAMICS OF ALGEBRAIC CURVES

ISAO NAKAI

Department of Mathematics
Hokkaido University

ABSTRACT. We investigate the dynamical properties of algebraic plane curves in $\mathbb{P}\times\mathbb{P}$, and discuss the relation with psudogroup of diffeomorphisms of the curves generated by the monodromy actions.

INTRODUCTION

Let Γ be a pseudogroup consisting of diffeomorphisms $f: U_f, 0 \to f(U_f), 0$ of open neighbourhoods U_f of the complex plane $\mathbb{C}$ which fix $0 \in \mathbb{C}$. We call the group Γ_0 of the germs of those $f \in \Gamma$ the *germ* of Γ, and call Γ a *representative* of Γ_0. Γ is *non-solvable* if its germ is non-solvable.

We say that a subset $A \subset \mathbb{C}$ is *invariant under* Γ if $f(A \cap U_f) = A \cap f(U_f)$ for all $f \in \Gamma$. We call a minimal invariant set an *orbit (which is not necessarily closed).* The orbit containing an x is unique and denoted $\mathcal{O}(x)$, which is the set of those $f(z)$ with $z \in U_f, f \in \Gamma$. Let B_f denote the set of those $z \in U_f$ such that $f^{(n)}(z) \to 0$ as $n \to \infty$, where $f^{(n)}$ stands for the n-iterated $f \circ \cdots \circ f$. If f has the indifferent linear term z (in other words *parabolic* or *flat* at the origin), $B_f \cup B_{f(-1)}$ is an open neighbourhood of 0 (Proposition 2.4). The *basin* B_Γ is the set of points z for which the closure of the orbit contains the origin. Proposition 2.5 asserts that the basin is an open neighbourhood of the origin if an $f \in \Gamma$ is flat.

Assume B_Γ is an open neighbourhood of 0. The *separatrix* $\Sigma(\Gamma)$ for Γ is a closed real semianalytic subset of B_Γ, which possesses the following properties.

(1) $\Sigma(\Gamma)$ is invariant under Γ and smooth off 0,

(2) The germ of $\Sigma(\Gamma)$ at 0 is holomorphically diffeomorphic to a union of 0 and some branches of the real analytic curve Im $z^k = 0$ for some k,

(3) Any orbit is dense or empty in each connected component of $B_\Gamma - \Sigma(\Gamma)$,

(4) Any orbit is dense or empty in each connected component of $\Sigma(\Gamma) - 0$.

Key words and phrases. Pseudogroup, group action, orbit, holomorphic diffeomorphism.

Theorem 1. (The separatrix theorem). *If the germ Γ_0 of a pseudogroup Γ is non-solvable, then the basin B_Γ is a neighbourhood of 0 and Γ admits the separatrix $\Sigma(\Gamma)$.*

By definition, the separatrix $\Sigma(\Gamma)$ is unique. From this theorem we obtain

Corollary 2. *If the germ Γ_0 is non-solvable and the subgroup Γ_0^0 of the germs of diffeomorphisms $h \in \Gamma_0$ with the indifferent linear term does not admit antiholomorphic involution, then $\Sigma(\Gamma) - 0$ and all orbits different from $0 \in \mathbb{C}$ are dense or empty on a neighbourhood of 0.*

Example 3. Let $f(z) = z/1 - z, U_f = \mathbb{C} - \{1 < \text{Re } z, \text{Im } z = 0\}, g(z) = \log(1+z) = z - 1/2\, z^2 + 1/3\, z^3 - \cdots$ and let U_g $(\ni 1)$ be a small neighbourhood of 0, on which g restricts to a diffeomorphism onto the image $g(U_g)$. Let Γ be the pseudogroup generated by f and g. Since on the $\tilde{z}$-plane, $\tilde{z} = 1/z$, f induces the translation by -1, the basin B_Γ is the whole plane $\mathbb{C}$. By Theorem 1.8, the group Γ_0 generated by the germs of f, g at 0 is non solvable. The real lime $\mathbb{R}$ is invariant under the group Γ_0 and there is no other invariant curves. If U_g is small enough: $U_g \cap \mathbb{R}$ is contained in the half line $\{-1 < \text{Re } z, \text{Im } x = 0\}$, then g maps the real line $U_g \cap \mathbb{R}$ into $\mathbb{R}$ hence $\mathbb{R}$ is invariant under Γ. Clearly Γ preserves the upper (respectively lower) half plane. Therefore we obtain

$$\Sigma(\Gamma) = \mathbb{R}.$$

Next extend g so that the domain of definition U_g intersects with the half line $\mathbb{R}^-_{-1} = \{\text{Re } z < -1, \text{Im } x = 0\}$. Then g maps the intersedtion $U_g \cap \mathbb{R}^-_{-1}$ into the complement of the real line, where all orbits are locally dense. The local density holds also at the intersedtion $U_g \cap \mathbb{R}^-_{-1}$ and propergates to the negative part of the real line $\mathbb{R}^-$. Therefore we obtain

$$\Sigma(\Gamma) = \mathbb{R}^+.$$

Let $C \subset \mathbb{C} \times \mathbb{C}$ be an irreducible algebraic curve and π_x, π_y be the natural projection onto the first and the second coordinates. Assume that C is nondegenerate i.e. C does not contain a fiber of the projections. The equivalence relation $\sim$ introduced on C is generated by the relations

$$p \sim q \quad \text{if} \quad \pi_x(p) = \pi_x(q) \quad \text{or} \quad \pi_y(p) = \pi_y(q).$$

Our problem is

Problem 4. *Study the structure of the equivalence classes.*

In order to observe the structure of the dynamics at infinity, we suppose $C \subset \mathbb{P} \times \mathbb{P}$. Level sets of a rational function on C form a pencil of zero divisors. So a morphism of C into $\mathbb{P} \times \mathbb{P}$ is determined by a pair of pencils (L, M) of zero divisors on C. Our problem is to study the geometry of the morphisms in terms of divisors in pencils.

In the paper [N1] the author proved

Theorem 5(Rigidity theorem). *Let E, F be non degenerate linear systems of effective divisors of Riemann surfaces C, C'. Assume that the image of the morphism of C*

into the dual spaces of E, F are not of degree 3. Let $h : C \to C'$ be an orientation preserving homeonorphism sending divisors in E to those in F. Then h is a holomorphic diffeomorphism.

This theorem tells that complex space curves are determined by the structure of hyperplane sections. Our problem can regarded as the spacial case of the above theorem. The theorem suggests

Question 6. *Let C, C' be irreducible and nondegenerate algebraic curves in $\mathbb{P} \times \mathbb{P}$. Let ϕ, ψ be orientation preserving homeomorphisms of $\mathbb{P}$ such that $\phi \times \psi(C) = C'$. Then are ϕ, ψ Möbius transformations?*

In some special cases the answer is affirmative. From now on we seek a method to study the structure of the equivalence classes.

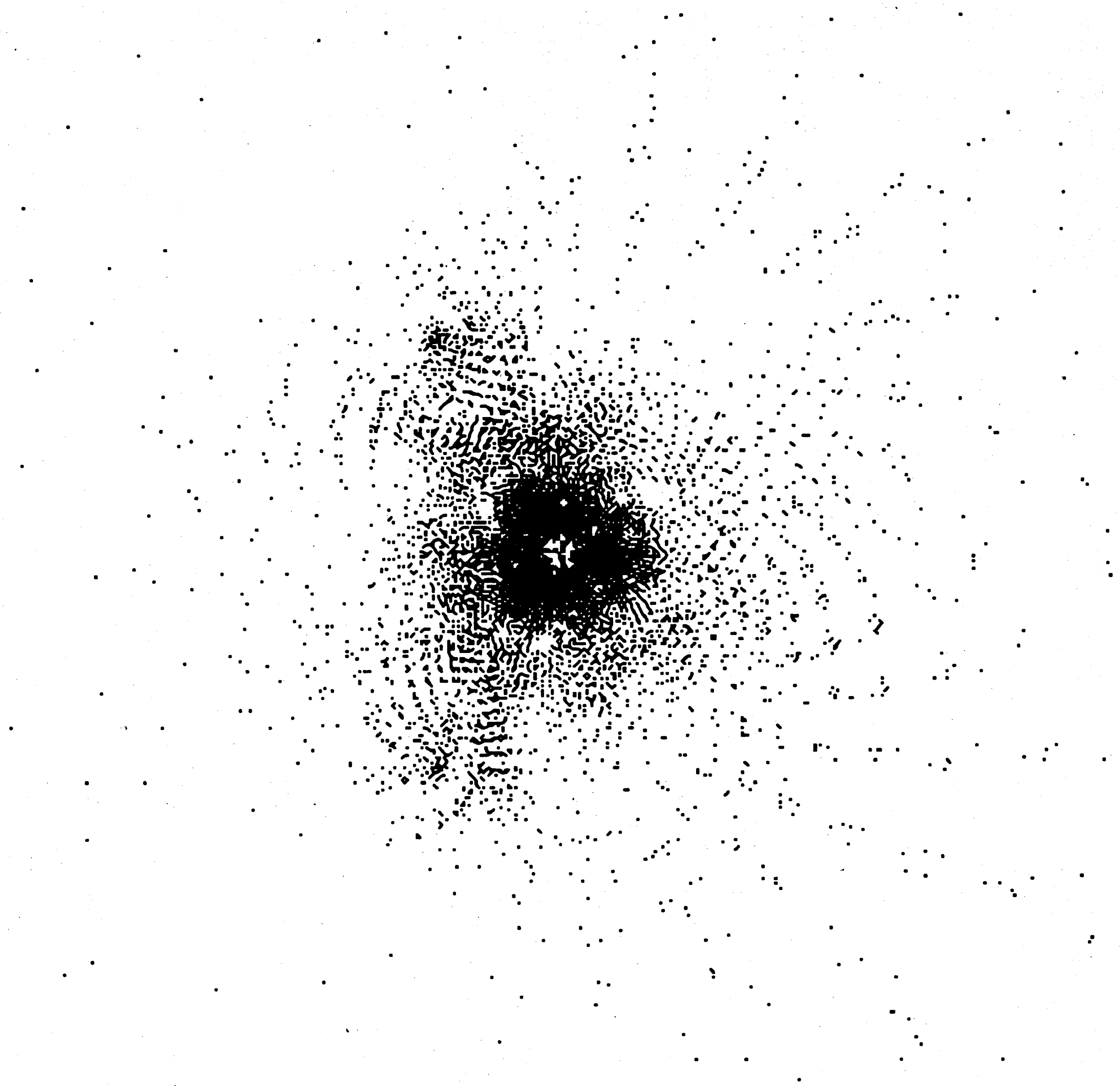
If these projections restrict respectively to d and e sheeted branched coverings of C, we say C is of degree (d, e). Two curves C, C' are *equivalent* if $C' = \phi \times \psi\ (C)$ with Möbius transformation ϕ, ψ of the coordinate lines. Let p be a singular point of C in the sense that either of the coordinate projections of C is not nonsingular and the local multiplicity of the first and second projections are respectively d', e'. Let t be a local coordinate of the curve centered at p. Then the monodromy action of the first and second projections are respectively order d', e' and generate a pseudogroup of diffeomorphisms of open neighbourhoods of 0 in the t line. The orbit of p under this pseudogroup Γ_p is contained in the equivalence class $\mathcal{O}(p)$ under the relation above introduced. If the germ G_p of the pseudogroup is non-solvable, the orbits under Γ_p are dense in some sectors of the basin B_{Γ_p} of the pseudogroup by Theorem 1. By Corollary 2 if the germ of the curve C at p does not admit a germ of antiholomorphic involution of type $\phi \times \psi$, the orbits under Γ_p are dense in a neighbourhood of B_p. Define the *basin* B_p of p by the set of those q such that the topological closure of the equivalence class $\mathcal{O}(q)$ contains p. It is easy to see that if the germ G_p is non-commutative, the basins $B_{\Gamma_p} \subset B_p$ are open. The above density of orbits propagates to the the basin B_p. If the germ is commutative, it is holomorphically conjugate with the cyclic subgroup of the linear S^1 action.

