

ON A SUBCLASS OF p -VALENT FUNCTIONS OF ORDER α

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ABSTRACT. Let $T_{n+p-1}(\alpha)$ denote the class of functions

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{p+k} z^{p+k} \quad (p \in \mathbb{N} = \{1, 2, \dots\})$$

which are regular in the unit disc $U = \{z: |z| < 1\}$ and satisfying

$$\operatorname{Re} \left\{ \frac{z(D^{n+p-1}f(z))'}{D^{n+p-1}f(z)} \right\} > \alpha, \quad z \in U$$

where $0 \leq \alpha < p$, n is any integer such that $n > -p$ and $D^{n+p-1}f(z)$ denote the $(n+p-1)$ th order Ruschewyh derivative of $f(z)$. We show that the functions in $T_{n+p-1}(\alpha)$ are p -valent. Further properties preserving integrals are considered.

KEY WORDS- Regular, p -valent, Ruschewyh derivative.

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1. Introduction.

Let $A(p)$ denote the class of functions

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{p+k} z^{p+k} \quad (p \in N = \{1, 2, \dots\}) \quad (1.1)$$

which are regular in the unit disc $U = \{z: |z| < 1\}$. Let $f(z)$ be in $A(p)$ and $g(z)$ be in $A(p)$. Then we denote by $f * g(z)$ the Hadamard product of $f(z)$ and $g(z)$, that is, if $f(z)$ is given by (1.1) and $g(z)$ is given by

$$g(z) = z^p + \sum_{k=1}^{\infty} b_{p+k} z^{p+k} \quad (p \in N), \quad (1.2)$$

then

$$f * g(z) = z^p + \sum_{k=1}^{\infty} a_{p+k} b_{p+k} z^{p+k}. \quad (1.3)$$

The $(n+p-1)$ th order Ruschewyh derivative $D^{n+p-1}f(z)$ of a function $f(z)$ of $A(p)$ is defined by

$$D^{n+p-1}f(z) = \frac{z^p (z^{n-1} f(z))^{(n+p-1)}}{(n+p-1)!} \quad (1.4)$$

where n is any integer such that $n > -p$. It is easy to see that

$$D^{n+p-1}f(z) = \frac{z^p}{(1-z)^{n+p}} * f(z). \quad (1.5)$$

In [6] Sohi introduced the classes T_{n+p-1} of functions $f(z)$ in $A(p)$ satisfying the condition

$$\operatorname{Re} \left\{ \frac{z(D^{n+p-1}f(z))'}{D^{n+p-1}f(z)} \right\} > 0, \quad z \in U. \quad (1.6)$$

Further Sohi [6] showed the basic property

$$T_{n+p} \subset T_{n+p-1}. \quad (1.7)$$

In this paper we consider the classes of functions $f(z) \in A(p)$ which satisfy the condition

$$\operatorname{Re} \left\{ \frac{z(D^{n+p-1}f(z))'}{D^{n+p-1}f(z)} \right\} > \alpha, \quad 0 \leq \alpha < p, \quad z \in U. \quad (1.8)$$

Using the identity

$$z(D^{n+p-1}f(z))' = (n+p)D^{n+p}f(z) - nD^{n+p-1}f(z) \quad (1.9)$$

condition (1.8) can be rewritten as

$$\operatorname{Re} \left\{ \frac{D^{n+p}f(z)}{D^{n+p-1}f(z)} \right\} > \frac{n+\alpha}{n+p}, \quad n > -p, \quad z \in U. \quad (1.10)$$

In this paper we prove that $T_{n+p}(\alpha) \subset T_{n+p-1}(\alpha)$. Since $T_0(\alpha)$ is the class of functions which satisfy the condition $\operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > \alpha$, $0 \leq \alpha < p$, $z \in U$, it follows that the functions in the classes $T_{n+p-1}(\alpha)$ are p -valent in U (see Umezawa [7]). Further properties preserving integrals are considered. Some known results of Bernardi [1,2], Goel [3], and Sohi [6] are extended.

2. Properties of the classes $T_{n+p-1}(\alpha)$.

In proving our main results (Theorem 1 and Theorem 2), we shall need the following lemma due to I. S. Jack [4].

Lemma. Let $w(z)$ be non-constant regular in U , $w(0) = 0$. If $|w(z)|$ attains its maximum value on the circle $|z| = r < 1$ at z_0 , we have $z_0 w'(z_0) = kw(z_0)$ where k is a real number, $k \geq 1$.

