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NEW CLASSES OF MEROMORPHICALLY MULTIVALENT FUNCTIONS

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Abstract. In this paper, we introduce new subclasses $C_{n,p}(\alpha)$ of meromorphically multivalent functions defined by the subordination relation. We also obtain the inclusion relations for the classes $C_{n,p}(\alpha)$ and investigate the integral preserving properties of functions in $C_{n,p}(\alpha)$.

1. Introduction

Let $\sum_p$ denote the class of functions of the form

$$f(z) = \frac{a_{-p}}{z^p} + \sum_{k=0}^{\infty} a_k z^k \quad (a_{-p} \neq 0, p \in N = \{1,2,...\}) \tag{1.1}$$

which are regular in the punctured disk $E = \{z : 0 < |z| < 1\}$. Following Uralegaddi and Somanatha [4], we define

$$D^0 f(z) = f(z), \tag{1.2}$$

$$D^1 f(z) = \frac{a_{-p}}{z^p} + (p+1)a_0 + (p+2)a_1 z + (p+3)a_2 z^2 + ... = \frac{(z^{p+1}f(z))'}{z^p}, \tag{1.3}$$

$$D^2 f(z) = D(D^1 f(z)), \tag{1.4}$$

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and for $n = 1, 2, \ldots,$

$$(1.5) \qquad \begin{aligned} D^n f(z) &= D(D^{n-1} f(z) \\ &= \frac{a_{-p}}{z^p} + \sum_{m=1}^{\infty} (p+m)^n a_{m-1} z^{m-1} \\ &= \frac{(z^{p+1} D^{n-1} f(z))'}{z^p}. \end{aligned}$$

Using the operator D^n, Cho and Lee [2] introduced the subclasses $B_{n,p}(\alpha)$ of $\sum_p$ whose members are characterized by the condition

$$(1.6) \qquad Re\left\{ z^{p+1}(D^n f(z))' \right\} < -p\frac{n+\alpha}{n+1} \ (z \in U = \{z : |z| < 1\})$$

for some $\alpha(0 \le \alpha < 1)$ and $n \in N_0 = N \cup \{0\}$. They proved that $B_{n+1,p}(\alpha) \subset B_{n,p}(\alpha)$, and since $B_{0,p}(\alpha)$ is the class of meromorphically p-valent functions [3]), all functions in $B_{n,p}(\alpha)$ are p-valent. Also they considered some properties in connection with certain integral transform.

In this paper, we introduce the new classes $C_{n,p}(\alpha)$ of meromorphically p-valent functions in U.

Let $C_{n,p}(\alpha)$ denote the class of functions $f \in \sum_p$ which satisfy the condition

$$(1.7) \qquad -z^{p+1}(D^n f(z))' \prec p + \frac{p(1-\alpha)}{n+2} z \ (0 \le \alpha < 1, z \in U),$$

where $\prec$ denotes the subordination relation. From (1.7), we have that $C_{n,p}(\alpha) \subset B_{n,p}(\alpha)$ for $n \in N_0$. Hence the classes $C_{n,p}(\alpha)$ are subclasses of meromorphically p-valent functions. Also we shall prove that $C_{n+1,p}(\alpha) \subset C_{n,p}(\alpha)$. Furthermore we consider certain integral transform of functions in $C_{n,p}(\alpha)$.

2. Properties of the classes $C_{n,p}(\alpha)$

For the proofs of comming theorems, we need the following lemma due to Jack [1].

