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CONVOLUTION OF CERTAIN STARLIKE FUNCTIONS WITH NEGATIVE COEFFICIENTS

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ABSTRACT. The object of the present paper is to determine several interesting results for the modified Hadamard product (convolution) of functions belonging to the class $T^{*}_{\lambda,n}(A,B)$ consisting of regular functions with negative coefficients.

KEY WORDS- Regular, Starlike, Ruscheweyh derivative, convolution.

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1. Introduction

Let T denote the class of functions of the form

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k \qquad (a_k \geq 0) \tag{1.1}$$

which are regular in the unit disc $U = \{z: |z| < 1\}$. If f(z) defined by (1.1) and g(z) defined by

$$g(z) = z - \sum_{k=2}^{\infty} b_k z^k \qquad (b_k \geq 0). \tag{1.2}$$

The modified Hadamard product (convolution) of f(z) and g(z) is defined by the power series

$$f * g(z) = z - \sum_{k=2}^{\infty} a_k b_k z^k. \tag{1.3}$$

The n-th order Ruscheweyh derivative of f(z), denoted by $D^n f(z)$, is defined by

$$D^n f(z) = \frac{z(z^{n-1} f(z))^{(n)}}{n!}, \tag{1.4}$$

where $n \in \mathbb{N}_o = \mathbb{N} \cup \{0\}$, $\mathbb{N} = \{1, 2, \ldots\}$. Ruscheweyh [3] determined that

$$D^n f(z) = \frac{z}{(1-z)^{n+1}} * f(z) \quad (n \in \mathbb{N}_o). \tag{1.5}$$

It is easy to see that

$$D^n f(z) = z - \sum_{k=2}^{\infty} \delta(n,k)\, a_k\, z^k, \tag{1.6}$$

where

$$\delta(n,k) = \binom{n+k-1}{n}. \tag{1.7}$$

A function $f(z) \in T$ is said to be in the class $T^*_{\lambda,n}(A,B)$ if it satisfies the following condition

$$\left| \frac{\dfrac{D^{n+1}f(z)}{D^n f(z)} - 1}{\lambda(A-B) - B\left(\dfrac{D^{n+1}f(z)}{D^n f(z)} - 1\right)} \right| < 1, \quad z \in U, \tag{1.8}$$

where $0 < \lambda \leq 1$, $-1 \leq B < A \leq 1$, and $-1 \leq B \leq 0$. The class $T^*_{\lambda,n}(A,B)$ was studied by Kumar and Chaudhary [1].

In order to prove our results for functions belonging to the class $T^*_{\lambda,n}(A,B)$, we shall require the following lemma given by Kumar and Chaudhary [1].

Lemma 1. Let the function $f(z)$ defined by (1.1). Then $f(z) \in T^*_{\lambda,n}(A,B)$ if and only if

$$\sum_{k=2}^{\infty} C_{k,\lambda}\, \delta(n,k)\, a_k \leq \lambda(A-B)(n+1), \tag{1.9}$$

where

$$C_{k,\lambda} = (1-B)(k-1) + \lambda(A-B)(n+1). \tag{1.10}$$

The result is sharp.

2. Modified Hadamard Product

Theorem 1. Let the function $f(z)$ defined by (1.1) and $g(z)$ defined by (1.2) be in the class $T^*_{\lambda,n}(A,B)$. Then $f * g(z) \in T^*_{\mu,n}(A,B)$, where

$$\mu = \frac{\lambda^2(A-B)(1-B)}{C^2_{2,\lambda} - \lambda^2(A-B)^2(n+1)}. \tag{2.1}$$

The result is sharp.

Proof. Employing the technique used earlier by Schild and Silverman [2]. We need to find the largest μ such that

$$\sum_{k=2}^{\infty} \frac{C_{k,\mu}\,\delta(n,k)}{\mu(A-B)(n+1)}\, a_k\, b_k \le 1. \tag{2.2}$$

Since

$$\sum_{k=2}^{\infty} \frac{C_{k,\lambda}\,\delta(n,k)}{\lambda(A-B)(n+1)}\, a_k \le 1 \tag{2.3}$$

and

$$\sum_{k=2}^{\infty} \frac{C_{k,\lambda}\ \delta(n,k)}{\lambda(A-B)(n+1)}\ b_k \leq 1, \tag{2.4}$$

by the Cauchy-Schwarz inequality we have

$$\sum_{k=2}^{\infty} \frac{C_{k,\lambda}\ \delta(n,k)}{\lambda(A-B)(n+1)} \sqrt{a_k\ b_k} \leq 1. \tag{2.5}$$