Theorem 1. $T_{n+p}(\alpha) \subset T_{n+p-1}(\alpha)$, $0 \leq \alpha < p$, $n > -p$.

Proof. Let $f(z) \in T_{n+p}(\alpha)$. Then

$$\operatorname{Re} \left\{ \frac{z(D^{n+p}f(z))'}{D^{n+p}f(z)} \right\} > \alpha, \quad z \in U. \quad (2.1)$$

We have to show that (2.1) implies the inequality

$$\operatorname{Re} \left\{ \frac{z(D^{n+p-1}f(z))'}{D^{n+p-1}f(z)} \right\} > \alpha, \quad z \in U. \quad (2.2)$$

Define a regular function $w(z)$ in U such that $w(0) = 0$, $w(z) \neq -1$ by

$$\frac{z(D^{n+p-1}f(z))'}{D^{n+p-1}f(z)} = \frac{p+(2\alpha-p)w(z)}{1+w(z)}. \quad (2.3)$$

Using (1.9), (2.3) may be written as

$$\frac{D^{n+p}f(z)}{D^{n+p-1}f(z)} = \frac{(p+n) + (2\alpha+n-p)w(z)}{1+w(z)}. \quad (2.4)$$

Logarithmic differentiation of (2.4) yields

$$\begin{aligned} \frac{z(D^{n+p}f(z))'}{D^{n+p}f(z)} &= \left[(p-\alpha) \frac{1-w(z)}{1+w(z)} + \alpha \right] \\ &\quad - \frac{2(p-\alpha)zw'(z)}{[(p+n)+(2\alpha+n-p)w(z)](1+w(z))}. \end{aligned} \quad (2.5)$$

We claim that $|w(z)| < 1$ in U . For otherwise (by Jack's lemma) there exists z_0 in U such that

$$z_0 w'(z_0) = k w(z_0) \quad (2.6)$$

where $|w(z_0)| = 1$ and $k \geq 1$. From (2.5) and (2.6), we obtain

$$\begin{aligned} \frac{z_0(D^{n+p}f(z_0))'}{D^{n+p}f(z_0)} &= \left[(p-\alpha) \frac{1-w(z_0)}{1+w(z_0)} + \alpha \right] \\ &\quad - \frac{2(p-\alpha)kw(z_0)}{[(p+n)+(2\alpha+n-p)w(z_0)](1+w(z_0))}. \end{aligned} \quad (2.7)$$

Since $\operatorname{Re} \left\{ \frac{1-w(z_0)}{1+w(z_0)} \right\} = 0$, $k \geq 1$,

$$2(p-\alpha)k \operatorname{Re} \left\{ \frac{w(z_0)}{[(p+n) + (2\alpha+n-p)w(z_0)](1+w(z_0))} \right\} \geq \frac{(p-\alpha)}{2(n+\alpha)},$$

we see that $\operatorname{Re} \left\{ \frac{z_0 (D^{n+p} f(z_0))'}{D^{n+p} f(z_0)} \right\} \leq \alpha - \frac{(p-\alpha)}{2(n+\alpha)} < \alpha$, which contradicts (2.1). Hence $|w(z)| < 1$, $z \in U$ and from (2.3) it follows that $f(z) \in T_{n+p-1}(\alpha)$.

Theorem 2. Let $f(z) \in A(p)$ satisfying the condition

$$\operatorname{Re} \left\{ \frac{z(D^{n+p-1} f(z))'}{D^{n+p-1} f(z)} \right\} > \alpha - \frac{(p-\alpha)}{2(c+\alpha)}, \quad c > 0, \quad 0 \leq \alpha < p, \quad (2.8)$$

then

$$F(z) = \frac{p+c}{z^c} \int_0^z t^{c-1} f(t) dt \quad (2.9)$$

belongs to $T_{n+p-1}(\alpha)$.

