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A direct approach to the planar graph presentations of the braid group

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0. Introduction

Recall that the classical braid group on n strings B_n can be considered as the fundamental group of the configuration space of unordered n points in the plane.

Given a planar finite graph whose vertices are n given points, one can define for each edge σ a braid, also denoted σ like in figure 1:

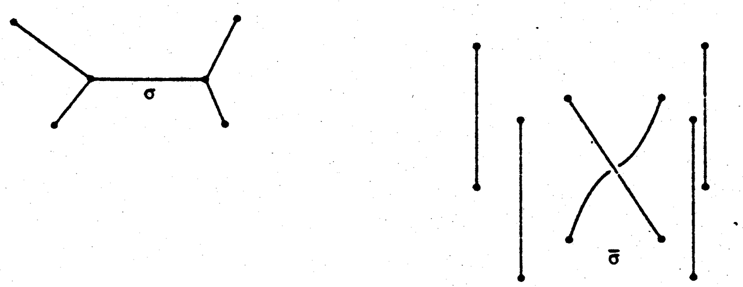


fig. 1

One just turns half around σ in a neighbourhood, the other strings being vertical.

If the graph is

fig. 2

one obtains the Artin generators of the braid group B_n, see [B].

Let us now suppose that the graph Γ is connected and without loops. In [S] we noted that the braids $\{\sigma\}$ corresponding to the edges verify the following relations :

(i) disjointness: if $\sigma_1 \cap \sigma_2 = \emptyset$ then $\sigma_1\sigma_2 = \sigma_2\sigma_1$.

(ii) adjacence: if $\sigma_1 \cap \sigma_2 =$ one vertex then $\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$.

(iii) *nodal*: if $\sigma_1, \sigma_2, \sigma_3$ have one common vertex like in figure 3; then $\sigma_1\sigma_2\sigma_3\sigma_1 = \sigma_2\sigma_3\sigma_1\sigma_2 = \sigma_3\sigma_1\sigma_2\sigma_3$

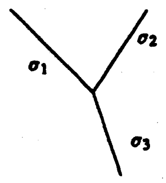


fig. 3

(iv) *cyclic*: if $\sigma_1 \cdots \sigma_n$ is a cycle such that $\sigma_1 \cdots \sigma_n$ bounds a disc without interior vertices, then $\sigma_1\sigma_2 \cdots \sigma_{n-1} = \sigma_2 \cdots \sigma_n = \sigma_n\sigma_1 \cdots \sigma_{n-2}$

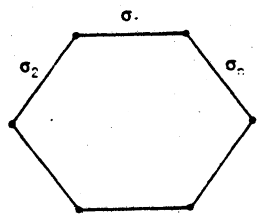


fig. 4

Moreover, we proved in [S] the

0.1. THEOREM. — *The braid group B_Γ on the vertex set $v(\Gamma)$ has a presentation $\langle X_\Gamma, R_\Gamma \rangle$ where X_Γ is the set of edges $\{\sigma\}$ and R_Γ the set of relations (i) – (iv).*

0.2. REMARK. — The above statement, which appears in [S] in a slightly more general context, was chosen here in order to keep notations simpler.

This theorem was presented at the Kyoto meeting together with some corollaries. The proof given in [S] used a recursive device using Artin's presentation as the starting point. Here I shall sketch a direct argument suggested by Fadell-Van Buskirt's proof, see [B], as modified by J. Morita [M].

I am grateful to Professors Suwa and Ito for the opportunity they gave me to participate to the R.I.M.S. meeting and for their warm hospitality.

1. The geometric argument

Let Γ be a finite *tree*, $v \in \Gamma$ an end vertex and $\Gamma' = \Gamma - \{v\}$ and v' the neighbour of v. Let P_Γ the kernel of the natural map $B_\Gamma \xrightarrow{\pi} \Sigma_\Gamma$, *i.e.* the pure braid group, where Σ_Γ is the permutation group of $v(\Gamma)$.

Forgetting the last string from v to v, one gets a natural map $P_\Gamma \to P_{\Gamma'}$. Think about this map as coming from the natural projection between configuration spaces. One easily sees that it's kernel is the free group $\pi_1(\mathbb{C} - v(\Gamma'))$ with $|v(\Gamma)| - 2$ generators.

Consider the subgroup $B_\Gamma^0 = \pi^{-1}(\Sigma_{\Gamma'})$ of B_Γ. Then $P_\Gamma \subset B_\Gamma^0$ and there is a natural map

$$\theta : B_\Gamma^0 \longrightarrow B_{\Gamma'}$$

which "forgets" the last string. The diagram

$$\begin{array}{ccc} P_\Gamma & \longrightarrow & P_{\Gamma'} \\ \downarrow & & \downarrow \\ B_\Gamma^0 & \longrightarrow & B_{\Gamma'} \end{array}$$

is commutative and the kernel of the horizontal maps is the same. One gets the

1.1. PROPOSITION. — *The kernel of the map $\theta : B_\Gamma^0 \longrightarrow B_{\Gamma'}$ is a free group of rang $|v(\Gamma)| - 2$.*

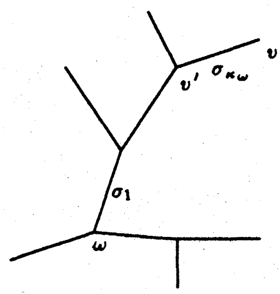
2. The inductive assertion

In this paragraph we will formulate the statement needed to prove theorem 0.1 for a *tree* Γ.

