数理解析研究所講究録
第881巻 1994年 20-22

One Criterion on a Class of Certain Analytic Functions

By Mamoru NUNOKAWA and Shinichi HOSHINO

Department of Mathematics, University of Gunma

(布川 護, 星野晋一, 群馬大学)

Let A denote the class of functions of the form

$$f(z)=z+\sum_{n=2}^{\infty}a_n z^n$$

which are analytic in the unit disk $U=\{z;|z|<1\}$.

A function belonging to A is said to be a member of the class $S(\alpha)$ if it satisfies

(1) $$\frac{zf'(z)}{f(z)} \qquad 1+(1-\alpha)z$$

for some $\alpha\,(0\leqq\alpha<1)$ and for all $z\in U$. The symbol denotes the subordination. It is easily confirmed that the condition (1) is equivalent to the following

(2) $$\left|\frac{zf'(z)}{f(z)}-1\right|<1-\alpha$$

for all $z\in U$.

In [1], Fukui obtained the following result

Theorem A. If $f(z)\in A$ satisfies

(3) $$\left|\beta\ \frac{zf'(z)}{f(z)}-1\ +(1-\beta)\frac{zf''(z)}{f(z)}\right|<1-\alpha$$

for some $\alpha\,(0\leqq\alpha<1)$, $\beta\,(0\leqq\beta<1)$, and for all $z\in U$, then $f(z)\in S(\alpha)$.

Making a lemma, we will improve Theorem A.

In order to derive our result, we need the following lemma due to Jack[2] (or Miller and Mocanu[3]).

Lemma 1. Let $w(z)$ be analytic in U with $w(0)=0$. If $|w(z)|$ attains its maximum value on the circle $|z|=r<1$ at a point z_0, then we have

$$z_0 w'(z_0)=k\,w(z_0)$$

where k is real and $k\geqq1$.

Applying Lemma 1, we have

Main Theorem. Let $p(z)$ be analytic in U, $p(0)=1$ and suppose that

(4) $$|\beta(p(z)-1)+(1-\beta)(p^2(z)-p(z)+zp'(z))|<(1-\alpha)(1+\alpha-\alpha\beta)$$

for some $\alpha\,(0\leqq\alpha<1)$, $\beta\,(0\leqq\beta<1)$ and for all $z\in U$. Then we have

$$| p(z)-1 | < 1-\alpha$$

for all $z \in U$.

Proof. Let us put

$$(1-\alpha)w(z)=(p(z)-1).$$

Then we have $w(0)=0$.

By an easy calculation, we have

$$\begin{aligned} &| \beta(p(z)-1)+(1-\beta)(p^2(z)-p(z)+zp'(z)) | \\ &= | \beta(1-\alpha)w(z)+(1-\beta)(1-\alpha)\ \{(1-\alpha)w^2(z)+w(z)+zw'(z)\} | \\ &= \left| (1-\alpha)w(z) \left\{ 1+(1-\alpha)(1-\beta)w(z)+(1-\beta)\frac{zw'(z)}{w(z)} \right\} \right| \end{aligned}$$

If there exists a point z_0 such that

$$\max_{z < z_0} | w(z) | = | w(z_0) | = 1,$$

then from Lemma 1, we have

$$\begin{aligned} &\left| (1-\alpha)w(z_0)\left\{ 1+(1-\beta)((1-\alpha)w(z_0)+\frac{z_0 w'(z_0)}{w(z_0)}) \right\} \right| \\ &=(1-\alpha) | 1+k(1-\beta)+(1-\alpha)(1-\beta)w(z_0) | \\ &\geqq (1-\alpha)(1+1-\beta-(1-\alpha)(1-\beta)) \\ &=(1-\alpha)(1+\alpha-\alpha\beta). \end{aligned}$$

This contradicts to (4). This shows that

$$| p(z)-1 | < 1-\alpha$$

for all $z \in U$. This completes our proof.

Putting

$$p(z)=\frac{zf'(z)}{f(z)}$$

then we have

$$p^2(z)-p(z)+zp'(z)=\frac{zf''(z)}{f(z)}.$$

Therefore, from the Main theorem, we have

Corollary 1. If $f(z) \in A$ satisfies

$$\left| \beta \frac{zf'(z)}{f(z)} - 1 + (1-\beta)\frac{z^2 f''(z)}{f(z)} \right| < (1-\alpha)(1+\alpha-\alpha\beta)$$

for some $\alpha\ (0 \leqq \alpha < 1)$, $\beta\ (0 \leqq \beta < 1)$ and for all $z \in U$, then we have $f(z) \in S(\alpha)$.

This is an improvement of Theorem A.

Taking $\beta=0$ in Corollary 1, we have

Corollary 2. If $f(z)\in A$ satisfies

$$\left| \frac{z^2 f''(z)}{f(z)} \right| < 1-\alpha^2$$

for some $\alpha\ (0\leqq\alpha<1)$ and for all $z\in U$, then we have $f(z)\in S(\alpha)$.

This is an improvement of [1, Corollary 1].

Taking $\beta=1/2$ in Corollary 1, we have

Corollary 3. If $f(z)\in A$ satisfies

$$\left| \frac{z f'(z)}{f(z)} - 1 + \frac{z^2 f''(z)}{f(z)} \right| < (2-\alpha+\alpha^2)$$

for some $\alpha\ (0\leqq\alpha<1)$ and for all $z\in U$, then we have $f(z)\in S(\alpha)$.

This is an improvement of [1, Corollary 2].

Taking $\beta=0$ in Main theorem, we have

Corollary 4. Let $p(z)$ be analytic in U, $p(0)=1$ and suppose that

$$| p^2(z) - p(z) + z p'(z) | < 1-\alpha^2$$

for all $z\in U$. Then we have

$$| p(z) - 1 | < 1-\alpha$$

for all $z\in U$.

References

[1] S. Fukui: A Remark on a Class of Certain Analytic Functions. Proc. Japan Acad., 66, Ser. A, 191-192(1990)

[2] I. S. Jack: Functions starlike and convex of order α. J. London Math. Soc., 3, 469-474 (1971)

[3] S. S. Miller and P. T. Mocanu: Second order differential inequalities in complex plane. J. Math. Anal. Appl., 65, 289-305 (1978)