Thus it is sufficient to show that

$$\frac{C_{k,\mu}\ \delta(n,k)}{\mu(A-B)(n+1)}\ a_k\ b_k \leq \frac{C_{k,\lambda}\ \delta(n,k)}{\lambda(A-B)(n+1)} \sqrt{a_k\ b_k} \quad (k \geq 2), \tag{2.6}$$

that is, that

$$\sqrt{a_k\ b_k} \leq \frac{\mu\ C_{k,\lambda}}{\lambda\ C_{k,\mu}}. \tag{2.7}$$

Note that

$$\sqrt{a_k\ b_k} \leq \frac{\lambda(A-B)(n+1)}{C_{k,\lambda}\ \delta(n,k)} \qquad (k \geq 2). \tag{2.8}$$

Consequently, we need only to prove that

$$\frac{\lambda(A-B)(n+1)}{C_{k,\lambda}\ \delta(n,k)} \leq \frac{\mu\ C_{k,\lambda}}{\lambda\ C_{k,\mu}} \quad (k \geq 2). \tag{2.9}$$

or, equivalently that,

$$\mu \le \frac{\lambda^2(A-B)(1-B)(n+1)(k-1)}{C_{k,\lambda}^2\ \delta(n,k) - \lambda^2(A-B)^2(n+1)^2} \tag{2.10}$$

Since

$$D(k) = \frac{\lambda^2(A-B)(1-B)(n+1)(k-1)}{C_{k,\lambda}^2\ \delta(n,k) - \lambda^2(A-B)^2(n+1)^2} \tag{2.11}$$

is an increasing function of $k (k \ge 2)$, letting $k = 2$ in (2.11) we obtain

$$\mu \le D(2) = \frac{\lambda^2(A-B)(1-B)}{C_{2,\lambda}^2 - \lambda^2(A-B)^2(n+1)}, \tag{2.12}$$

which completes the proof of Theorem 1.

Finally, by taking the functions

$$f(z) = g(z) = z - \frac{\lambda(A-B)}{C_{2,\lambda}} z^2 \tag{2.13}$$

we can see that the result in Theorem 1 is sharp.

Corollary 1. For $f(z)$ and $g(z)$ as in Theorem 1, we have

$$h(z) = z - \sum_{k=2}^{\infty} \sqrt{a_k\ b_k}\ z^k \tag{2.14}$$

belongs to the class $T^*_{\lambda,n}(A,B)$.

The result follows from the inequality (2.5). It is sharp for the same functions as in Theorem 1.

Theorem 2. Let the function $f(z)$ defined by (1.1) be in the class $T^*_{\lambda,n}(A,B)$ and the function $g(z)$ defined by (1.2) be in the class $T^*_{\beta,n}(A,B)$, then $f * g(z) \in T^*_{\zeta,n}(A,B)$, where

$$\zeta = \frac{\lambda\beta(A-B)(1-B)}{C_{2,\lambda}C_{2,\beta} - \lambda\beta(A-B)^2(n+1)}. \tag{2.15}$$

The result is sharp.

Proof. Proceeding as in the proof of Theorem 1, we get

$$\zeta \le E(k) = \frac{\lambda\beta(A-B)(1-B)(n+1)(k-1)}{C_{k,\lambda}C_{k,\beta}\delta(n,k) - \lambda\beta(A-B)^2(n+1)^2}. \tag{2.16}$$

Since the function $E(k)$ is an increasing function of $k(k \ge 2)$, letting $k = 2$ in (2.16) we obtain

$$\zeta \le E(2) = \frac{\lambda\beta(A-B)(1-B)}{C_{2,\lambda}C_{2,\beta} - \lambda\beta(A-B)^2(n+1)}, \tag{2.17}$$

which evidently proves Theorem 2.

Further, taking

$$f(z) = z - \frac{\lambda(A-B)}{C_{2,\lambda}} z^2 \tag{2.18}$$

and

$$g(z) = z - \frac{\beta(A-B)}{C_{2,\beta}} z^2 \tag{2.19}$$

we can show that the result of Theorem 2 is sharp.

Corollary 2. Let the functions f(z), g(z), h(z) be in the class $T^*_{\lambda,n}(A,B)$, then $f * g * h(z) \in T^*_{\eta,n}(A,B)$, where

$$\eta = \frac{\lambda^3(A-B)^2(1-B)}{C^3_{2,\lambda} - \lambda^3(A-B)^3(n+1)}. \tag{2.20}$$

The result is best possible for the functions

$$f(z) = g(z) = h(z) = z - \frac{\lambda(A-B)}{C_{2,\lambda}} z^2. \tag{2.21}$$

Proof. From Theorem 1, we have $f * g(z) \in T^*_{\mu,n}(A,B)$, where μ is given by (2.1). We use now Theorem 2, we get $f * g * h(z) \in T^*_{\eta,n}(A,B)$, where

$$\eta = \frac{\lambda\mu(A-B)(1-B)}{C_{2,\lambda}C_{2,\mu} - \lambda\mu(A-B)^2(n+1)} = \frac{\lambda^3(A-B)^2(1-B)}{C^3_{2,\lambda} - \lambda^3(A-B)^3(n+1)}.$$

This completes the proof of Corollary 2.

