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SOME EXAMPLES OF DEFORMATIONS OF COMPLEX MANIFOLDS

ÉTIENNE GHYS

Let Γ be a discrete cocompact subgroup in $SL(n,\mathbb{C})$. Recall that if $n \geq 2$, a (special case of) a theorem of A. Weil shows that any homomorphism from Γ to $SL(n,\mathbb{C})$ close enough to the embedding is conjugate to this embedding. Moreover, Raghunathan has shown that if $n \geq 3$ then the compact complex manifold $SL(n,\mathbb{C})/\Gamma$ is rigid as a complex manifold. The purpose of this note is to describe explicit examples of non trivial deformations of the complex manifold $\mathrm{SL}(2,\mathbb{C})/\Gamma$.

This note is extracted from [Gh] which will be published elsewhere: it corresponds to the talk I gave in the meeting 'Singularities of holomorphic vector fields and related topics' at RIMS, Kyoto, in november 1993.

Observe that, up to a $\mathbb{Z}/2\mathbb{Z}$-extension, $\mathrm{SL}(2,\mathbb{C})$ is the isometry group of the real hyperbolic 3-dimensional space $\mathbb{H}^3$ so that Γ is the fundamental group of a hyperbolic 3-dimensional orbifold. Many examples have nonvanishing first Betti number, *i.e.*, are such that there exist nontrivial homomorphsims $u : \Gamma \to \mathbb{C}^*$ (see [Th]).

If u is such a homomorphism, we consider the right action of Γ on $\mathrm{SL}(2,\mathbb{C})$ defined by:

$$(x,\gamma) \in \mathrm{SL}(2,\mathbb{C}) \times \Gamma \mapsto x \bullet \gamma = \begin{pmatrix} u(\gamma) & 0 \\ 0 & u(\gamma)^{-1} \end{pmatrix} x\gamma.$$

If this action is free, proper and totally discontinuous, we denote by $\mathrm{SL}(2,\mathbb{C})/\!\!/_u\Gamma$ the quotient, and we say that u is admissible. We noted that there is a natural $\mathbb{C}^*$-action on this quotient, coming from left translations by matrices $\left(\begin{smallmatrix} T & 0 \\ 0 & T^{-1} \end{smallmatrix}\right)$.

Let H_+ and H_- be the right invariant holomorphic vector fields in $\mathrm{SL}(2,\mathbb{C})$ corresponding to the elements $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$ of the Lie algebra of $\mathrm{SL}(2,\mathbb{C})$ and denote by $\mathcal{H}^+$ and $\mathcal{H}^-$ the one-dimensional holomorphic foliations generated by H^+ and H^-. It is easy to check that the differential of the right action by γ in $\mathrm{SL}(2,\mathbb{C})$ maps H^+ and H^- to $u(\gamma)^2H^+$ and $u(\gamma)^{-2}H^-$ so that H^+ and H^- are not invariant (unless u^2 is trivial) but $\mathcal{H}^+$ and $\mathcal{H}^-$ are invariant. In other words, on the compact manifold $\mathrm{SL}(2,\mathbb{C})/\!\!/_u\Gamma$, we have two natural foliations $\mathcal{H}^+$ and $\mathcal{H}^-$ which are invariant under the $\mathbb{C}^*$-action. When u^2 is trivial, $\mathcal{H}^+$ and $\mathcal{H}^-$ are parametrized by vector fields H^+ and H^-.

In order to simplify our description of these examples, we shall assume that Γ is torsion-free (this can always be achieved by replacing Γ by a finite index subgroup by a theorem of Selberg). In particular, Γ injects into $\mathrm{PSL}(2,\mathbb{C}) = \mathrm{SL}(2,\mathbb{C})/\{\pm id\}$.

Note that if $\epsilon : \Gamma \to \{\pm 1\}$ is a homomorphism, the map $\tau : \gamma \in \Gamma \mapsto \epsilon(\gamma)\gamma \in \mathrm{SL}(2,\mathbb{C})$ is an injective homomorphism whose image is another discrete subgroup Γ' of $\mathrm{SL}(2,\mathbb{C})$. In such a situation, we shall write $\Gamma = \pm\Gamma'$. This happens precisely when Γ and Γ' have the same projection in $\mathrm{PSL}(2,\mathbb{C})$. Of course, $u : \Gamma \to \mathrm{SL}(2,\mathbb{C})$ is admissible if and only if $\epsilon . u \circ \tau^{-1} : \Gamma' \to \mathbb{C}^*$ is admissible and the corresponding actions of $\mathbb{C}^*$ are conjugate.