Let $\widetilde{B}_\Gamma$ be the group given by a presentation $\langle X_\Gamma, R_\Gamma \rangle$ as in theorem 0.1. Our task is to prove that the natural map $\widetilde{B}_\Gamma \longrightarrow B_\Gamma$ is an isomorphism. We use induction on $|v(\Gamma)|$.

For each vertex $\omega \in \Gamma'$ let $\sigma_1 \cdots \sigma_{\kappa_\omega}$ be the simple path from ω to v, $\rho_\omega = \sigma_1 \cdots \sigma_{\kappa_\omega}$ the corresponding braid and $\tau_\omega = \sigma_{\kappa_\omega} \cdots \sigma_2 \sigma_1^2 \sigma_2^{-1} \cdots \sigma_{\kappa_\omega}^{-1}$ if $\omega \neq v'$

and $\tau_\omega = \sigma_1^2$ if $\omega = v'$. Note that ρ_ω and τ_ω make sense in B_Γ and in $\widetilde{B}_\Gamma$.



Let $\widetilde{B}_\Gamma^0$ be the subgroup of $\widetilde{B}_\Gamma$ generated by $\{\sigma | \sigma \in \Gamma'\} \cup \{\tau_\omega | \omega \in \Gamma'\}$.

One has a natural diagram :

$$\begin{array}{ccc} \widetilde{B}_\Gamma^0 & \xrightarrow{\tilde{\theta}} & \widetilde{B}_{\Gamma'} \\ \downarrow & & \downarrow \\ B_\Gamma^0 & \xrightarrow{\theta} & B_{\Gamma'} \end{array}$$

Note that the map $\tilde{\theta}$ is well defined because the right map is an isomorphism by the inductive assumption.

In the next paragraph we shall prove that the left side map $\widetilde{B}_\Gamma^0 \longrightarrow B_\Gamma^0$ is an isomorphism and show how this implies that the map $\widetilde{B}_\Gamma \longrightarrow B_\Gamma$ is an isomorphism.

3. Proof of the inductive step

The map $\tilde{\theta} : \widetilde{B}_\Gamma^0 \longrightarrow \widetilde{B}_{\Gamma'}$ has an obvious section. The kernel of $\tilde{\theta}$ is the subgroup generated by the $\{\tau_\omega\}$: this follows using the section and the fact that the τ_ω's generate a normal subgroup.

Direct checking shows that the τ_ω's, when considered in B_Γ^0 freely generate the kernel of θ (see 1.1). This implies that the map from $\ker\tilde{\theta}$ to $\ker\theta$ is an isomorphism and by the five lemma and the inductive assumption the same is true for the map from $\widetilde{B}_\Gamma^0$ to B_Γ^0.

In order to deduce that the map from $\widetilde{B}_\Gamma$ to B_Γ is an isomorphism we first note that it is surjective : it's image contains $P_\Gamma \subset B_\Gamma^0$ and it obviously surjects onto Σ_Γ.

$$\begin{array}{ccccc} & & \widetilde{B}_\Gamma & & \\ & & \downarrow & & \\ P_\Gamma & \rightarrowtail & B_\Gamma & \twoheadrightarrow & \Sigma_\Gamma \end{array}$$

As B_Γ^0 is a subgroup of index $|v(\Gamma)|$ of B_Γ by it's very definition, it will be sufficient to show the same thing about the index of $\widetilde{B}_\Gamma^0$ in $\widetilde{B}_\Gamma$.

Consider the set $\widetilde{X} = \bigcup_{\omega \in v(\Gamma)} \rho_\omega \widetilde{B}_\Gamma^0$ (where we put $\rho_v = e$). We leave to the reader to prove that $\widetilde{X}$ is a *subgroup* of $\widetilde{B}_\Gamma$. One then deduces that the index of $\widetilde{B}_\Gamma^0$ in $\widetilde{X}$ is $|v(\Gamma)|$ as $\rho_{\omega_1}^{-1}\rho_{\omega_2} \notin \widetilde{B}_\Gamma^0$ if $\omega_1 \neq \omega_2$. Finally, as $\widetilde{B}_\Gamma$ is generated by $\widetilde{B}_\Gamma^0$ together with any ρ_ω, $\omega \neq v$, one has $\widetilde{B}_\Gamma = \widetilde{X}$ and so the index of $\widetilde{B}_\Gamma^0$ in $\widetilde{B}_\Gamma$ is $|v(\Gamma)|$. This completes the argument when Γ is a tree.

4. End of the proof

We now take Γ to be any graph like in theorem 0.1 and $b(\Gamma)$ it's first Betti number. If $b(\Gamma) = 0$, Γ is a tree on the result is true.

Let us suppose that the theorem is true for all graphs whose first Betti number is less than $b(\Gamma)$. We chose an edge α on a cycle of Γ which does not bound a second cycle on the other side. The theorem is then true for the graph $\Gamma - \alpha$ and it is easily seen that this implies it is true for Γ: any cyclic relation is true in $B_{\Gamma-\{\alpha\}} = B_\Gamma$ and it defines implicitely the element $\alpha \in B_\Gamma$ (see [S] for more details).

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[B] BIRMAN J.S. — *Braids, links and mapping class groups*, Ann. Math. Studies 82, Princeton Univ. Press, Princeton, 1975.

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