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ON NEW SUBCLASSES OF UNIVALENT FUNCTIONS WITH NEGATIVE COEFFICIENTS

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ABSTRACT. The object of the present paper is to obtain coefficient estimates, some properties, distortion theorems and closure theorems for the classes $R_n^*(A,B)$ of analytic and univalent functions with negative coefficients, defined by using the n-th order Ruscheweyh derivative. We also obtain several interesting results for the modified Hadamard product of functions belonging to the class $R_n^*(A,B)$. Further, we obtain radii of close-to-convexity, starlikeness and convexity and integral operators for the classes $R_n^*(A,B)$.

KEY WORDS- Analytic, Ruscheweyh derivative, modified Hadamard product.

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1. Introduction

Let S denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1.1}$$

which are analytic and univalent in the unit disc $U = \{z: |z| < 1\}$. Let

$$D^n = \frac{z(z^{n-1}f(z))^{(n)}}{n!} \tag{1.2}$$

for $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, where $\mathbb{N} = \{1,2,\dots\}$. This symbol $D^n f(z)$ was named the n-th order Ruscheweyh derivative of f(z) by Al-Amiri [1]. We note that $D^0 f(z) = f(z)$ and $D^1 f(z) = zf'(z)$.

The Hadamard product of two functions $f(z) \in S$ and $g(z) \in S$ will be denoted by $f * g(z)$, that is, if f(z) is given by (1.1) and g(z) is given by

$$g(z) = z + \sum_{k=2}^{\infty} b_k z^k, \tag{1.3}$$

Then

$$f * g(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k. \tag{1.4}$$

By using the Hadamard product, Ruscheweyh [4] defined that

$$D^{\beta}f(z) = \frac{z}{(1-z)^{\beta+1}} * f(z) \qquad (\beta \geq -1\ ) \tag{1.4}$$

which implies (1.2) for $\beta \in \mathbb{N}_o$.

It is easy to see that

$$D^n f(z) = z + \sum_{k=2}^{\infty} \delta(n,k)\ a_k\ z^k, \tag{1.5}$$

where

$$\delta(n,k) = \binom{n+k-1}{n}. \tag{1.6}$$

Let T denote the subclass of S consisting of functions of the form

$$f(z) = z - \sum_{k=2}^{\infty} a_k\ z^k \qquad (a_k \geq 0). \tag{1.7}$$

Let $R_n^*(A,B)$ denote the class of functions $f(z) \in T$ such that

$$\left| \frac{(n+1)\left(\dfrac{D^{n+1}f(z)}{D^n f(z)} - 1\right)}{B(n+1)\dfrac{D^{n+1}f(z)}{D^n f(z)} - (Bn+A)} \right| < 1 \tag{1.8}$$

for $z \in U$, where $-1 \leq A < B \leq 1$, $0 < B \leq 1$, and $n \in \mathbb{N}_o$.

We note that:

(1) $R_n^*(-1,1) = R_n^*$ (Owa [3]);

(2) $R_o^*(2\alpha-1,1) = T^*(\alpha)$ $(0 \le \alpha < 1)$ (Silverman [5]);

(3) $R_o^*((2\alpha-1)\beta,\beta) = S^*(\alpha,\beta)$ $(0 \le \alpha < 1,\ 0 < \beta \le 1)$ (Gupta and Jain [2]).

2. Coefficient Estimates

Theorem 1. Let the function f(z) be defined by (1.7). Then $f(z) \in R_n^*(A,B)$ if and only if

$$\sum_{k=2}^{\infty} \Big[(1+B)k - (A+1)\Big]\delta(n,k)a_k \le B - A. \tag{2.1}$$

The result is sharp.

Proof. Assume that the inequality (2.1) holds and let $|z| = 1$. Then we get

$$\Big|(n+1)\Big(D^{n+1}f(z)-D^n f(z)\Big)\Big| - \Big|B(n+1)D^{n+1}f(z)-(Bn+A)D^n f(z)\Big|$$

$$= \Big|-\sum_{k=2}^{\infty}(k-1)\delta(n,k)a_k z^k\Big| - \Big|(B-A)z-\sum_{k=2}^{\infty}(Bk-A)\delta(n,k)a_k z^k\Big|$$

$$\le \sum_{k=2}^{\infty}(k-1)\delta(n,k)a_k - (B-A) + \sum_{k=2}^{\infty}(Bk-A)\delta(n,k)a_k$$

$$= \sum_{k=2}^{\infty}\Big[(1+B)k - (A+1)\Big]\delta(n,k)a_k - (B-A)$$

$\leq 0,$ by hypotheses.