Lemma 1. *Let w be non-constant regular in $U = \{z : |z| < 1\}$, $w(0) = 0$. If $|w|$ attains its maximum value on the circle $|z| = r < 1$ at z_0, we have $z_0 w'(z_0) = kw(z_0)$ where k is a real number, $k \geq 1$.*

Theorem 1. *$C_{n+1,p}(\alpha) \subset C_{n,p}(\alpha)$ for each $n \in N_0$.*

Proof. Let $f \in C_{n+1,p}(\alpha)$. Then

$$-z^{p+1}(D^{n+1}f(z))' \prec p + \frac{p(1-\alpha)}{n+3}z. \tag{2.1}$$

Define $w(z)$ in $U = \{z : |z| < 1\}$ by

$$-z^{p+1}(D^n f(z))' = p + \frac{p(1-\alpha)}{n+2}w(z). \tag{2.2}$$

Clearly $w(0) = 0$. Using the identity

$$z(D^n f(z))' = D^{n+1}f(z) - (p+1)D^n f(z), \tag{2.3}$$

the equation (2.2) may be written as

$$-z^p(D^{n+1}f(z) - (p+1)D^n f(z)) = p + \frac{p(1-\alpha)}{n+2}w(z). \tag{2.4}$$

Differentiating (2.4), we obtain

$$-z^{p+1}(D^{n+1}f(z))' = p + \frac{p(1-\alpha)}{n+2}\{w(z) + zw'(z)\}. \tag{2.5}$$

We claim that $|w(z)| < 1$ in U. Suppose that there exisrs a point $z_0 \in U$ such that $\max_{|z|<|z_0|} |w(z)| = |w(z_0)| = 1(w(z_0) \neq 1)$. Then, by Lemma 1, we have

$$z_0 w'(z_0) = kw(z_0), \tag{2.6}$$

where $k \geq 1$. The equation (2.5) in conjuction with (2.6) yields

$$\begin{aligned} |z_0^{p+1}(D^{n+1}f(z_0))' + p| &= |\frac{p(1-\alpha)}{n+2}(1+k)| \\ &> \frac{p(1-\alpha)}{n+3}, \end{aligned} \tag{2.7}$$

which is a contradiction to (2.1). Hence $|w(z)| < 1$ in U and from (2.2) it follows that $f \in C_{n,p}(\alpha)$.

Theorem 2. *Let $f \in C_{n,p}(\alpha)$. Then*

$$F(z) = \frac{c}{z^{c+p}} \int_0^z t^{c+p-1} f(t)dt \ \ (c > 0) \tag{2.8}$$

belongs to $C_{n,p}(\alpha)$.

Proof. Let $f \in C_{n,p}(\alpha)$. Define $w(z)$ in U by

$$-z^{p+1}(D^n F(z))' = p + \frac{p(1-\alpha)}{n+2} w(z). \tag{2.9}$$

Clearly $w(0) = 0$. Using the equation

$$z(D^n F(z))' = cD^n f(z) - (c+p)D^n F(z), \tag{2.10}$$

the equation (2.9) may be written as

$$-z^p(cD^n f(z) - (c+p)D^n F(z)) = p + \frac{p(1-\alpha)}{n+2}w(z). \tag{2.11}$$

Differentiating (2.11), we have

$$-z^{p+1}(D^n f(z))' = p + \frac{p(1-\alpha)}{n+2}w(z) + \frac{p(1-\alpha)}{c(n+2)}zw'(z). \tag{2.12}$$

We claim that $|w(z)| < 1$ in U. For otherwise, by Lemma 1, there exists z_0, $|z_0| < 1$ such that $z_0 w'(z_0) = kw(z_0)$, where $|w(z_0)| = 1$ and $k \geq 1$. Applying this result to (2.12), we obtain

$$\begin{aligned} \left|z_0^{p+1}(D^n f(z_0))' + p\right| &= \left|\frac{p(1-\alpha)}{n+2} + \frac{p(1-\alpha)k}{c(n+2)}\right| \\ &> \frac{p(1-\alpha)}{n+2}, \end{aligned} \tag{2.13}$$

which contradicts our assumption. Hence $|w(z)| < 1$ in U and from (2.12) we have that $F \in C_{n,p}(\alpha)$.

Taking $n = 0$ and $c = 1$ in Theorem 2, we have the following

Corollary 1. *Let $f \in C_{0,p}(\alpha)$. Then*

$$F(z) = \frac{1}{z^{1+p}} \int_0^z t^p f(t)dt \tag{2.14}$$

belongs to $C_{0,p}(\alpha)$.

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