Proposition. *Let Γ be a discrete torsion-free cocompact subgroup of $\mathrm{SL}(2,\mathbb{C})$. Then homomorphisms $u : \Gamma \to \mathbb{C}^*$ which are close enough to the trivial homomorphism are admissible.*

Let Γ_1 and Γ_2 be two discrete torsion-free cocompact subgroups of $\mathrm{SL}(2,\mathbb{C})$. Then $\mathrm{SL}(2,\mathbb{C})/\!\!/_{u_1}\Gamma_1$ and $\mathrm{SL}(2,\mathbb{C})/\!\!/_{u_2}\Gamma_2$ are homeomorphic if and only if there is a continuous automorphism θ of $\mathrm{SL}(2,\mathbb{C})$ such that $\theta(\Gamma_1) = \pm\Gamma_2$. In such a case, there is a C^∞-diffeomorphism between $\mathrm{SL}(2,\mathbb{C})/\!\!/_{u_1}\Gamma_1$ and $\mathrm{SL}(2,\mathbb{C})/\!\!/_{u_2}\Gamma_2$ sending orbits of the first $\mathbb{C}^*$-action to orbits of the second (without necessarily commuting with the actions).

Proof. The first property follows from a very general fact. Let G be a Lie group acting analytically on a manifold V and let $\Gamma \to G$ be a homeomorphism such that the induced action of Γ on V is free, proper and totally discontinuous. Then any perturbation of the homomorphism $\Gamma \to G$ has the same property (see, for instance [Th]). Assertion i) follows from the special case where $V = \mathrm{SL}(2,\mathbb{C})$ and $G = \mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C})$ acting by left and right translations.

Assume $\mathrm{SL}(2,\mathbb{C})/\!\!/_{u_1}\Gamma_1$ and $\mathrm{SL}(2,\mathbb{C})/\!\!/_{u_2}\Gamma_2$ are homeomorphic. Then Γ_1 and Γ_2 are isomorphic as abstract groups and it follows from Mostow's rigidity theorem that there is a continuous automorphism θ of $\mathrm{SL}(2,\mathbb{C})$ such that $\theta(\Gamma_1) = \pm\Gamma_2$. Note that, up to conjugacy, the only nontrivial continuous automorphism of $\mathrm{SL}(2,\mathbb{C})$ is given by $\theta(x) = {}^t x^{-1}$.

We now show that if $\Gamma_2 = \pm\theta(\Gamma_1)$ then $\mathrm{SL}(2,\mathbb{C})/\!\!/_{u_1}\Gamma_1$ and $\mathrm{SL}(2,\mathbb{C})/\!\!/_{u_2}\Gamma_2$ are diffeomorphic. We can of course assume that $\theta = id$, and that $\Gamma_1 = \Gamma_2 = \Gamma$. Let us consider first of all the quotients $M_i = \mathrm{U}(1)\backslash \mathrm{SL}(2,\mathbb{C})/\!\!/_{u_i}\Gamma_i$ $(i = 1,2)$. These are manifolds since we assumed that Γ is torsion free. Note that if u_i is trivial, then $\mathrm{SL}(2,\mathbb{C})/\Gamma_i$ is the 2-fold (spin)-cover of the orthonormal frame bundle of the 3-manifold V which is the quotient of the hyperbolic 3-space by the action of Γ and M_i is the unit tangent bundle of V.

On M_i, we have a real one-parameter flow f_i^t coming from the complex one-parameter flow on $\mathrm{SL}(2,\mathbb{C})/\!\!/_{u_i}\Gamma$. Of course when u_i is trivial the flow f_i^t is nothing but the geodesic flow of V.

The quotient $\mathbb{C}^*\backslash \mathrm{SL}(2,\mathbb{C})$ of $\mathrm{SL}(2,\mathbb{C})$ by the diagonal subgroup is isomorphic to the complement of the diagonal in $\mathbb{CP}^1 \times \mathbb{CP}^1$. The universal cover $\widetilde{M_i}$ of M_i, naturally identified with $\mathrm{U}(1)\backslash \mathrm{SL}(2,\mathbb{C})$, fibres over the complement of the diagonal Δ in $\mathbb{CP}^1 \times \mathbb{CP}^1$:

$$D_i : \widetilde{M_i} \to \mathbb{CP}^1 \times \mathbb{CP}^1 - \Delta$$

and this fibration is equivariant under the diagonal embedding:

$$H_i : \gamma \in \Gamma \mapsto (\gamma, \gamma) \in \mathrm{PSL}(2,\mathbb{C}) \times \mathrm{PSL}(2,\mathbb{C}).$$

The fibres of D_i are the orbits of the lifted flow $\tilde{f}_i^t$. We therefore observe that both flows f_1^t and f_2^t have the same transverse structure, *i.e.*, equivalent holonomy pseudogroups. It follows from [Ha] (see also [Gr], [Ba]) that *there is a C^∞-diffeomorphism between M_1 and M_2 sending orbits of f_1^t to orbits of f_2^t and, in particular, that M_1 and M_2 are diffeomorphic.*