Hence by the maximum modulus theorem, $f(z) \in R_n^*(A,B)$.

Conversely, suppose that

$$\left|\frac{(n+1)\left(\dfrac{D^{n+1}f(z)}{D^n f(z)} - 1\right)}{B(n+1)\dfrac{D^{n+1}f(z)}{D^n f(z)} - (Bn+A)}\right|$$

$$= \left|\frac{-\sum_{k=2}^{\infty}(k-1)\delta(n,k)a_k z^{k-1}}{(B-A) - \sum_{k=2}^{\infty}(Bk-A)\delta(n,k)a_k z^{k-1}}\right| \leq 1, \quad z \in U; \qquad (2.2)$$

Since $|Re(z)| \leq |z|$ for all z, we have

$$Re\left\{\frac{\sum_{k=2}^{\infty}(k-1)\delta(n,k)a_k z^{k-1}}{(B-A) - \sum_{k=2}^{\infty}(Bk-A)\delta(n,k)a_k z^{k-1}}\right\} < 1. \qquad (2.3)$$

Choose values of z on the real axis so that $\dfrac{D^{n+1}f(z)}{D^n f(z)}$ is real, upon clearing the denominator in (2.3) and letting $z \longrightarrow 1^-$ through real values, we obtain

$$\sum_{k=2}^{\infty} (k-1)\delta(n,k)a_k \le (B-A) - \sum_{k=2}^{\infty} (Bk-A)\delta(n,k)a_k .$$

This gives the required condition.

Finally, the function

$$f(z) = z - \frac{B - A}{[(1+B)k-(A+1)]\delta(n,k)} z^k \quad (k \ge 2) \qquad (2.4)$$

is an extremal function for the theorem.

Corollary 1. Let the function $f(z)$ defined by (1.7) be in the class $R_n^*(A,B)$, then

$$a_k \le \frac{B - A}{[(1+B)k-(A+1)]\delta(n,k)} \qquad (k \ge 2). \qquad (2.5)$$

The result is sharp for the function $f(z)$ given by (2.4).

3. Some Properties of the Class $R_n^*(A,B)$

Theorem 2. Let $-1 \le A_1 \le A_2 < B_1 \le B_2 \le 1$, $0 < B_1 \le B_2 \le 1$, and $n \in \mathbb{N}_0$. Then we have

$$R_n^*(A_1,B_2) \supset R_n^*(A_2,B_1).$$

Proof. Let the function f(z) defined by (1.7) be in the class $R_n^*(A_2,B_1)$, $B_2 = B_1 + \varepsilon_1$ and $A_2 = A_1 + \varepsilon_2$. Then, by Theorem 1, we get

$$\sum_{k=2}^{\infty} \Big[(1+B_1)k - (A_2+1)\Big]\delta(n,k)\ a_k \le B_1 - A_2. \qquad (3.1)$$

Hence

$$\sum_{k=2}^{\infty} \Big[(1+B_2)k - (A_1+1)\Big]\delta(n,k)\ a_k$$

$$= \sum_{k=2}^{\infty} \Big[(1+B_1+\varepsilon_1)k - (A_2-\varepsilon_2+1)\Big]\delta(n,k)\ a_k$$

$$= \sum_{k=2}^{\infty} \Big[(1+B_1)k - (A_2+1)\Big]\delta(n,k)\ a_k$$

$$+ \varepsilon_1 \sum_{k=2}^{\infty} k\ \delta(n,k)\ a_k + \varepsilon_2 \sum_{k=2}^{\infty} \delta(n,k)\ a_k$$

$$\le (B_1 - A_2) + \varepsilon_1 \frac{B_1 - A_2}{2[2B_1-A_2+1]} + \varepsilon_2 \frac{B_1 - A_2}{[2B_1-A_2+1]}$$

$$\le (B_1 - A_2) + \varepsilon_1 + \varepsilon_2 = (B_1 + \varepsilon_1) - (A_2 - \varepsilon_2)$$

$$= B_2 - A_1 \qquad (3.2)$$

which gives that $f(z) \in R_n^*(A_1,B_2)$. This completes the proof of Theorem 2.