In the paper [N1], the author classified the commutative and nonsolvable groups of germs of holomorphic diffeomorphisms of the complex plane at the origin. It would be a good exercise to classify all algebraic curves singular at the origin $p = 0 \times 0$ where the group G_p is nonsolvable [N2]. A subset $K \subset C$ is *invariant* if K is a union of equivalence classes. Clearly the basin is invariant.

proclaimProblem Study the complement of the basins of the singular points of algebraic curves.

Example 7. *Assume C is defined by $(y - x^2 - c)(y - x) = 0$, which is the union of the diagonal line $y = x$ and the parabola $y = x^2 + c$. By the equivalence relation $\sim$ the points (x, y) on the parabola are identified with those points (x, x) on the diagonal line. Therefore the first projection of the orbits are generated by the relations $x \sim x^2 + c$ and $x \sim -x$, which are the union of the foreword orbit and its backword orbits. It is classically known that the cluster point set of any backword orbit is Julia set of the dynamics $x \to x^2 + c$.*

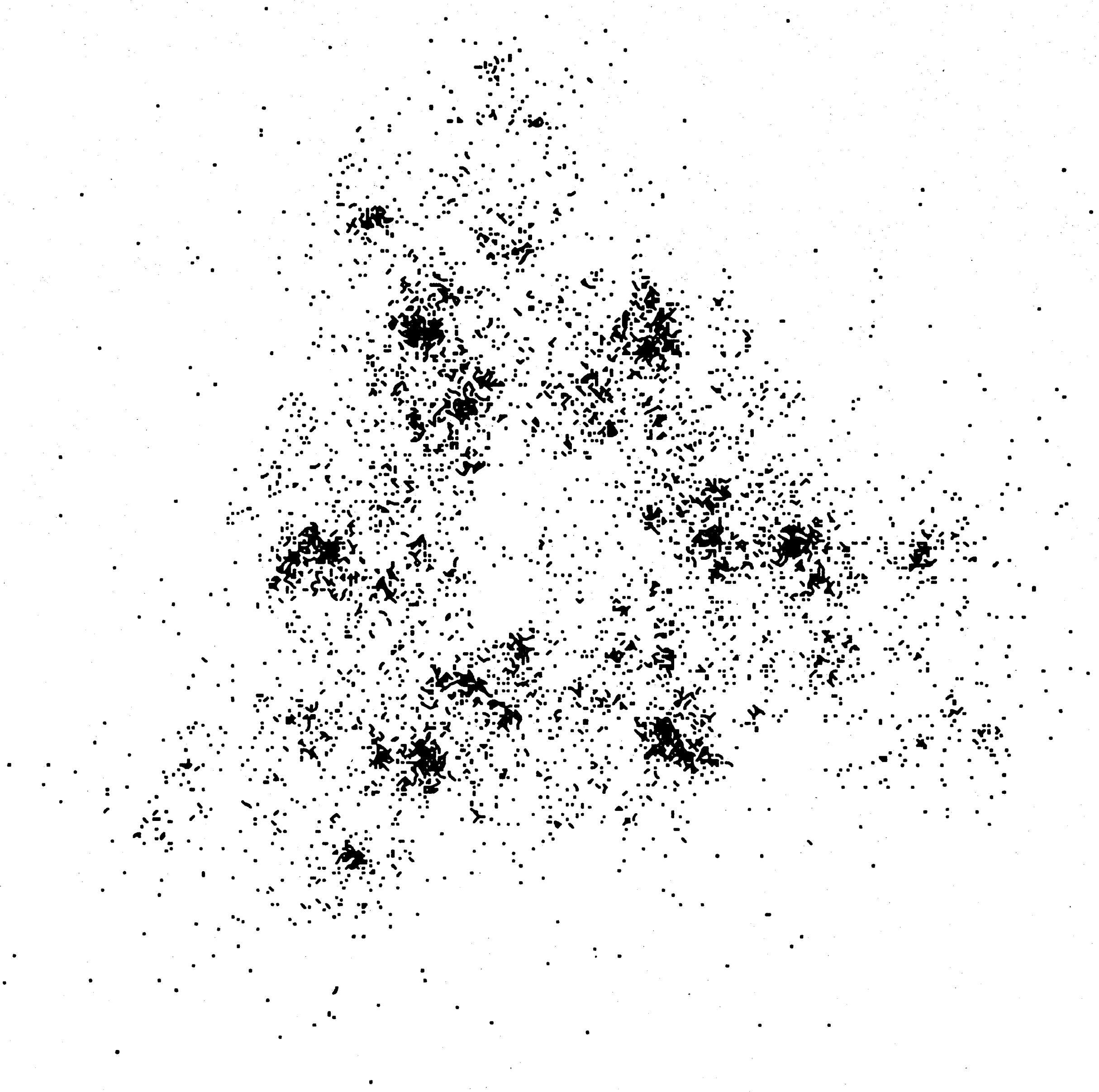
In the following we present some generic orbits of the dynamics on the various algebraic curves.



The projection of an orbit on the curve

$$x^3 + x^2 + 0.1xy + y^2 + x = 0$$

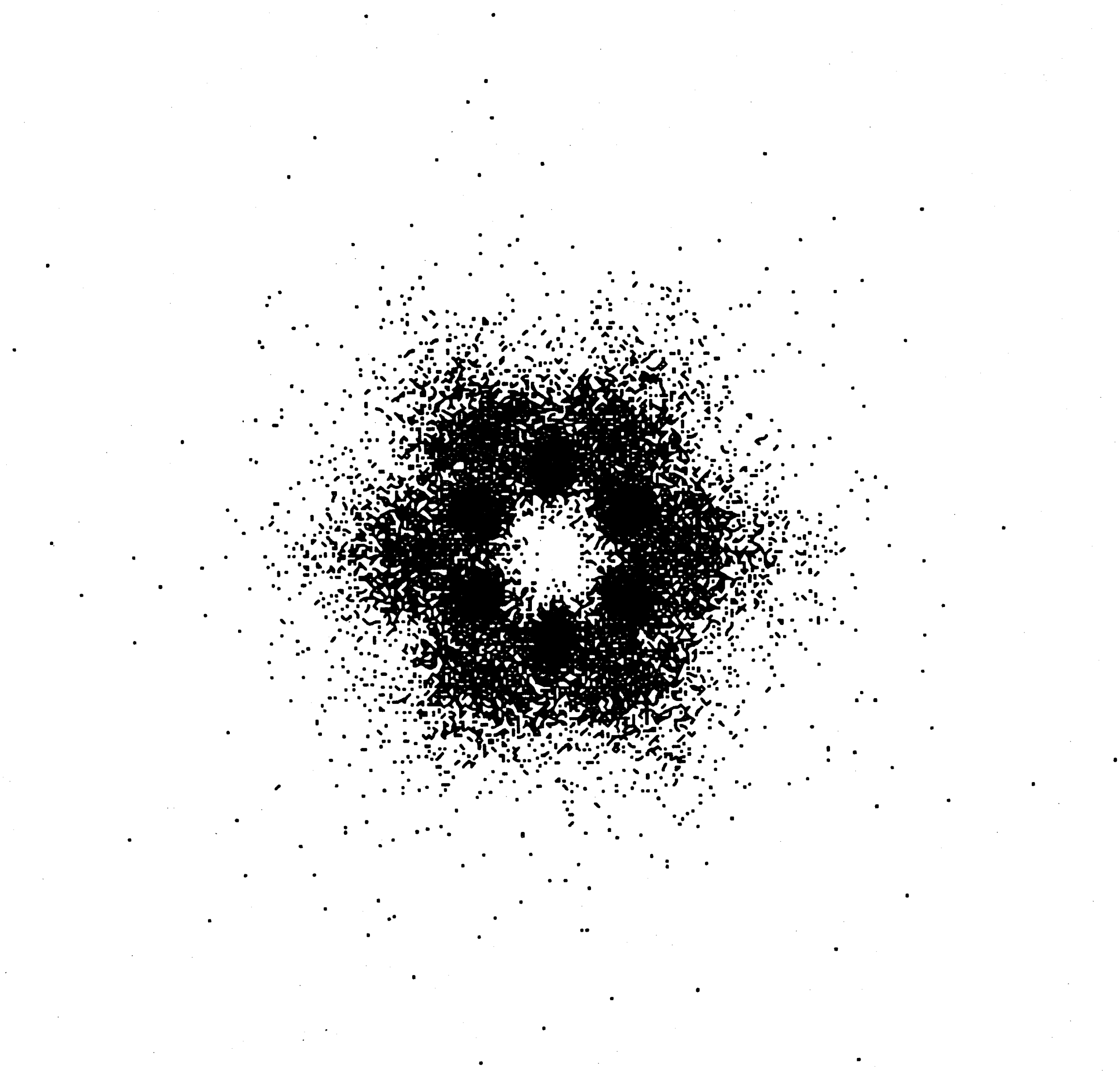
onto the $\tilde{x}$-plane, $\tilde{x} = 1/x$.



The projection of an orbit on the curve

$$(1+2i)x^3 + (3+2i)x^2y + (1+2i)xy^2 + (3+2i)y^3 \ + \ 1 = 0$$

onto the $\tilde{x}$-plane, $\tilde{x} = 1/x$.