We claim that the circle fibrations $\mathrm{SL}(2,\mathbb{C})/\!\!/_{u_i}\Gamma \to M_i$ are trivial fibrations. This follows from the fact that orientable closed 3-manifolds are parallelizable and from the fact that the space of homomorphisms from Γ to $\mathbb{C}^*$ is connected. Indeed, choose a path u_t ($t \in [0,1]$) connecting the trivial homomorphism to u_1 and consider the right action of Γ on $\mathrm{SL}(2,\mathbb{C}) \times \mathbb{H}^3$ (where $\mathbb{H}^3$ is the hyperbolic 3-space) given by:

$$(x,p) \bullet_t \gamma = \left(\begin{pmatrix} u_t(\gamma) & 0 \\ 0 & u_t(\gamma)^{-1} \end{pmatrix} x\gamma, \gamma^{-1}(p) \right) \in \mathrm{SL}(2,\mathbb{C}) \times \mathbb{H}^3 .$$

The second factor has been introduced in such a way that the action is free, proper, and totally discontinuous for each $t \in [0,1]$. The quotient spaces are homotopy equivalent to $\mathrm{SL}(2,\mathbb{C})/\Gamma$ and $\mathrm{SL}(2,\mathbb{C})/\!\!/_{u_1}\Gamma$ for $t = 0$ and $t = 1$. Moreover, for each t, the right-action of Γ commutes with left translations by $\mathrm{U}(1)$ so that each quotient space is the total space of circle bundle. Since we noticed that this circle bundle is trivial of $t = 0$, we deduce that it is also trivial for $t = 1$. Hence the circle bundles $\mathrm{SL}(2,\mathbb{C})/\!\!/_{u_i}\Gamma \to M_i$ are trivial and the diffeomorphism between M_1 and M_2 sending orbits of f_1^t to orbits of f_2^t can be lifted to a diffeomorphism between $\mathrm{SL}(2,\mathbb{C})/\!\!/_{u_1}\Gamma$ and $\mathrm{SL}(2,\mathbb{C})/\!\!/_{u_2}\Gamma$ sending orbits of the first $\mathbb{C}^*$-action to orbits of the second one.

This completes the proof of proposition 6.1. □

Proposition. *If $u : \Gamma \to \mathbb{C}^*$ is an admissible homomorphism such that u^2 is non trivial, then the space of holomorphic vector fields on $\mathrm{SL}(2,\mathbb{C})/\!\!/_u\Gamma$ has complex dimension 1 and is generated by the vector field corresponding to the $\mathbb{C}^*$-action.*

Proof. We have already noticed that there are two holomorphic one dimensional foliations $\mathcal{H}^+$ and $\mathcal{H}^-$ on $V = \mathrm{SL}(2,\mathbb{C})/\!\!/_u\Gamma$ which are invariant under the $\mathbb{C}^*$-action and which provide, together with the tangent bundle to the orbits of $\mathbb{C}^*$, a splitting of $T_{\mathbb{C}}V$ as a sum of three line-bundles. In order to show the proposition, it is enough to show that there is no nonzero holomorphic vector field in V tangent to $\mathcal{H}^+$ (or to $\mathcal{H}^-$) if u is nontrivial. Assume there is such a vector field ξ. Using the fact that the $\mathbb{C}^*$-action preserves $\mathcal{H}^+$ and that the space of holomorphic vector fields is finite dimensional, one can choose ξ such that the $\mathbb{C}^*$- action $\phi(T)$ ($T \in \mathbb{C}^*$) satisfies, for some $k \in \mathbb{Z}$:

$$d\phi(T)(\xi) = T^k \xi \text{ for all } T \in \mathbb{C}^* .$$

If one lifts ξ to $\mathrm{SL}(2,\mathbb{C})$, one gets a vector field $\tilde{\xi}$ which is of the form $f \cdot H^+$ where f is holomorphic on $\mathrm{SL}(2,\mathbb{C})$. Taking into account the invariance of $\tilde{\xi}$ under the action of Γ and the non-invariance of H^+ already observed, we get:

$$(1) \qquad f(x \bullet \gamma) = u(\gamma)^{-2} f(x) \text{ for } \gamma \in \Gamma \text{ and } x \in \mathrm{SL}(2,\mathbb{C}).$$

Moreover, we have:

$$(2) \qquad f\left(\begin{pmatrix} T & 0 \\ 0 & T^{-1} \end{pmatrix} \cdot x \right) = T^k f(x) \text{ for all } T \in \mathbb{C}^* \text{ and } x \in \mathrm{SL}(2,\mathbb{C})$$