Theorem 3. $R_{n+1}^*(A,B) \subset R_n^*(A,B)$ for $-1 \leq A < B \leq 1$, $0 < B \leq 1$ and $n \in \mathbb{N}_o$.

Proof. Let the function $f(z)$ defined by (1.7) be in the class $R_{n+1}^*(A,B)$. Then, by Theorem 1, we have

$$\sum_{k=2}^{\infty} \Big[(1+B)k - (A+1)\Big]\delta(n+1,k)\ a_k \leq B - A \tag{3.3}$$

and since

$$\delta(n,k) \leq \delta(n+1,k) \qquad \text{for} \quad k \geq 2, \tag{3.4}$$

we have

$$\sum_{k=2}^{\infty} \Big[(1+B)k - (A+1)\Big]\delta(n,k)\ a_k$$

$$\leq \sum_{k=2}^{\infty} \Big[(1+B)k - (A+1)\Big]\delta(n+1,k)\ a_k \leq B - A. \tag{3.5}$$

The result follows from Theorem 1.

4. Distortion Theorems

Theorem 4. Let the function f(z) defined by (1.7) be in the class $R_n^*(A,B)$. Then we have

$$|f(z)| \geq |z| - \frac{B-A}{[2B-A+1](n+1)}|z|^2 \tag{4.1}$$

and

$$|f(z)| \leq |z| + \frac{B-A}{[2B-A+1](n+1)}|z|^2 \tag{4.2}$$

for $z \in U$. The result is sharp.

Proof. Since $f(z) \in R_n^*(A,B)$, in view of Theorem 1, we obtain

$$[2B-A+1](n+1)\sum_{k=2}^{\infty} a_k \leq \sum_{k=2}^{\infty}\Big[(1+B)k-(A+1)\Big]\delta(n,k)\ a_k$$

$$\leq B - A \tag{4.3}$$

which implies that

$$\sum_{k=2}^{\infty} a_k \leq \frac{B-A}{[2B-A+1](n+1)}. \tag{4.4}$$

Therefore we can show that

$$|f(z)| \geq |z| - |z|^2 \sum_{k=2}^{\infty} a_k \geq |z| - \frac{B - A}{[2B-A+1](n+1)} |z|^2 \quad (4.5)$$

and

$$|f(z)| \leq |z| + |z|^2 \sum_{k=2}^{\infty} a_k \leq |z| + \frac{B - A}{[2B-A+1](n+1)} |z|^2 \quad (4.6)$$

for $z \in U$. This completes the proof of Theorem 4. Finally, by taking the function

$$f(z) = z - \frac{B - A}{[2B-A+1](n+1)} z^2, \quad (4.7)$$

we can show that the results of Theorem 4 are sharp.

Corollary 2. Let the function $f(z)$ defined by (1.7) be in the class $R_n^*(A,B)$. Then $f(z)$ is included in a disc with its center at the origin and radius r_1 given by

$$r_1 = \frac{(B-A)(n+2)+(B+1)(n+1)}{[2B-A+1](n+1)}. \quad (4.8)$$

Theorem 5. Let the function $f(z)$ defined by (1.7) be in the class $R_n^*(A,B)$. Then we have

$$|f'(z)| \geq 1 - \frac{2(B-A)}{[2B-A+1](n+1)}|z| \qquad (4.9)$$

and

$$|f'(z)| \leq 1 + \frac{2(B-A)}{[2B-A+1](n+1)}|z| \qquad (4.10)$$

for $z \in U$. The result is sharp.