The projection of an orbit on the curve

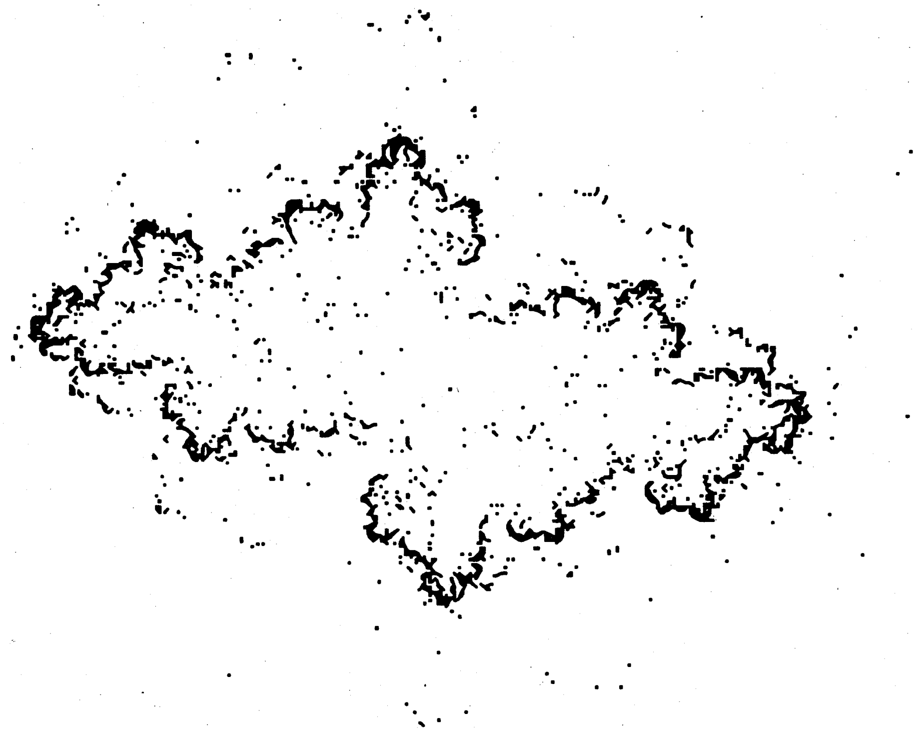
$$x^2y + y^3 + 1 = 0$$

onto the $\tilde{x}$-plane, $\tilde{x} = 1/x$.

The projection of an orbit on the curve (Julia set)

$$(y - x^2 + 0.7 - 0.3i)(y - x) = 0$$

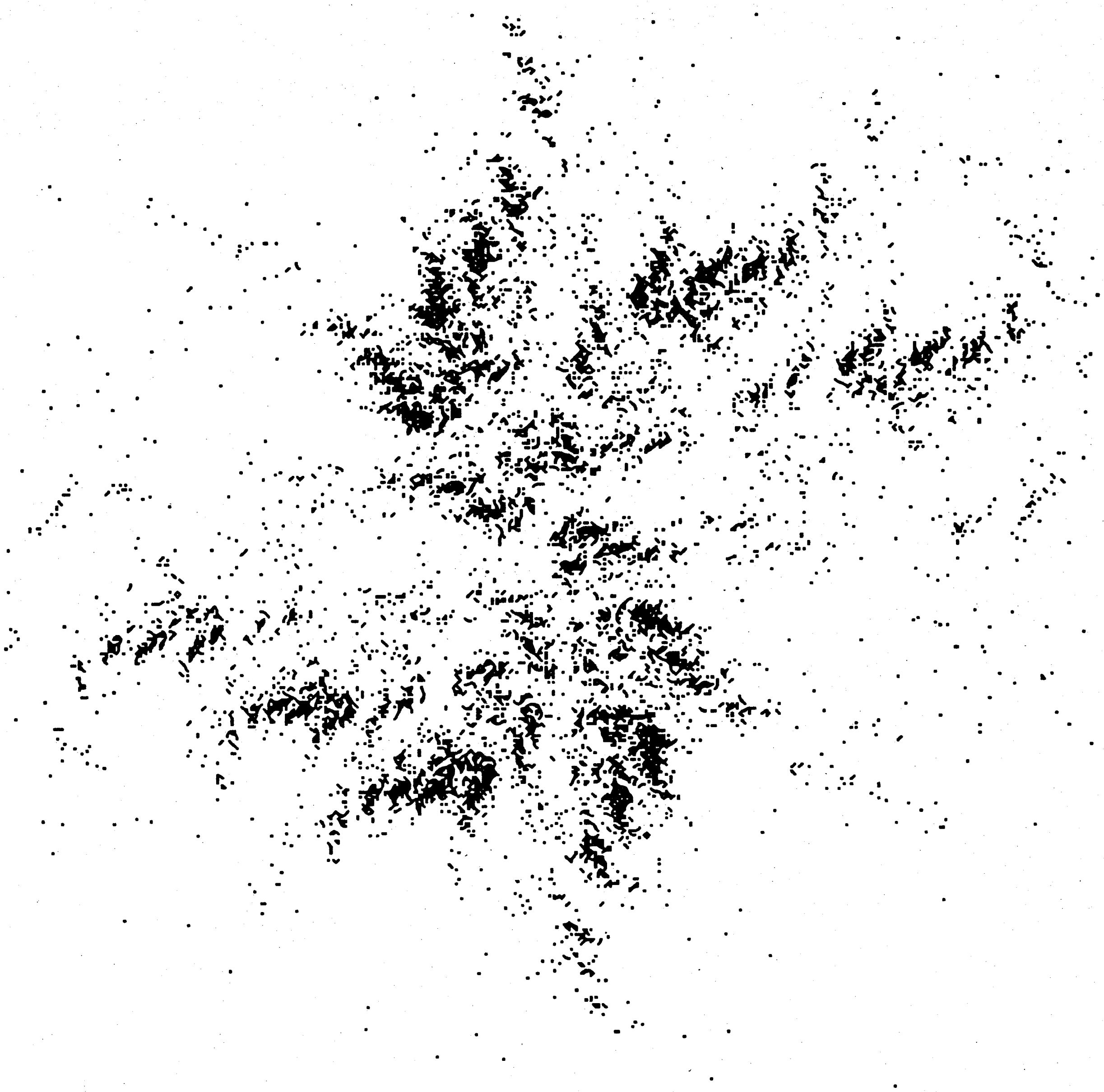
onto the $\tilde{x}$-plane, $\tilde{x} = 1/x$.



The projection of an orbit on the curve

$$(y - x^2 + 0.7 - 0.3i)(y - x) + 0.01 = 0$$

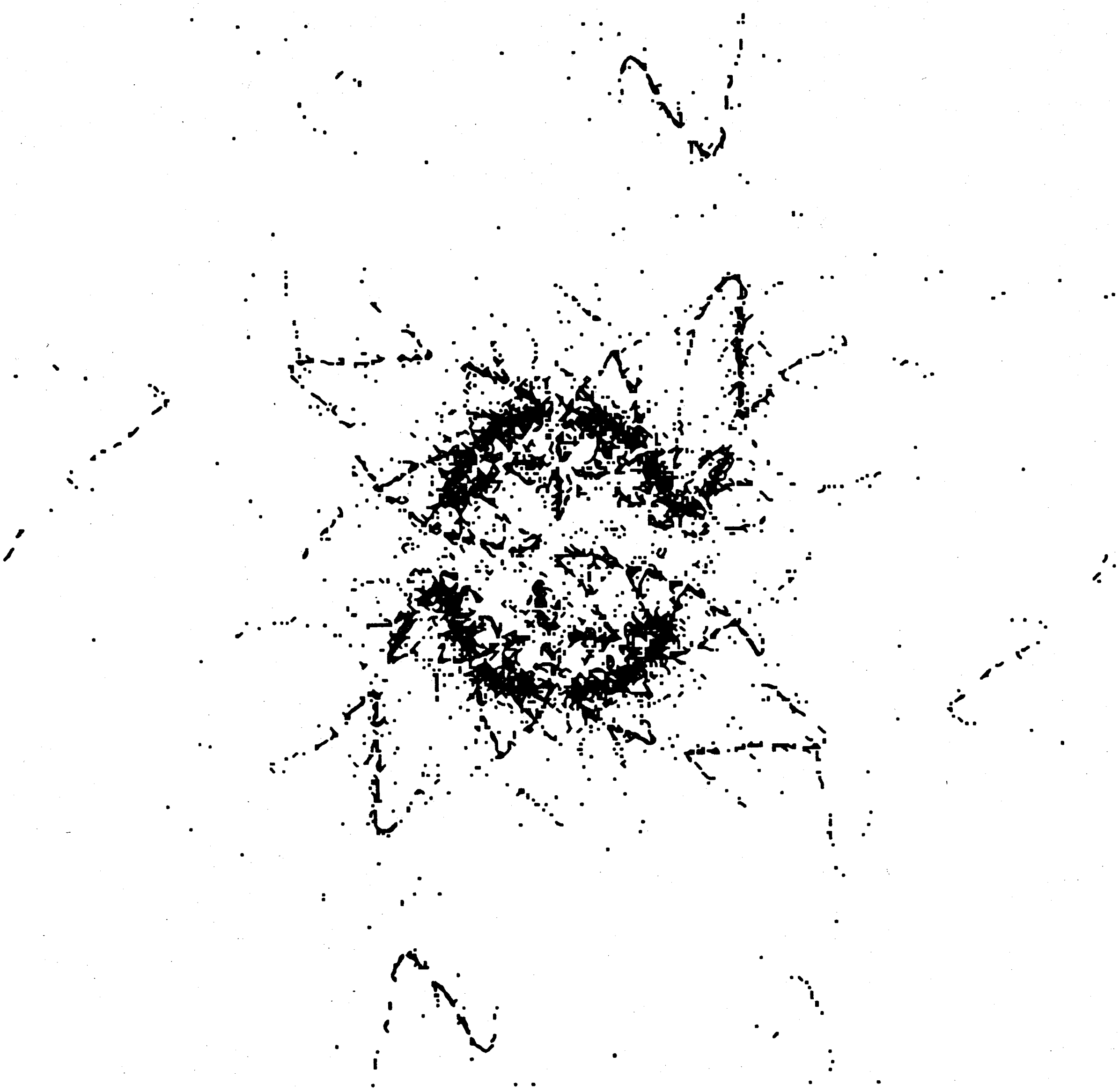
onto the $\tilde{x}$-plane, $\tilde{x} = 1/x$.