THEOREM 3. Let the functions $f(z)$, $g(z)$ defined by (1.1) and (1.2), respectively, be in the class $T^{*}_{\lambda,n}(A,B)$. Then the function $h(z)$ defined by

$$h(z) = z - \sum_{k=2}^{\infty} \left(a_k^2 + b_k^2\right) z^k \qquad (2.22)$$

belongs to the class $T^{*}_{\varphi,n}(A,B)$, where

$$\varphi = \frac{2\lambda^2 (A-B)(1-B)}{C_{2,\lambda}^2 - 2\lambda^2 (A-B)^2 (n+1)}. \qquad (2.23)$$

The result is sharp for the functions $f(z) = g(z)$ defined by (2.13).

PROOF. By virtue of Lemma 1, we obtain

$$\sum_{k=2}^{\infty} \left\{\frac{C_{k,\lambda}\ \delta(n,k)}{\lambda(A-B)(n+1)}\right\}^2 a_k^2 \le \left\{\sum_{k=2}^{\infty} \frac{C_{k,\lambda}\ \delta(n,k)}{\lambda(A-B)(n+1)} a_k\right\}^2 \le 1 \quad (2.24)$$

and

$$\sum_{k=2}^{\infty} \left\{\frac{C_{k,\lambda}\ \delta(n,k)}{\lambda(A-B)(n+1)}\right\}^2 b_k^2 \le \left\{\sum_{k=2}^{\infty} \frac{C_{k,\lambda}\ \delta(n,k)}{\lambda(A-B)(n+1)} b_k\right\}^2 \le 1. \quad (2.25)$$

It follows from (2.24) and (2.25) that

$$\sum_{k=2}^{\infty} \frac{1}{2}\left\{\frac{C_{k,\lambda}\ \delta(n,k)}{\lambda(A-B)(n+1)}\right\}^2 \left[a_k^2 + b_k^2\right] \leq 1. \tag{2.26}$$

Therefore, we need to find the largest φ sush that

$$\frac{C_{k,\varphi}\ \delta(n,k)}{\varphi(A-B)(n+1)} \leq \frac{1}{2}\left\{\frac{C_{k,\lambda}\ \delta(n,k)}{\lambda(A-B)(n+1)}\right\}^2 (k \geq 2),$$

that is, that

$$\varphi \leq \frac{2(k-1)\lambda^2(A-B)(1-B)(n+1)}{C_{k,\lambda}^2\ \delta(n,k)\ -\ 2\lambda^2(A-B)^2(n+1)^2}\ . \tag{2.27}$$

Since

$$G(k) = \frac{2(k-1)\lambda^2(A-B)(1-B)(n+1)}{C_{k,\lambda}^2\ \delta(n,k)\ -\ 2\lambda^2(A-B)^2(n+1)^2} \tag{2.28}$$

is an increasing function of $k(k \geq 2)$, we readily have

$$\varphi \leq G(2) = \frac{2\lambda^2(A-B)(1-B)}{C_{2,\lambda}^2 - 2\lambda^2(A-B)^2(n+1)}, \tag{2.29}$$

which completes the proof of Theorem 3.

Theorem 4. Let the function $f(z) = z - \sum_{k=2}^{\infty} a_k z^k$ be in the class $T^*_{\lambda,n}(A,B)$ and $g(z) = z - \sum_{k=2}^{\infty} b_k z^k$ with $|b_k| \le 1$, $k = 2,3,\ldots$, then $f * g(z) \in T^*_{\lambda,n}(A,B)$.

Proof. Since

$$\sum_{k=2}^{\infty} C_{k,\lambda}\, \delta(n,k)\, |a_k b_k| = \sum_{k=2}^{\infty} C_{k,\lambda}\, \delta(n,k)\, a_k |b_k|$$

$$\le \sum_{k=2}^{\infty} C_{k,\lambda}\, \delta(n,k)\, a_k \le \lambda(A-B)(n+1)$$

by Lemma 1 it follows that $f * g(z) \in T^*_{\lambda,n}(A,B)$.

Corollary 3. If $f(z) \in T^*_{\lambda,n}(A,B)$ and $g(z) = z - \sum_{k=2}^{\infty} b_k z^k$ with $0 \le b_k \le 1$, $k = 2,3,\ldots$, then $f * g(z) \in T^*_{\lambda,n}(A,B)$.

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