PROOF. In view of Theorem 1, we obtain

$$\frac{1}{2}[2B-A+1](n+1)\sum_{k=2}^{\infty} k\, a_k \leq \sum_{k=2}^{\infty}\Big[(1+B)k-(A+1)\Big]\delta(n,k)\, a_k$$

$$\leq B - A \qquad (4.11)$$

which implies that

$$\sum_{k=2}^{\infty} k\, a_k \leq \frac{2(B-A)}{[2B-A+1](n+1)}. \qquad (4.12)$$

Hence, with the aid of (4.12), we have

$$|f'(z)| \geq 1 - |z|\sum_{k=2}^{\infty} k\, a_k \geq 1 - \frac{2(B-A)}{[2B-A+1](n+1)}|z| \qquad (4.13)$$

and

$$|f'(z)| \le 1 + |z|\sum_{k=2}^{\infty} k\, a_k \le 1 + \frac{2(B-A)}{[2B-A+1](n+1)}|z| \qquad (4.14)$$

for $z \in U$. Further the results of Theorem 5 are sharp for the function $f(z)$ given by (4.7).

Corollary 3. Let the function $f(z)$ defined by (1.7) be in the class $R_n^*(A,B)$. Then $f(z)$ is included in a disc with its center at the origin and radius r_2 given by

$$r_2 = \frac{2(B-A)(n+2)+(A+1)(n+1)}{[2B-A+1](n+1)}. \qquad (4.15)$$

5. Closure Theorems

Let the functions $f_i(z)$ be defined, for $i = 1,2,\ldots, m$, by

$$f_i(z) = z - \sum_{k=2}^{\infty} a_{k,i}\, z^k \quad (a_{k,i} \ge 0) \qquad (5.1)$$

for $z \in U$.

We shall prove the following results for the closure of functions in the class $R_n^*(A,B)$.

Theorem 6. Let the functions $f_i(z)$ $(i = 1, 2, \ldots, m)$ defined by (5.1) be in the class $R_n^*(A,B)$. Then the function $h(z)$ defined by

$$h(z) = \sum_{i=1}^{m} c_i f_i(z) \qquad (c_i \geq 0) \tag{5.2}$$

is also in the same class $R_n^*(A,B)$, where

$$\sum_{i=1}^{m} c_i = 1. \tag{5.3}$$

Proof. By means of the definition of $h(z)$, we obtain

$$h(z) = z - \sum_{k=2}^{\infty} \left[\sum_{i=1}^{m} c_i a_{k,i} \right] z^k. \tag{5.4}$$

Further, since $f_i(z)$ are in $R_n^*(A,B)$ for every $i = 1,2,\ldots,m$, we get

$$\sum_{k=2}^{\infty} \Big[(1+B)k-(A+1)\Big]\delta(n,k)\ a_{k,i} \leq B - A \tag{5.5}$$

for every $i = 1,2,\ldots,m$. Hence we can see that

$$\sum_{k=2}^{\infty} \Big[(1+B)k-(A+1)\Big]\delta(n,k) \left[\sum_{i=1}^{m} c_i a_{k,i} \right]$$

$$= \sum_{i=1}^{m} c_i \left[\sum_{k=2}^{\infty} \left[(1+B)k-(A+1) \right] \delta(n,k) a_{k,i} \right]$$

$$\leq \left[\sum_{i=1}^{m} c_i \right] (B - A) = B - A \qquad (5.6)$$

with the aid of (5.5). This proves that the function h(z) is in the class $R_n^*(A,B)$ by means of Theorem 1. Thus we have the theorem

Theorem 7. Let the functions $f_i(z)$ $(i = 1,2,\ldots, m)$ defined by (5.1) be in the class $R_n^*(A,B)$. Then the function h(z) defined by

$$h(z) = z - \sum_{k=2}^{\infty} b_k z^k \qquad (5.7)$$

also belongs to the class $R_n^*(A,B)$, where

$$b_k = \frac{1}{m} \sum_{i=1}^{m} a_{k,i}. \qquad (5.8)$$

Proof. Since $f_i(z) \in R_n^*(A,B)$, it follows from Theorem 1, that

$$\sum_{k=2}^{\infty}\Big[(1+B)k-(A+1)\Big]\delta(n,k)a_{k,i} \le B-A, \quad i = 1,2,\ldots,m. \qquad (5.9)$$

Therefore,

$$\sum_{k=2}^{\infty}\Big[(1+B)k-(A+1)\Big]\delta(n,k)b_k$$

$$= \sum_{k=2}^{\infty}\Big[(1+B)k-(A+1)\Big]\delta(n,k)\left[\frac{1}{m}\sum_{i=1}^{m} a_{k,i}\right]$$

$$\le B - A. \qquad (5.10)$$

Hence by Theorem 1, $h(z) \in R_n^*(A,B)$. Thus we have the theorem.