Assume first that $k = 0$ so that f actually defines a function $\bar{f}$ on

$$\mathbb{C}^* \setminus \mathrm{SL}(2, \mathbb{C}) \cong \mathbb{CP}^1 \times \mathbb{CP}^1 - \Delta.$$

Then, by (1), $\bar{f}$ is invariant under the action of the first commutator group Γ' of Γ (on which u is obviously trivial). Now this action of Γ' on $\mathbb{CP}^1 \times \mathbb{CP}^1$ is topologically transitive. This is equivalent to the fact that the geodesic flow of the homology cover of a compact hyperbolic manifold is topologically transitive. Indeed all non trivial normal subgroups of a discrete group of isometries of a hyperbolic space have the same limit set and all non elementary groups act topologically transitively on the square of their limit set (see [Th] and [G-H] page 123). Therefore f is constant—but this is impossible if u and f are not trivial.

Now, assume that $k \neq 0$. Consider the function $f : V' = \mathrm{SL}(2, \mathbb{C})/\Gamma' \to \mathbb{C}$. According to (2), f has to vanish on periodic orbits of the $\mathbb{C}^*$-action on V'. But, on any compact hyperbolic manifold the union of closed geodesics homologous to zero is dense (as follows also from [G-H]). This shows that the union of closed orbits of the $\mathbb{C}^*$-action on V' is dense V'. It follows that f is zero. □

Corollary. *Let Γ be a discrete torsion free cocompact subgroup of* $\mathrm{SL}(2, \mathbb{C})$ *and $u_1, u_2 : \Gamma \to C^*$ be two admissible homomorphisms. Then the compact complex manifolds* $\mathrm{SL}(2, \mathbb{C})/\!\!/_{u_i}\Gamma$ *$(i = 1, 2)$ are holomorphically diffeomorphic if and only if there is an automorphism θ of Γ such that $u_2^{\pm 1} = u_1 \circ \theta$.*

Proof. If u_1^2 is trivial, then $\mathrm{SL}(2, \mathbb{C})/\!\!/_{u_1}\Gamma$ is a homogeneous space of $\mathrm{SL}(2, \mathbb{C})$ and therefore admits three linearly independent holomorphic vector fields. According to 6.2, on deduces that u_2^2 is also trivial if $\mathrm{SL}(2, \mathbb{C})/\!\!/_{u_2}\Gamma$ is holomorphically diffeomorphic to $\mathrm{SL}(2, \mathbb{C})/\!\!/_{u_1}\Gamma$. The corresponding complex manifolds are therefore of the form $\mathrm{SL}(2, \mathbb{C})/\Gamma_i$ $(i = 1, 2)$ and $\Gamma_1 = \pm\Gamma_2$. Any holomorphic diffeomorphism between these two homogeneous spaces induces an isomorphism between the Lie algebras of holomorphic vector fields which are themselves isomorphic to the Lie algebra of $\mathrm{SL}(2, \mathbb{C})$. The corollary follows in this special case.

Now, assume that u_1^2 and u_2^2 are nontrivial and that there is a holomorphic diffeomorphism F between the corresponding compact complex manifolds. Proposition 6.2 implies that F conjugates the $\mathbb{C}^*$-actions or one with the inverse of the other. Let γ be a nontrivial element of Γ and denote by $\lambda(\gamma)$, $\lambda(\gamma)^{-1}$ its two eigenvalues. The $\mathbb{C}^*$-action on $\mathrm{SL}(2, \mathbb{C})/\!\!/_{u_i}\mathbb{C}$ contains precisely two closed orbits containing a loop freely homotopic to $\gamma^{\pm 1}$, whose "periods" are $\lambda(\gamma)u_i(\gamma)$ and $\lambda^{-1}(\gamma)u_i(\gamma)$. Note that periods of closed orbits related under F should be equal or inverse. If θ denotes the automorphism of Γ (defined up to conjugacy) induced by F, it follows that either $u_2 = u_1 \circ \theta$ or $u_2^{-1} = u_1 \circ \theta$. □

Corollary. *Let Γ be a discrete torsion free cocompact subgroup of* $\mathrm{SL}(2, \mathbb{C})$ *and $u_1, u_2 : \Gamma \to \mathbb{C}^*$ be two admissible homomorphisms. Then the $\mathbb{C}^*$-actions on* $\mathrm{SL}(2, \mathbb{C})/\!\!/_{u_i}\Gamma$ *are conjugate by a homeomorphism if and only if there is an automorphism θ of Γ such that $u_2 = u_1 \circ \theta$.*

Proof. This is the same proof as that of the previous Corollary since we only used preservation of periods of closed orbits. □

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ÉCOLE NORMALE SUPÉRIEURE DE LYON, UMR 128 CNRS, 46 ALLÉE D'ITALIE 69364 LYON, FRANCE