The projection of an orbit on the curve

$$(y - x^2 - 0.3)(y - x) + 0.3iy^2 = 0$$

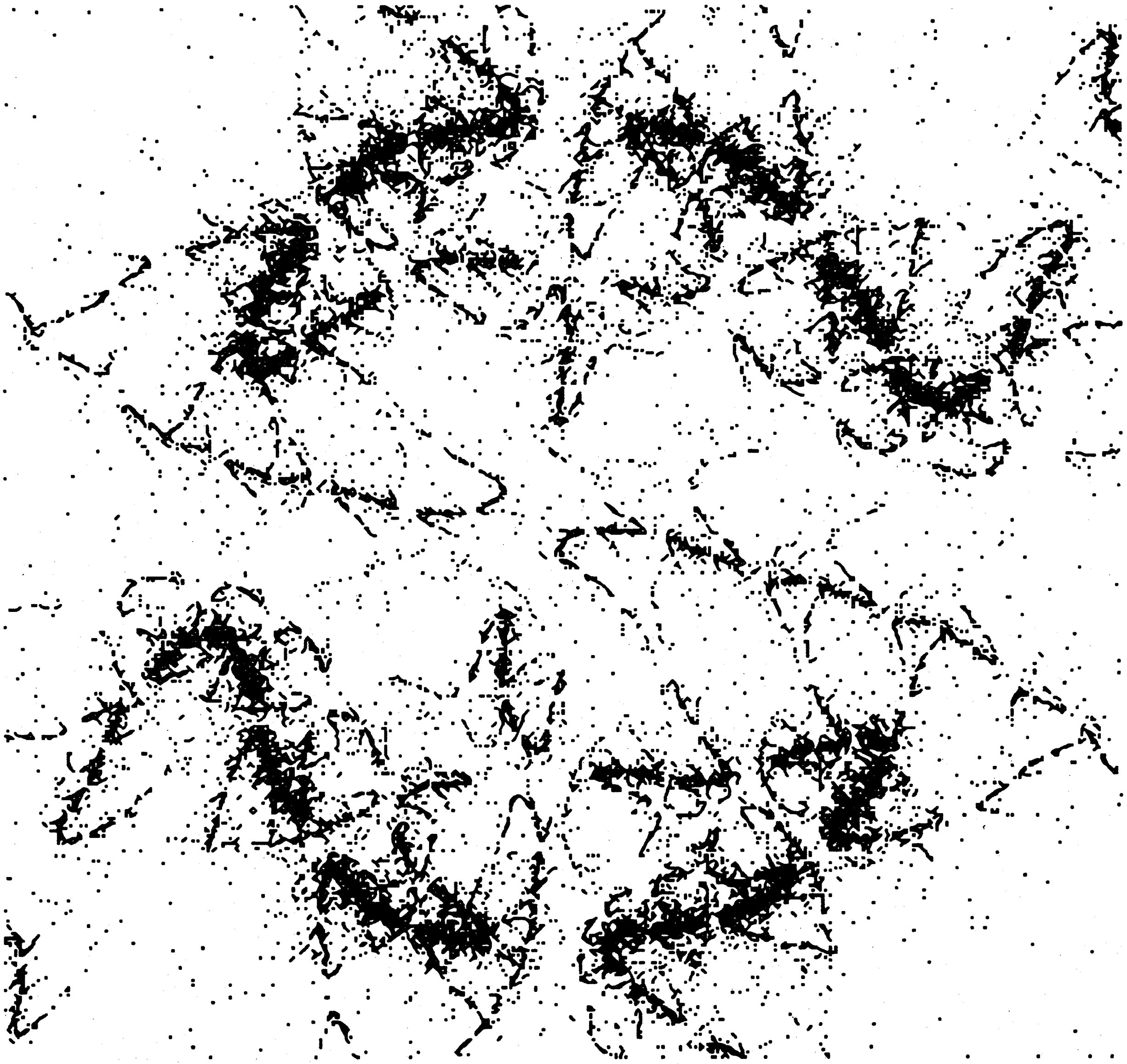
onto the x-plane.



The projection of an orbit on the curve

$$(y - x^2 - 0.3)(y - x) - 0.03 + 0.03i = 0$$

onto the x-plane.



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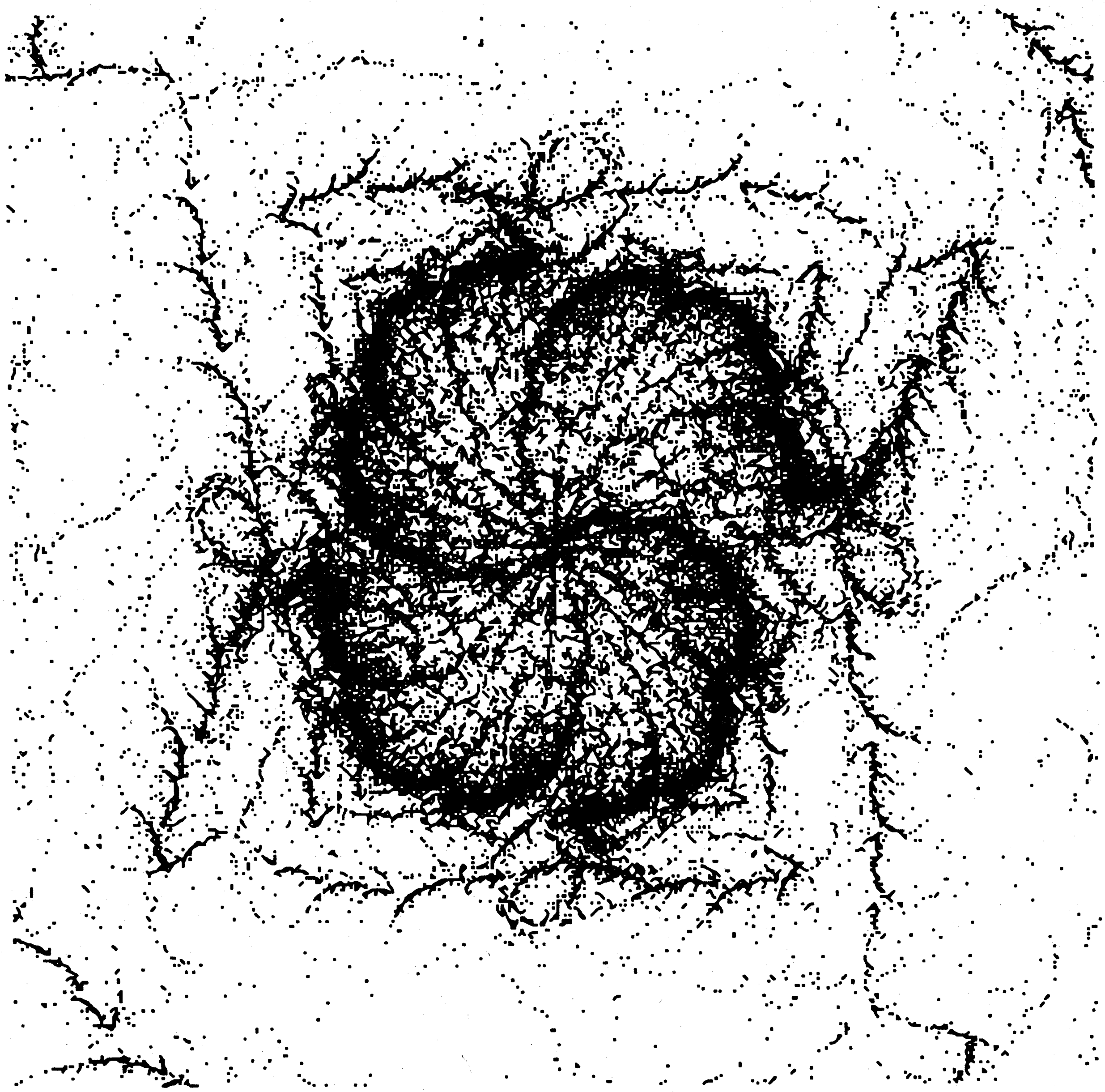