Theorem 8. The class $R_n^*(A,B)$ is closed under convex linear combination.

Proof. Let the functions $f_i(z)$ $(i = 1,2)$ defined by (5.1) be in the class $R_n^*(A,B)$. Then it is sufficient to show that the function

$$h(z) = \lambda f_1(z) + (1-\lambda)f_2(z) \quad (0 \le \lambda \le 1) \qquad (5.11)$$

is in the class $R_n^*(A,B)$. Since, for $0 \le \lambda \le 1$,

$$h(z) = z - \sum_{k=2}^{\infty} \left\{\lambda\ a_{k,1} + (1-\lambda)a_{k,2}\right\} z^k, \qquad (5.12)$$

with the aid of Theorem 1, we have

$$\sum_{k=2}^{\infty} \left[(1+B)k-(A+1)\right]\delta(n,k)\left\{\lambda\ a_{k,1}+(1-\lambda)a_{k,2}\right\} \le (B-A), \quad (5.13)$$

which implies that $h(z) \in R_n^*(A,B)$.

As a consequence of Theorem 8, there exists the extreme points of the class $R_n^*(A,B)$.

Theorem 9. Let $f_1(z) = z$ and

$$f_k(z) = z - \frac{B-A}{[(1+B)k-(A+1)]\delta(n,k)} z^k \quad (k \ge 2) \qquad (5.14)$$

for $-1 \le A < B \le 1$, $0 < B \le 1$, and $n \in \mathbb{N}_0$. Then $f(z)$ is in the class $R_n^*(A,B)$ if and only if it can be expressed in the form

$$f(z) = \sum_{k=1}^{\infty} \lambda_k\ f_k(z), \qquad (5.15)$$

where $\lambda_k \ge 0$ $(k \ge 1)$ and $\sum_{k=1}^{\infty} \lambda_k = 1$.

PROOF. Suppose that

$$f(z) = \sum_{k=1}^{\infty} \lambda_k f_k(z)$$

$$= z - \sum_{k=2}^{\infty} \frac{(B-A)\lambda_k}{[(1+B)k-(A+1)]\delta(n,k)} z^k. \qquad (5.16)$$

Then it follows that

$$\sum_{k=2}^{\infty} \frac{[(1+B)k-(A+1)]\delta(n,k)}{(B-A)} \cdot \frac{(B-A)\lambda_k}{[(1+B)k-(A+1)]\delta(n,k)}$$

$$= \sum_{k=2}^{\infty} \lambda_k = 1 - \lambda_1 \leq 1. \qquad (5.17)$$

So by Theorem 1, $f(z) \in R_n^*(A,B)$.

Conversely, assume that the function $f(z)$ defined by (1.7) belongs to the class $R_n^*(A,B)$. Then

$$a_k \leq \frac{(B-A)}{[(1+B)k-(A+1)]\delta(n,k)} \quad (k \geq 2). \qquad (5.18)$$

Setting

$$\lambda_k = \frac{[(1+B)k-(A+1)]\delta(n,k)}{(B-A)} a_k \qquad (5.19)$$

and

$$\lambda_1 = 1 - \sum_{k=2}^{\infty} \lambda_k. \tag{5.20}$$

Hence, we can see that f(z) can be expressed in the form (5.14). This completes the proof of Theorem 9.

Corollary 4. The extreme points of the class $R_n^*(A,B)$ are the functions $f_k(z)$ $(k \geq 1)$ given by Theorem 9.

6. Integral Operators

Theorem 10. Let the function f(z) defined by (1.7) be in the class $R_n^*(A,B)$, and let c be a real number such that $c > -1$. Then the function F(z) defined by

$$F(z) = \frac{c+1}{z^c} \int_0^z t^{c-1} f(t)\, dt \tag{6.1}$$

also belongs to the class $R_n^*(A,B)$.

