数理解析研究所講究録
第 881 巻 1994 年 17-19

On Another Easy Proof of Miller and Mocanu's Theorem

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In [1], Miller and Mocanu obtained the following theorem:

Theorem. Let $g(z) = g_n z^n + g_{n+1} z^{n+1} + \cdots$ be analytic in $E = \{ z \mid |z| < 1 \}$ with $g_n \neq 0$ and $n \geqq 1$. If $z_0 = r_0 e^{i\theta_0}$ ($0 < r_0 < 1$) and

$$|g(z_0)| = \max_{|z| \leqq r_0} |g(z)| = R$$

then

(1)
$$\frac{z_0 g'(z_0)}{g(z_0)} = m$$

and

(2)
$$1 + \operatorname{Re} \frac{z_0 g''(z_0)}{g'(z_0)} \geqq m$$

where $m \geqq n \geqq 1$.

The proof of (2) was given in [1, p.290-291] and it is not easy and simple.

It is a purpose of the present author to give an easy and elemental proof of (2).

Applying the same method as the proof of [2], we put

$$p(z) = \frac{R - g(z)}{R + g(z)} = 1 + c_n z^n + \cdots, \quad c_n \neq 0$$

or

(3)
$$g(z) = R \frac{1 - p(z)}{1 + p(z)}, \quad p(0) = 1$$

then $p(z)$ is analytic in $|z| < |z_0|$.

From the assumption of the theorem, we have

$$\operatorname{Re} p(z) > 0 \text{ for } |z| < |z_0|$$

and

$$\operatorname{Re} p(z_0) = 0.$$

From (1) and (3), we have

$$\frac{z_0 g'(z_0)}{g(z_0)} = m \geqslant n$$

and

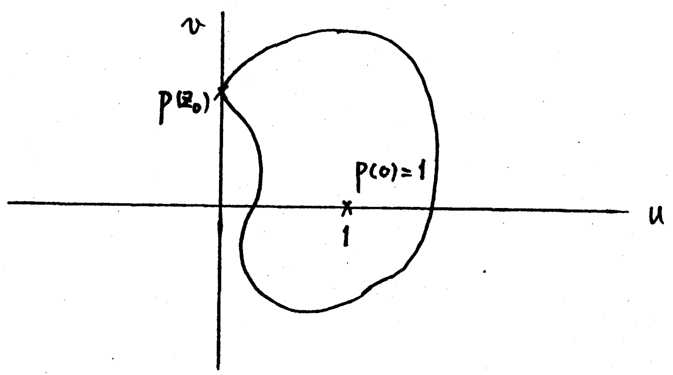
(4) $$\frac{zg'(z)}{g(z)} = -\frac{2zp'(z)}{1 - p(z)^2}.$$

By logarithmic differentiation of (4), we have

(5) $$1 + \frac{zg''(z)}{g'(z)} = \frac{zg'(z)}{g(z)} + 1 + \frac{zp''(z)}{p'(z)} + \frac{2zp'(z)p(z)}{1 - p(z)^2}.$$

From the geometrical property, the tangent vector of $p(z)$ at $z = z_0$ on the circle $|z| = |r_0 e^{i\theta}| = r_0$, $0 \leqslant \theta \leqslant 2$ moves to positive direction or

$$1 + \operatorname{Re}\frac{z_0 p''(z_0)}{p'(z_0)} \geqslant 0$$



Putting $z = z_0$ in (5), then we have

$$1 + \operatorname{Re}\frac{z_0 g''(z_0)}{g'(z_0)} = \operatorname{Re}\frac{z_0 g'(z_0)}{g(z_0)} + 1 + \operatorname{Re}\frac{z_0 p''(z_0)}{p'(z_0)}$$

$$\geqslant m$$

This completes the proof of (2) and the proof is easy and elemental.

References

[1] S. S. Miller and P. T. Mocanu: Second order differential inequalities in the complex plane, J. Math. Anal. Appl., 65, 289-305(1978).

[2] M. Nunokawa: On properties of non-Carathéodory functions, Proc. Japan Acad., 68, No.6, 152-153(1992).

This paper was obtained under the support of the Institute of Mathematics, University of Kyoto on March 15-18, 1994.