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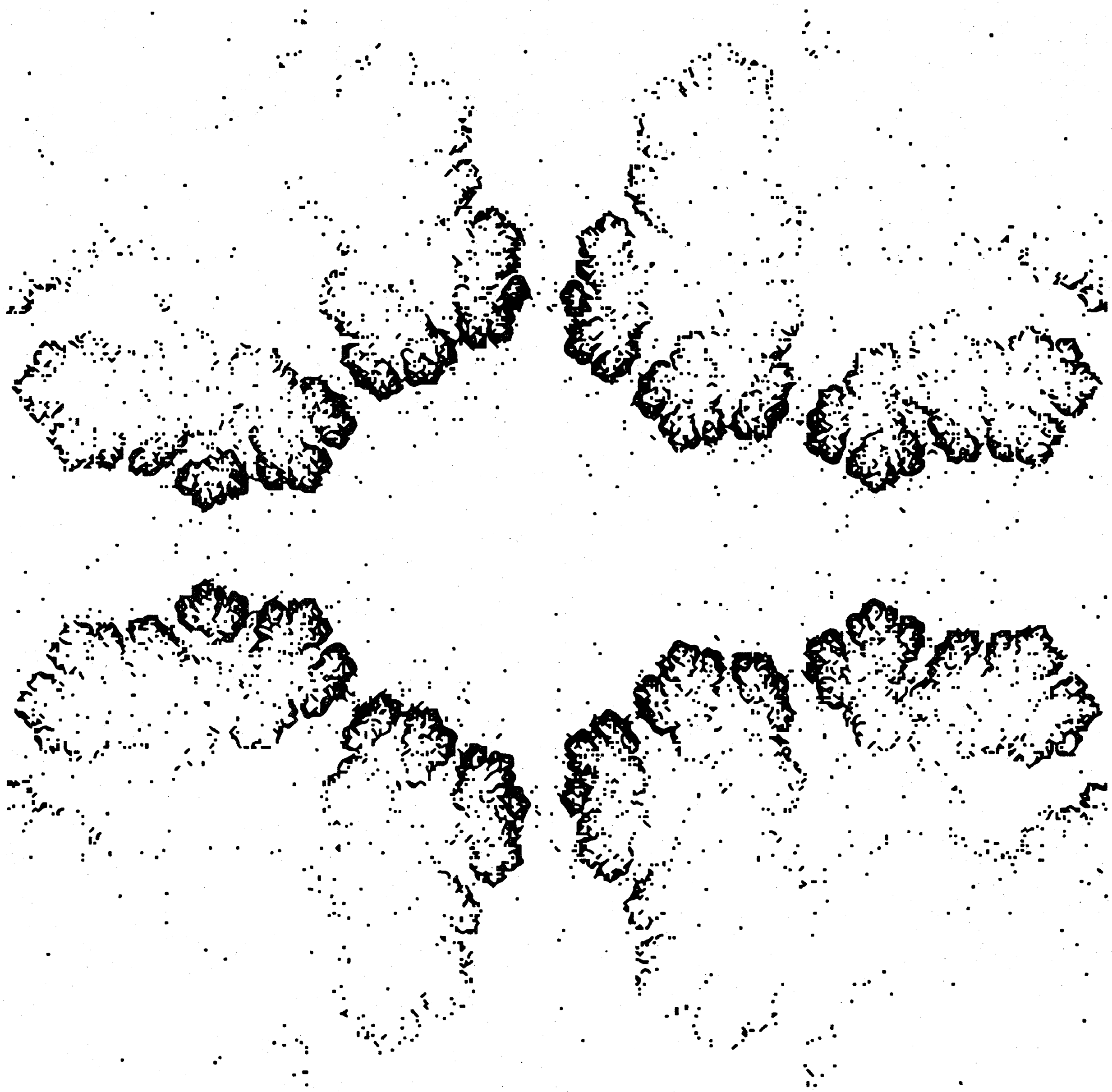
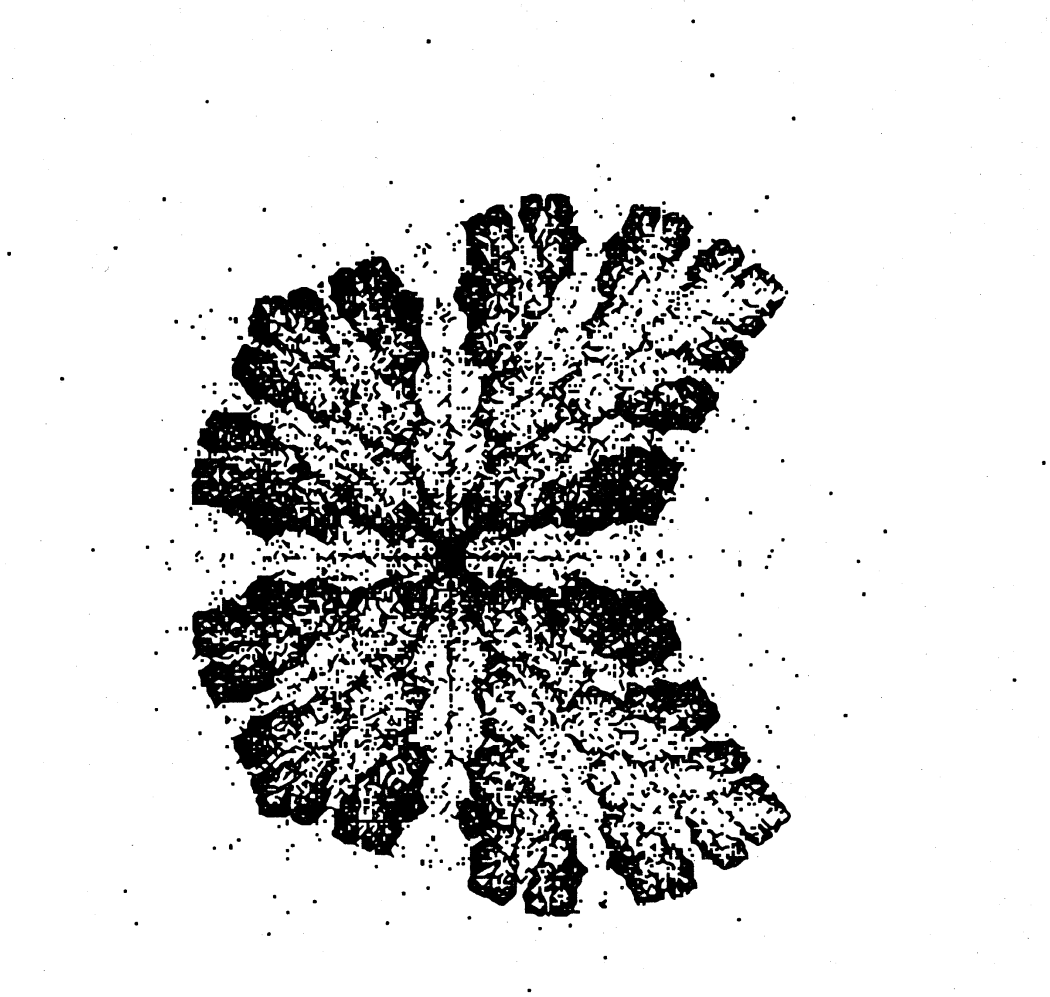


The projection of an orbit on the curve

$$(y - x^2 + 0.3)(y - x) - 0.3 = 0$$

onto the x-plane.



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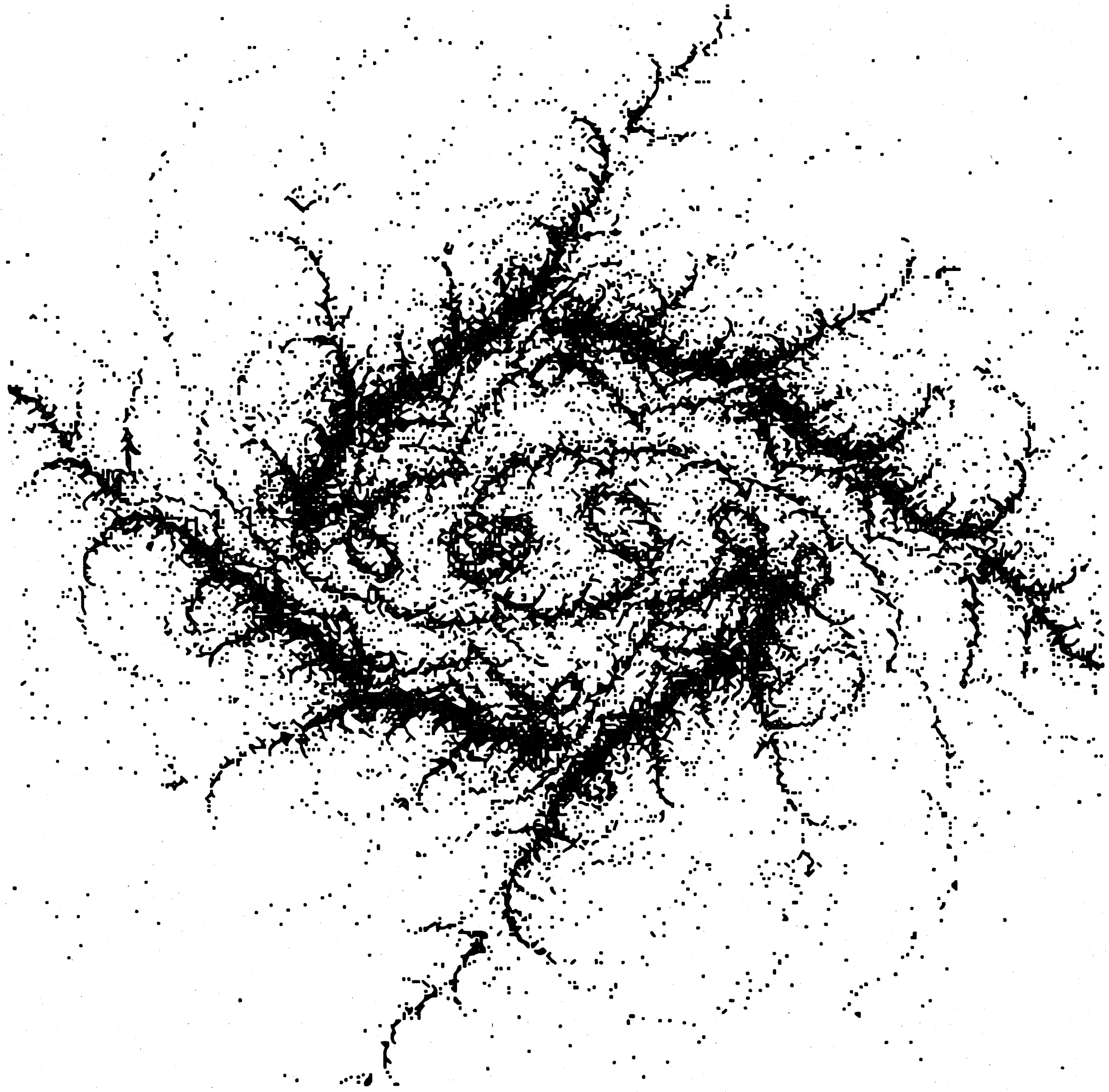
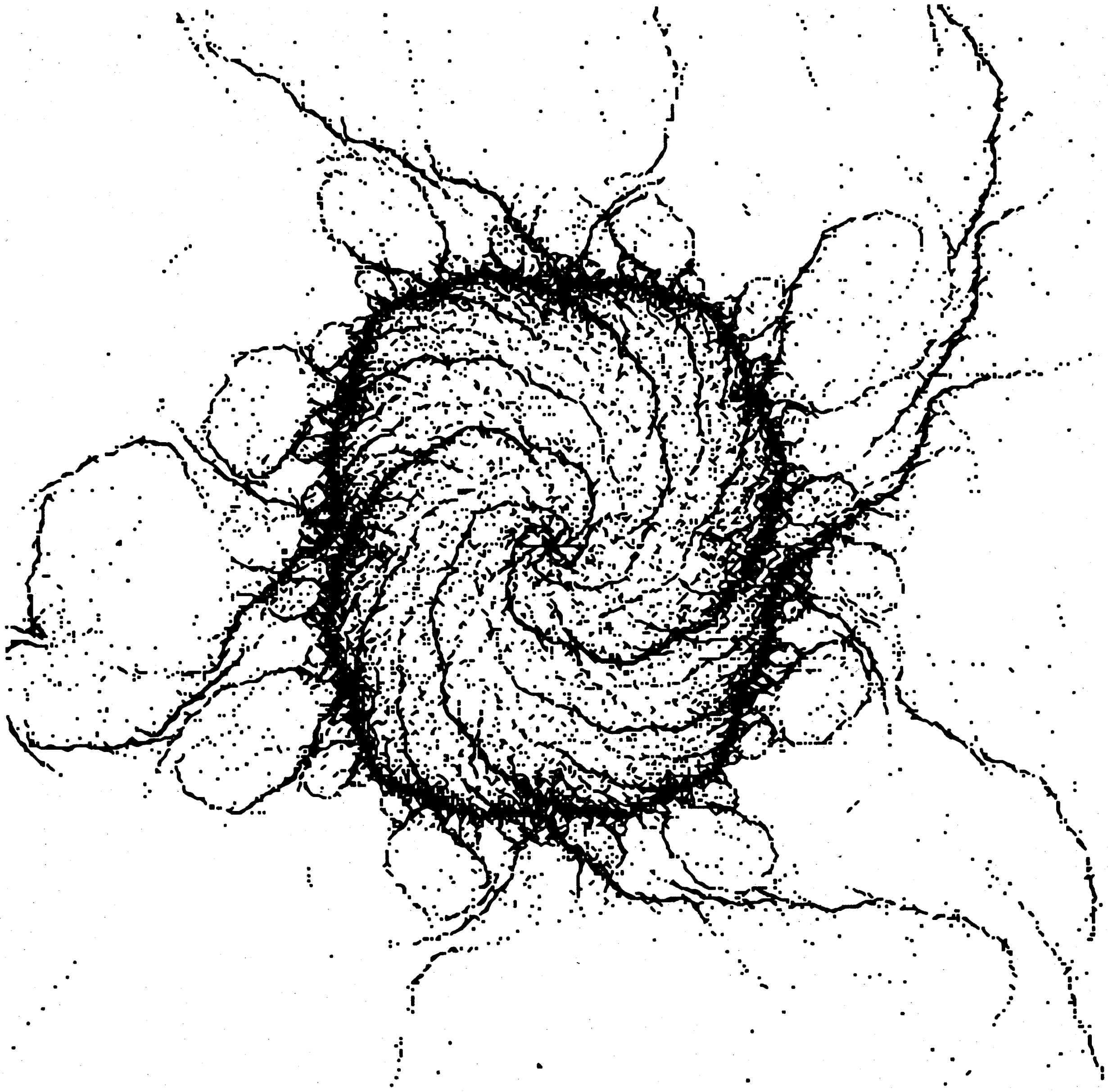