Proof. From the representation of F(z), it follows that

$$F(z) = z - \sum_{k=2}^{\infty} b_k z^k, \tag{6.2}$$

where

$$b_k = \left(\frac{c+1}{c+k}\right) a_k. \tag{6.3}$$

Therefore,

$$\sum_{k=2}^{\infty}\Big[(1+B)k - (A+1)\Big]\delta(n,k)\; b_k$$

$$= \sum_{k=2}^{\infty}\Big[(1+B)k - (A+1)\Big]\delta(n,k)\;\left(\frac{c+1}{c+k}\right) a_k$$

$$\leq \sum_{k=2}^{\infty}\Big[(1+B)k - (A+1)\Big]\delta(n,k)\; a_k \leq B - A, \tag{6.4}$$

since $f(z) \in R_n^*(A,B)$. Hence, by Theorem 1, $F(z) \in R_n^*(A,B)$.

THEOREM 11. Let the function $F(z) = z - \sum_{k=2}^{\infty} a_k z^k$ ($a_k \geq 0$) be in the class $R_n^*(A,B)$, and let c be a real number such that $c > -1$. Then the function $f(z)$ defined by (6.1) is univalent in $|z| < R^*$, where

$$R^* = \inf_k \left\{\frac{[(1+B)k-(A+1)]\delta(n,k)(c+1)}{k(B-A)(c+k)}\right\}^{\frac{1}{k-1}} \quad (k \geq 2). \tag{6.5}$$

The result is sharp.

Proof. From (6.1), we have

$$f(z) = \frac{z^{1-c}\,(z^c F(z))'}{(c+1)} \quad (c > -1)$$

$$= z - \sum_{k=2}^{\infty} \left(\frac{c+k}{c+1}\right) a_k\, z^k. \tag{6.6}$$

In order to obtain the required result it suffices to show that

$$|f'(z) - 1| < 1 \text{ in } |z| < R^*.$$

Now

$$|f'(z) - 1| \leq \sum_{k=2}^{\infty} \frac{k(c+k)}{(c+1)}\, a_k\, |z|^{k-1}.$$

Thus $|f'(z) - 1| < 1$, if

$$\sum_{k=2}^{\infty} \frac{k(c+k)}{(c+1)}\, a_k\, |z|^{k-1} < 1. \tag{6.7}$$

But Theorem 1 confirms that

$$\sum_{k=2}^{\infty} \frac{[(1+B)k-(A+1)]\delta(n,k)}{(B-A)}\, a_k \leq 1. \tag{6.8}$$

Hence (6.7) will be satisfied if

$$\frac{k(c+k)\,|z|^{k-1}}{(c+1)} < \frac{[(1+B)k-(A+1)]\delta(n,k)}{(B-A)}$$

or if

$$|z| < \left\{\frac{[(1+B)k-(A+1)]\delta(n,k)(c+1)}{k(B-A)(c+k)}\right\}^{\frac{1}{k-1}} \quad (k \geq 2). \qquad (6.9)$$

Therefore f(z) is univalent in $|z| < R^*$. Sharpness follows if we take

$$f(z) = z - \frac{(B-A)(c+k)}{[(1+B)k-(A+1)]\delta(n,k)(c+1)} z^k \quad (k \geq 2). \qquad (6.10)$$

7. Radii of Close-to-Convexity, Starlikeness and Convexity

Theorem 12. Let the function f(z) defined by (1.7) be in the class $R_n^*(A,B)$, then f(z) is close-to-convex of order $\rho(0 \leq \rho < 1)$ in $|z| < r_1(n,A,B,\rho)$, where

$$r_1(n,A,B,\rho) = \inf_k \left\{\frac{(1-\rho)[(1+B)k-(A+1)]\delta(n,k)}{k(B-A)}\right\}^{\frac{1}{k-1}} (k \geq 2). \qquad (7.1)$$

The result is sharp, with the extremal function f(z) given by (2.4).