Proof. Let $w(z)$ be a regular function in U , $w(0) = 0$, $w(z) \neq -1$ defined by

$$\frac{z(D^{n+p-1} F(z))'}{D^{n+p-1} F(z)} = \frac{p-(p-2\alpha)w(z)}{1+w(z)}. \quad (2.10)$$

Using the identity

$$z(D^{n+p-1}F(z))' = (c+p)D^{n+p-1}f(z) - c D^{n+p-1}F(z) \quad (2.11)$$

(2.10) can be rewritten as

$$\frac{D^{n+p-1}f(z)}{D^{n+p-1}F(z)} = \frac{(c+p)+(c+2\alpha-p)w(z)}{(1+w(z))(c+p)} \quad (2.12)$$

Taking logarithmic differentiation of (2.12), we get after a simple computation

$$\begin{aligned} \frac{z(D^{n+p-1}f(z))'}{D^{n+p-1}f(z)} &= \left[(p-\alpha) \frac{1-w(z)}{1+w(z)} + \alpha \right] \\ &\quad - \frac{2(p-\alpha)zw'(z)}{[(c+p)+(c+2\alpha-p)w(z)](1+w(z))} \end{aligned} \quad (2.13)$$

Now we claim that $|w(z)| < 1$ for $z \in U$, for otherwise (by Jack's lemma) there exists a $z_0, z_0 \in U$, such that

$$z_0 w'(z_0) = k w(z_0) \quad (2.14)$$

with $|w(z_0)| = 1$ and $k \geq 1$.

From (2.13) and (2.14) we have

$$\begin{aligned} \frac{z_0(D^{n+p-1}f(z_0))'}{D^{n+p-1}f(z_0)} &= \left[(p-\alpha) \frac{1-w(z_0)}{1+w(z_0)} + \alpha \right] \\ &\quad - \frac{2(p-\alpha)kw(z_0)}{[(c+p)+(c+2\alpha-p)w(z_0)](1+w(z_0))} \end{aligned} \quad (2.15)$$

Since $\operatorname{Re} \left\{ \frac{1-w(z_0)}{1+w(z_0)} \right\} = 0, \quad k \geq 1,$

$$2(p-\alpha)k \operatorname{Re} \left\{ \frac{w(z_0)}{[(c+p)+(c+2\alpha-p)w(z_0)](1+w(z_0))} \right\} \geq \frac{(p-\alpha)}{2(c+\alpha)},$$

it follows from (2.15) that

$$\operatorname{Re} \left\{ \frac{z_0 (D^{n+p-1} f(z_0))'}{D^{n+p-1} f(z_0)} \right\} \leq \alpha - \frac{(p-\alpha)}{2(c+\alpha)},$$

which contradicts (2.8). Hence $|w(z)| < 1$ in U and from (2.10) it follows that $F(z) \in T_{n+p-1}(\alpha)$. This completes the proof of Theorem 2.

Putting $p = 1$ and taking $n = 0$ and $n = 1$ in Theorem 2, we obtain the following extensions of the earlier results of Bernardi [1] and Sohi [6].

Corollary 1. If $f(z)$ is starlike of order $\alpha - \frac{1-\alpha}{2(c+\alpha)}$, then the function

$$F(z) = \frac{1+c}{z^c} \int_0^z t^{c-1} f(t) dt$$

is starlike of order $\alpha, 0 \leq \alpha < 1.$

Corollary 2. If $f(z)$ is convex of order $\alpha - \frac{1-\alpha}{2(c+\alpha)}$, then the function

$$F(z) = \frac{1+c}{z^c} \int_0^z t^{c-1} f(t) dt$$

is convex of order α , $0 \leq \alpha < 1$.

Theorem 3. If $F(z) \in T_{n+p-1}(\alpha)$, $0 \leq \alpha < p$, and $f(z)$ is defined by (2.9), then $f(z) \in T_{n+p-1}(\alpha)$ for

$$|z| < \frac{c+p}{\sqrt{2p+1+c^2+\alpha(\alpha+2c-2)} + (p+1-\alpha)}$$

the result is sharp.

Proof. Since $F(z) \in T_{n+p-1}(\alpha)$, $0 \leq \alpha < p$, therefore we can write

$$\frac{z(D^{n+p-1}F(z))'}{D^{n+p-1}F(z)} - \alpha = \frac{p-\alpha}{p-\alpha} = u(z) \quad (2.16)$$

where $u(z)$ is regular in U and satisfies the conditions $\operatorname{Re}\{u(z)\} > 0$ and $u(0) = 1$.