The projection of an orbit on the curve

$$(y - x^2 + 0.3)(y - x) + 0.1iy^2 = 0$$

onto the x-plane.



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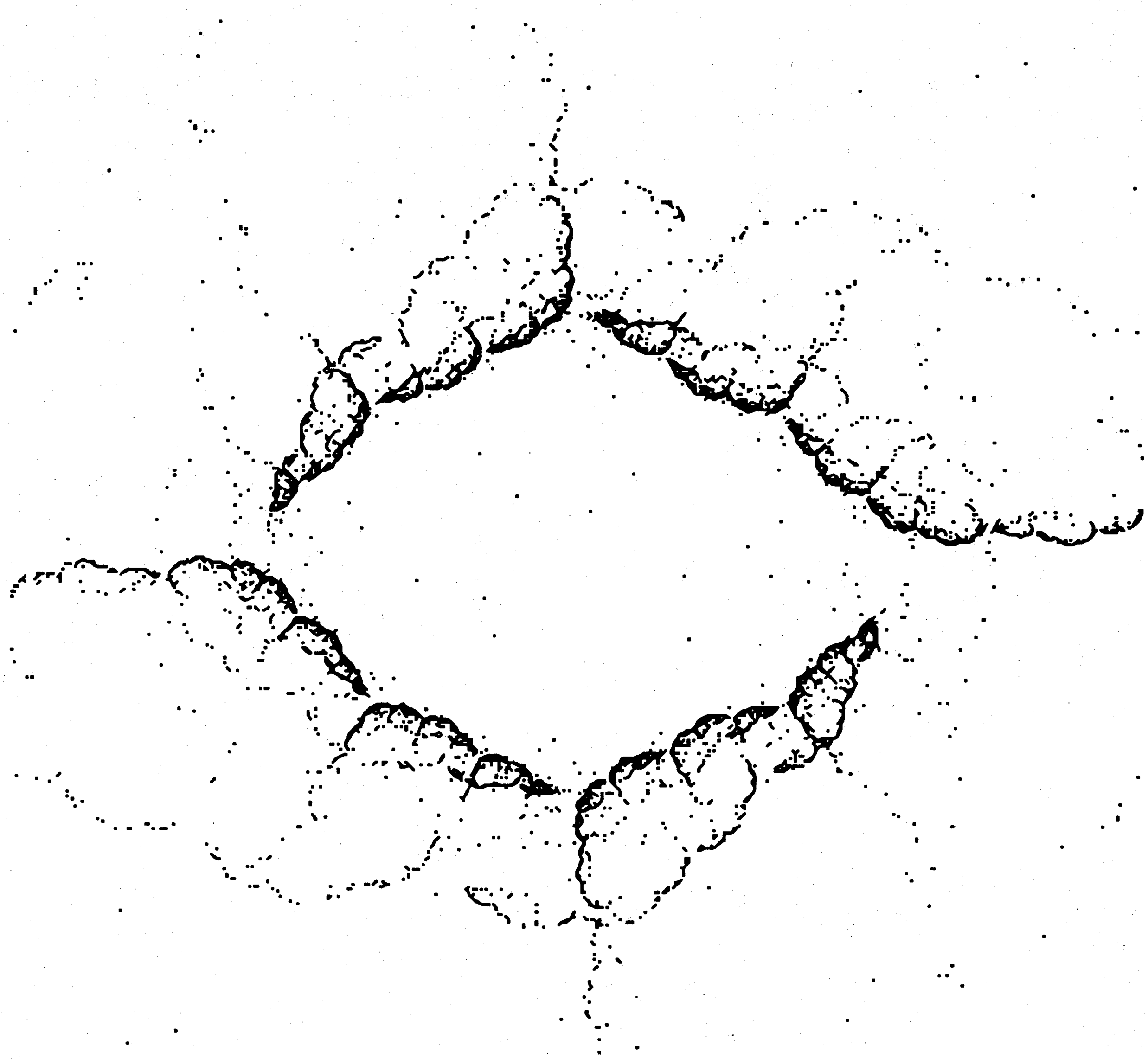

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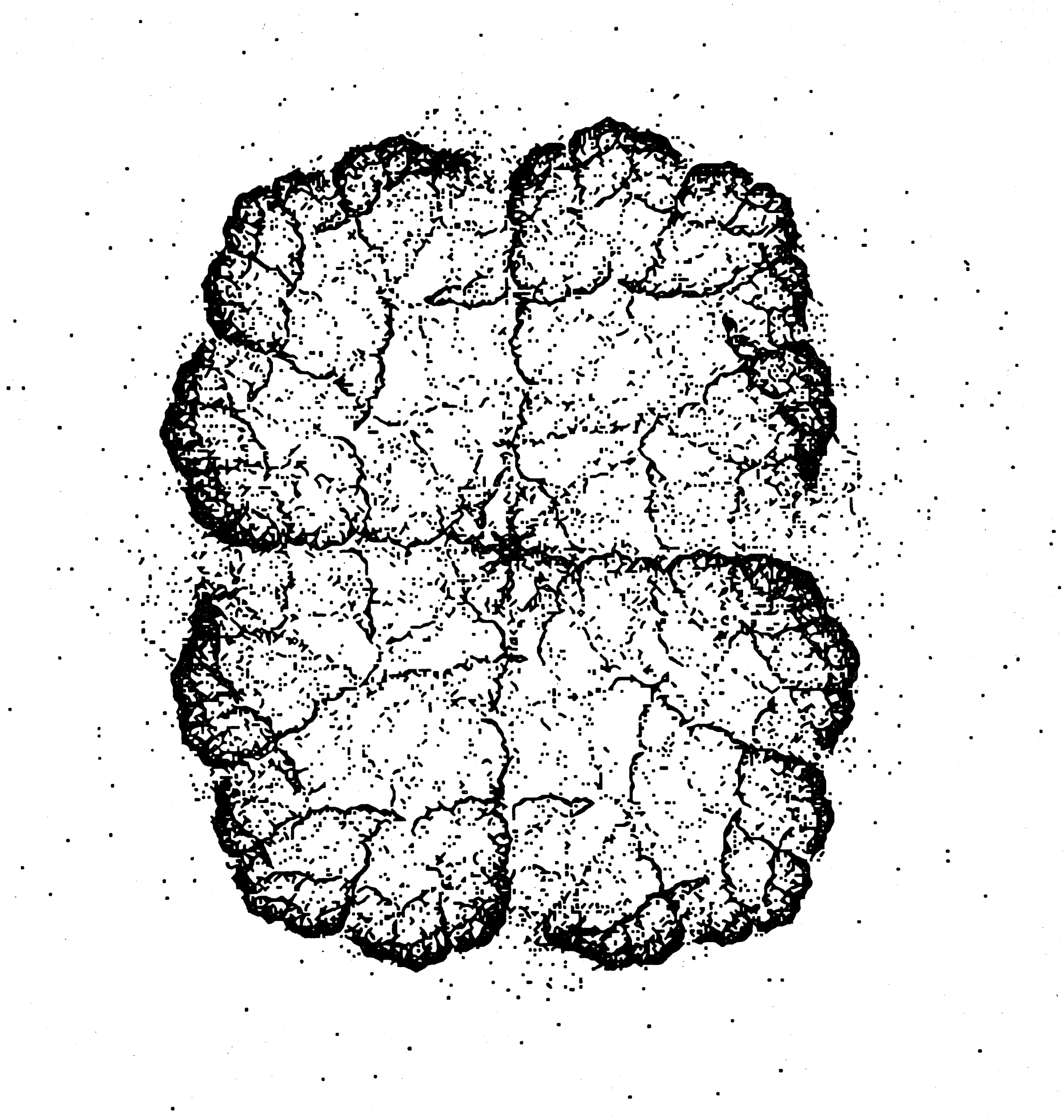