From (2.11) and (2.16) we have

$$\frac{D^{n+p-1}f(z)}{D^{n+p-1}F(z)} = \frac{(p-\alpha)u(z) + (\alpha+c)}{(c+p)}. \quad (2.17)$$

Taking logarithmic differentiation of (2.17) and using (2.16), we get after a simple computation

$$\frac{z(D^{n+p-1}f(z))'}{D^{n+p-1}f(z)} - \alpha = (p-\alpha) \left\{ u(z) + \frac{z u'(z)}{(p-\alpha)u(z) + (\alpha+c)} \right\}. \quad (2.18)$$

It is well known [5] that for $|z| = r < 1$,

$$|z u'(z)| \leq \frac{2r}{1-r^2} \operatorname{Re}\{u(z)\}. \quad (2.19)$$

Thus from (2.18) and (2.19) we have for $|z| = r < 1$,

$$\operatorname{Re} \left\{ \frac{z(D^{n+p-1}f(z))'}{D^{n+p-1}f(z)} - \alpha \right\} \geq (p-\alpha) \left\{ 1 - \frac{2r}{(1-r)[c+p+(c+2\alpha-p)r]} \right\} \operatorname{Re}\{u(z)\}. \quad (2.20)$$

Thus $\operatorname{Re} \left\{ \frac{z(D^{n+p-1}f(z))'}{D^{n+p-1}f(z)} - \alpha \right\} > 0$

if

$$|z| < \frac{c+p}{\sqrt{2p+1+c^2+\alpha(\alpha+2c-2)} + (p+1-\alpha)}$$

The result is sharp for the function

$$f(z) = \frac{z^{1-c}}{p+c} (z^c F(z))',$$

where $F(z)$ is given by

$$\frac{z(D^{n+p-1}F(z))'}{D^{n+p-1}F(z)} = (p-\alpha) \frac{1-z}{1+z} + \alpha, \quad 0 \leq \alpha < p.$$

Putting $n + p = 1 = c$ in Theorem 3, we obtain

Corollary 3. If $F(z) \in S^*(p, \alpha)$ (the class of p -valent starlike functions of order α ($0 \leq \alpha < p$)) then $f(z) = \frac{1}{p+1} (zF(z))' \in S^*(p, \alpha)$ for $|z| < \frac{p+1}{\sqrt{2+2p+\alpha^2} + (p+1-\alpha)}$. The result is sharp.

Remark 1. Putting $\alpha = 0$ in Corollary 3, we get the result obtained by Goel [3].

Putting $p = 1$ and $n = 0$ in Theorem 3, we obtain

Corollary 4. If $F(z) \in S^*(\alpha)$ (the class of starlike functions of order α ($0 \leq \alpha < 1$)) then $f(z) = \frac{1}{c+1} (zF(z))' \in S^*(\alpha)$ for $|z| < \frac{c+1}{\sqrt{3+c^2+\alpha(\alpha+2c-2)}+(2-\alpha)}$. The result is sharp.

Remark 2. Putting $\alpha = 0$ in Corollary 4, we get the result obtained by Bernardi [2].

Theorem 4. If $f(z) \in T_{n+p-1}(\alpha)$, and $F(z)$ is defined by

$$F(z) = \frac{n+p}{z^n} \int_0^z t^{n-1} f(t) dt, \quad (2.21)$$

then $F(z) \in T_{n+p}(\alpha)$.

Proof. From (2.21) we have

$$n F(z) + z F'(z) = (n+p)f(z).$$

Therefore

$$n D^{n+p-1} F(z) + D^{n+p-1}(zF'(z)) = (n+p)D^{n+p-1}f(z)$$

or

$$n D^{n+p-1} F(z) + z(D^{n+p-1} F(z))' = (n+p)D^{n+p-1}f(z). \quad (2.22)$$

Using (1.10), we conclude from (2.22) that

$$D^{n+p} F(z) = D^{n+p-1} f(z). \quad (2.23)$$

Taking logarithmic differentiation of (2.23) and using the fact that $f(z) \in T_{n+p-1}(\alpha)$ we have

$$\operatorname{Re} \left\{ \frac{z(D^{n+p} F(z))'}{D^{n+p} F(z)} \right\} = \operatorname{Re} \left\{ \frac{z(D^{n+p-1} f(z))'}{D^{n+p-1} f(z)} \right\} > \alpha, \quad 0 \leq \alpha < p, \quad z \in U.$$

Hence $F(z) \in T_{n+p}(\alpha)$.

Remark 3. Taking $\alpha = 0$ in the above theorems, we get the results obtained by Sohi [6].

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