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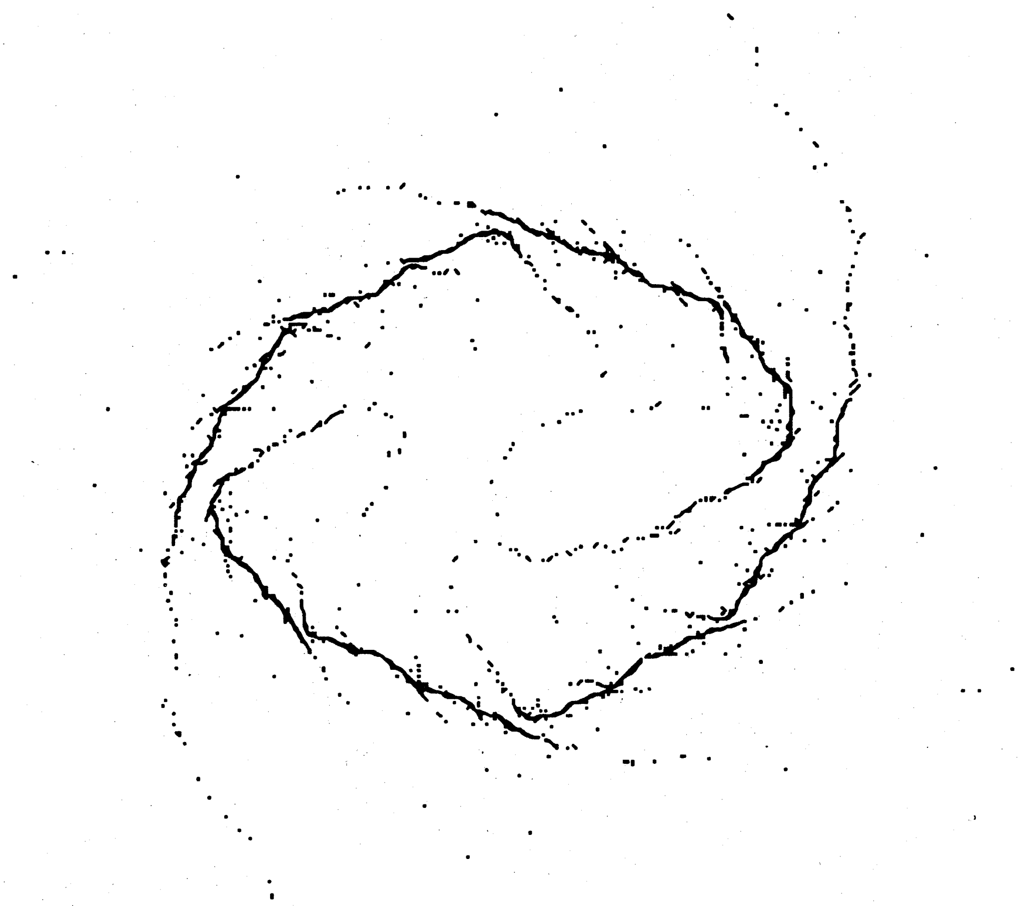

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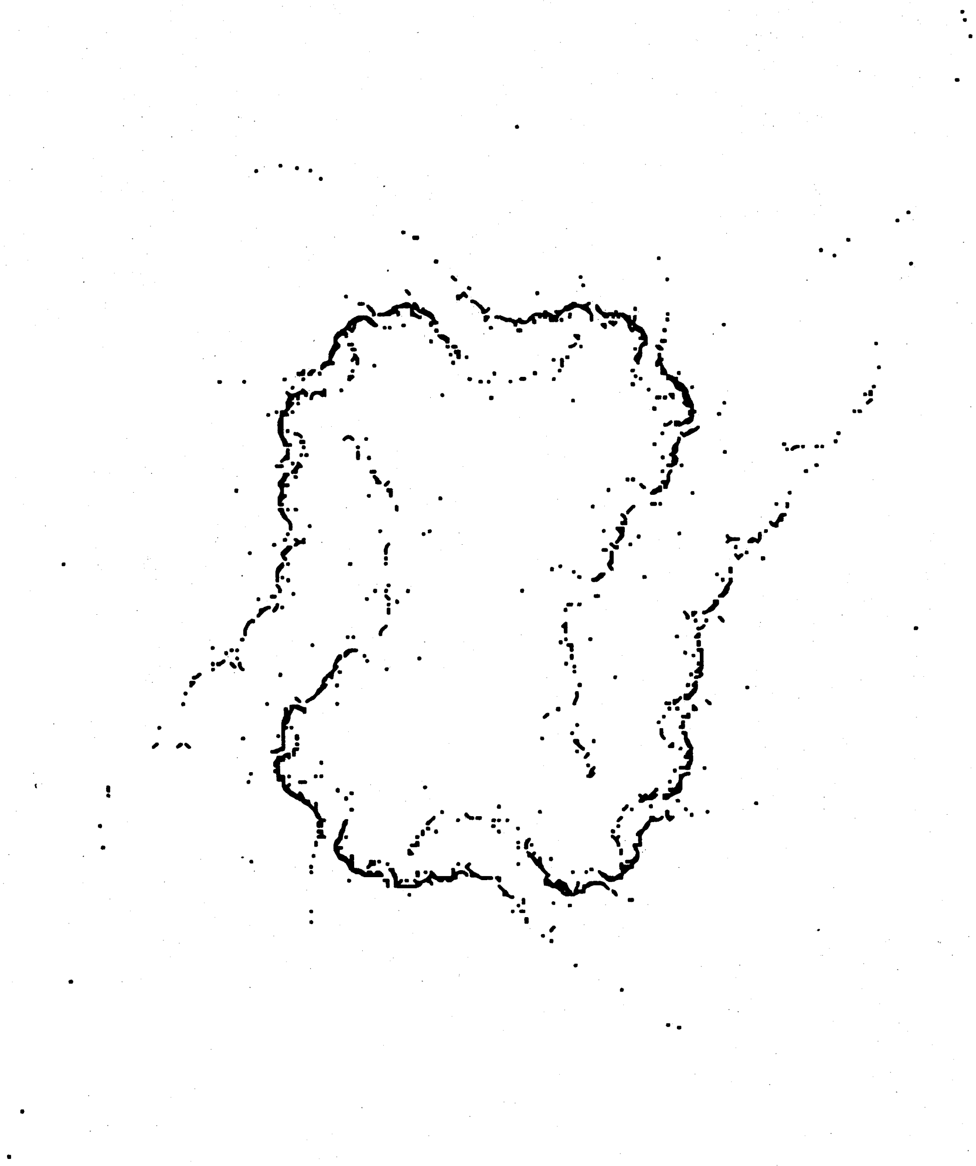
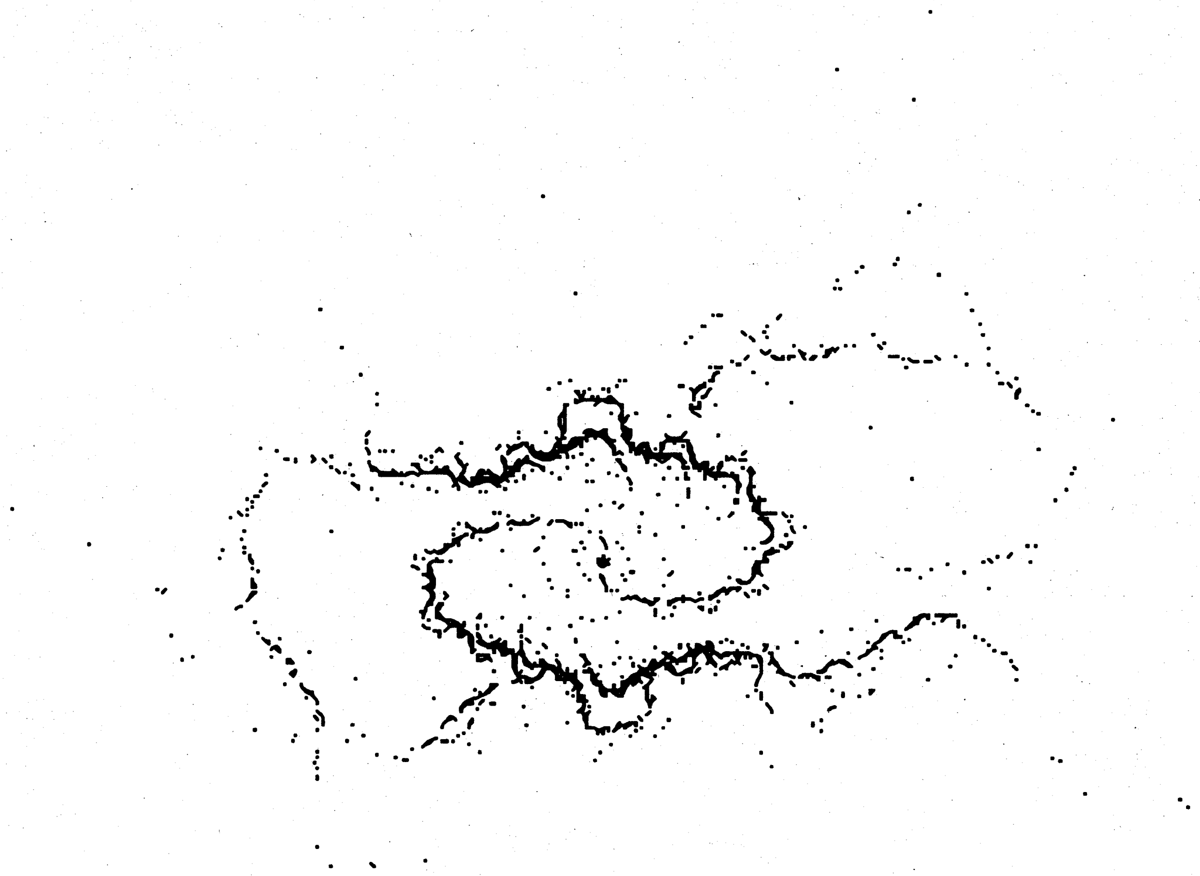


The projection of an orbit on the curve

$$(y - x^2 - 0.3)(y - x) + 0.03 - 0.03i = 0$$

onto the x-plane.



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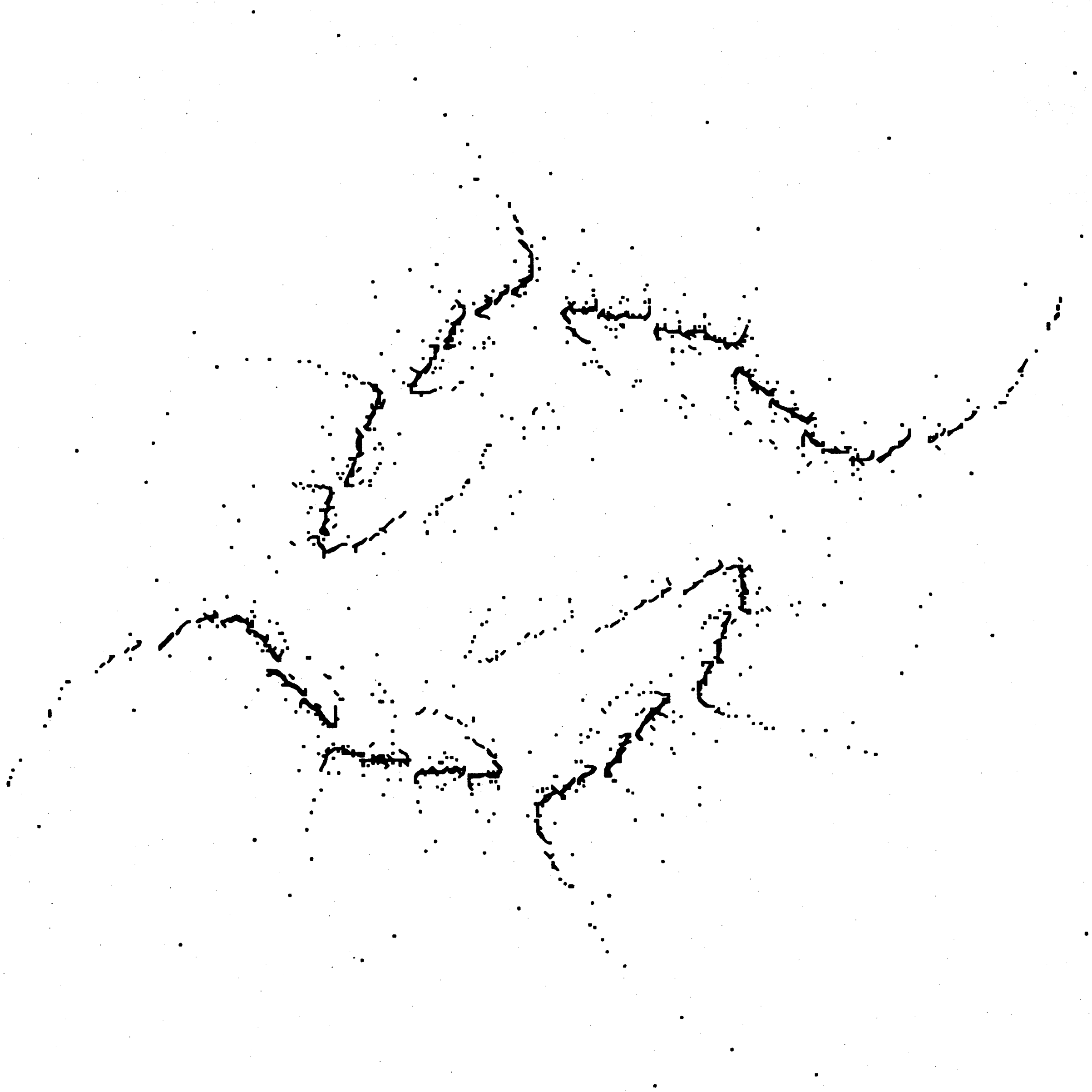
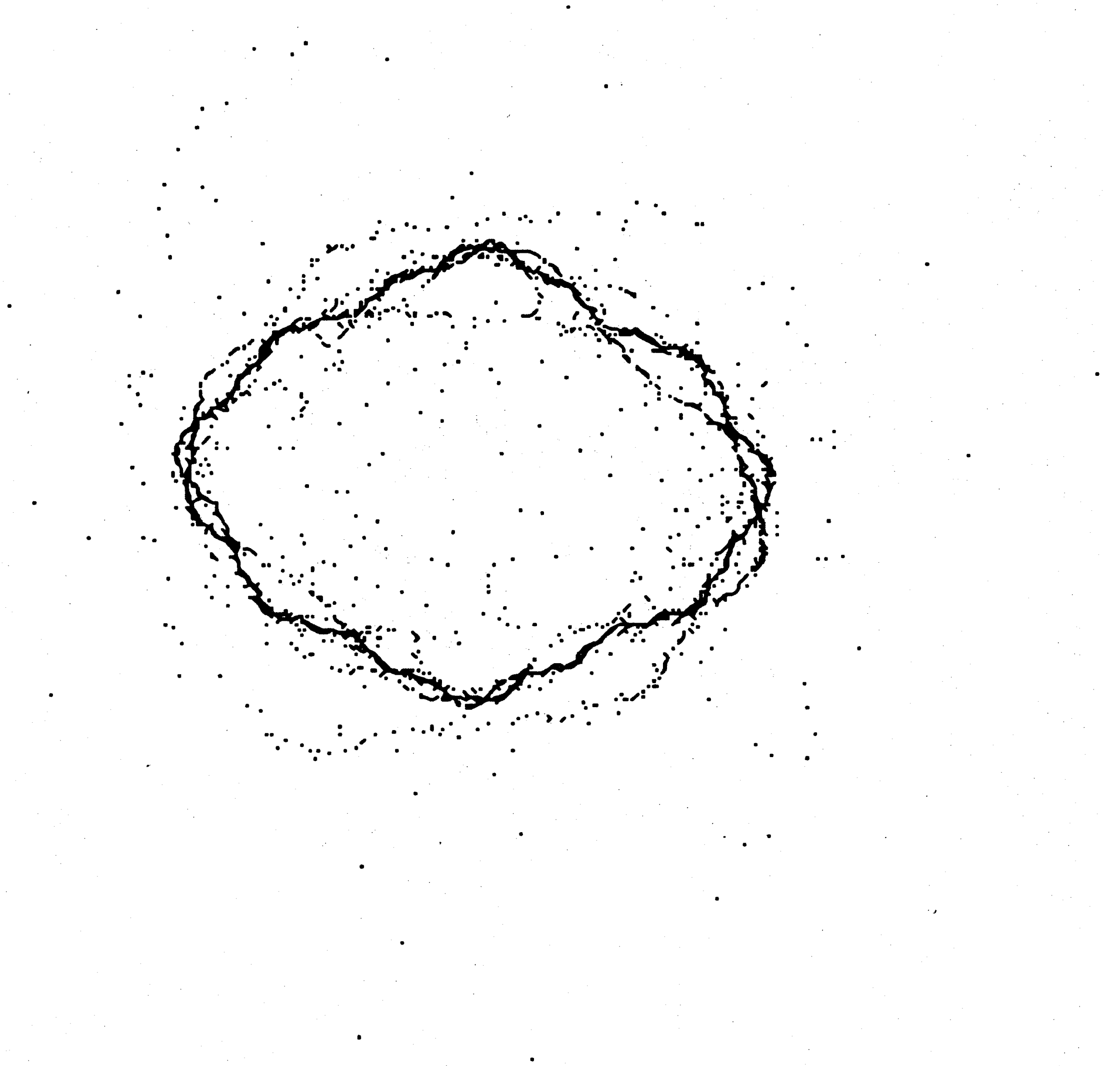


The projection of an orbit on the curve

$$(y - x^2 + 0.3i)(y - x) - 0.03 + 0.03i = 0$$

onto the x-plane.



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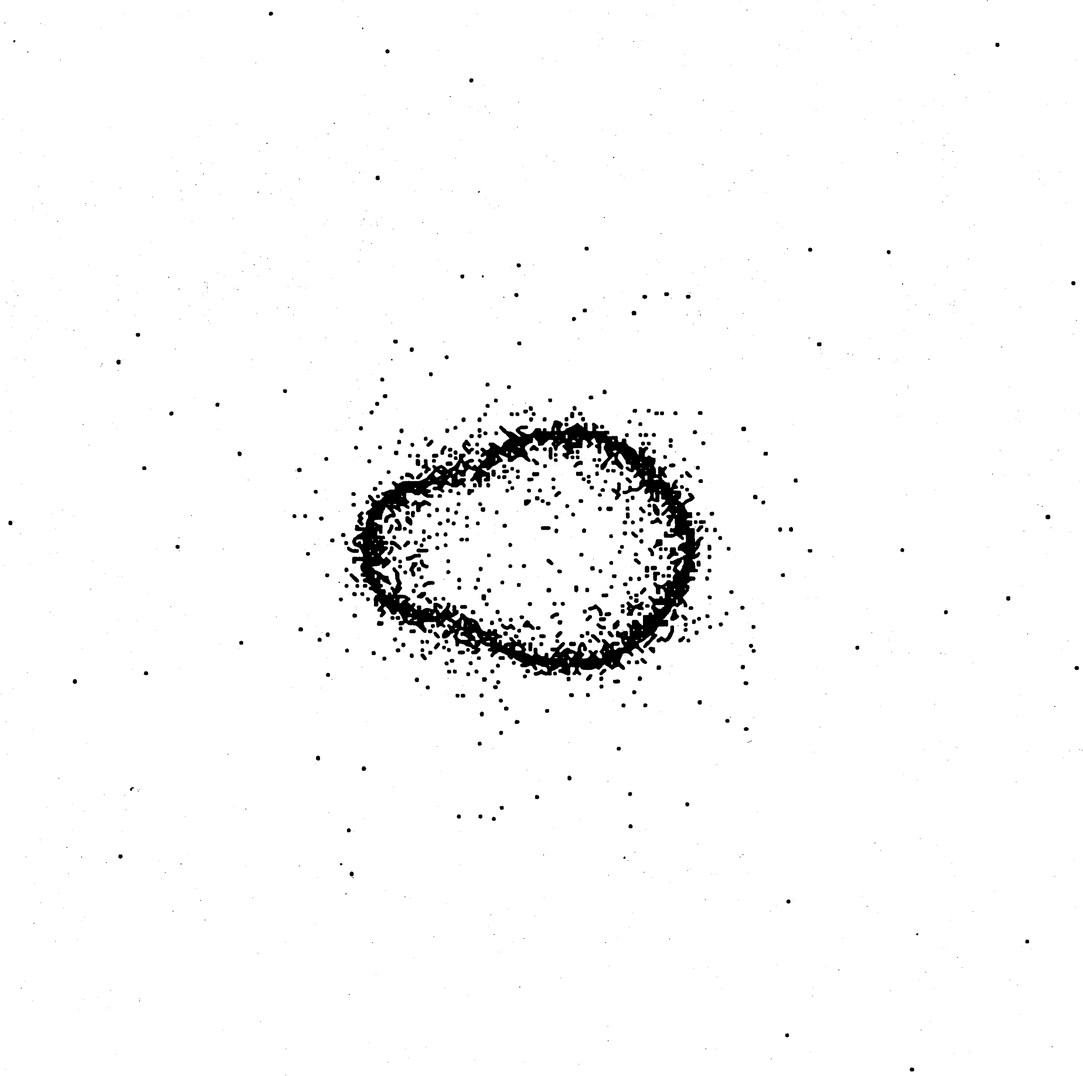
onto the x-plane.



The projection of an orbit on the curve

$$(y - x^2)(y - x) + 0.7x^2 = 0$$

onto the x-plane.

REFERENCES

1. I. Nakai, *Separatrices for non solvable dynamics on C.O*, Ann. l'Inst. Fourier **44** (**2**).
2. I. Nakai, *Topology of complex webs of codimension one and geometry of projective space curves*, Topology **26**, 475–504.
3. preparation .
4. I. Naka: Preprint IHES July, Topology of webs of divisors in a linear system of a complex manifold..

SAPPORO 060 JAPAN

E-mail: nakai@math.hokudai.ac.jp