Proof. It is sufficient to show that $|f'(z) - 1| \leq 1 - \rho$ $(0 \leq \rho < 1)$ for $|z| < r_1(n,A,B,\rho)$. We have

$$|f'(z)-1| \le \sum_{k=2}^{\infty} k\, a_k\, |z|^{k-1}.$$

Thus $|f'(z)-1| \le 1-\rho$ if

$$\sum_{k=2}^{\infty} \left(\frac{k}{1-\rho}\right) a_k\, |z|^{k-1} \le 1. \tag{7.2}$$

Hence, by using (6.8), (7.2) will be true if

$$\frac{k|z|^{k-1}}{(1-\rho)} \le \frac{[(1+B)k-(A+1)]\delta(n,k)}{(B-A)}$$

or if

$$|z| \le \left\{\frac{(1-\rho)[(1+B)k-(A+1)]\delta(n,k)}{k(B-A)}\right\}^{\frac{1}{k-1}} \quad (k \ge 2). \tag{7.3}$$

The theorem follows easily from (7.3).

Theorem 13. Let the function $f(z)$ defined by (1.7) be in the class $R_n^*(A,B)$, then $f(z)$ is starlike of order $\rho(0 \le \rho < 1)$ in $|z| < r_2(n,A,B,\rho)$, where

$$r_2(n,A,B,\rho) = \inf_k \left\{\frac{(1-\rho)[(1+B)k-(A+1)]\delta(n,k)}{(k-\rho)(B-A)}\right\}^{\frac{1}{k-1}} \quad (k \ge 2). \tag{7.4}$$

The result is sharp, with the extremal function f(z) given by (2.4).

Proof. We must show that $\left|\frac{zf'(z)}{f(z)} - 1\right| \leq 1 - \rho$ $(0 \leq \rho < 1)$ for $|z| < r_2(n,A,B,\rho)$. We have

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq \frac{\sum_{k=2}^{\infty} (k-1)\ a_k\ |z|^{k-1}}{1 - \sum_{k=2}^{\infty} a_k\ |z|^{k-1}}$$

Thus $\left|\frac{zf'(z)}{f(z)} - 1\right| \leq 1 - \rho$ if

$$\sum_{k=2}^{\infty} \frac{(k-\rho)a_k\ |z|^{k-1}}{(1-\rho)} \leq 1. \tag{7.5}$$

Hence, by using (6.8), (7.5) will be true if

$$\frac{(k-\rho)\,|z|^{k-1}}{(1-\rho)} \leq \frac{[(1+B)k-(A+1)]\delta(n,k)}{(B-A)}$$

or if

$$|z| \leq \left\{\frac{(1-\rho)[(1+B)k-(A+1)]\delta(n,k)}{(k-\rho)(B-A)}\right\}^{\frac{1}{k-1}} \quad (k \geq 2). \tag{7.6}$$

The theorem follows easily from (7.6).

Corollary 5. Let the function f(z) defined by (1.7) be in the class $R_n^*(A,B)$, then f(z) is convex of order $\rho(0 \le \rho < 1)$ in $|z| < r_3(n,A,B,\rho)$, where

$$r_3(n,A,B,\rho) = \inf_k \left\{ \frac{(1-\rho)[(1+B)k-(A+1)]\delta(n,k)}{k(k-\rho)(B-A)} \right\}^{\frac{1}{k-1}} \quad (k \ge 2). \tag{7.7}$$

The result is sharp, with the extremal function f(z) given by (2.4).

8. Modified Hadamard Product

Let the functions $f_i(z)$ $(i = 1,2)$ be defined by (5.1). The modified Hadamard product of $f_1(z)$ and $f_2(z)$ is defined by

$$f_1 * f_2(z) = z - \sum_{k=2}^{\infty} a_{k,1}\, a_{k,2}\, z^k. \tag{8.1}$$

Theorem 14. Let the function $f_1(z)$ defined by (5.1) be in the class $R_n^*(A,B)$ and the function $f_2(z)$ defined by (5.1) be in the class $R_n^*(C,D)$ $(-1 \le C < D \le 1,\ 0 < D \le 1)$. Then the modified Hadamard product $f_1 * f_2(z)$ belongs to the class

$$\cdot R_n^*\left[1 - \frac{2(B-A)(D-C)}{[2B-A+1][2D-C+1](n+1)-(B-A)(D-C)},\ 1\right].$$

The result is sharp.

Proof. From Theorem 1, we have

$$\sum_{k=2}^{\infty} \frac{[(1+B)k-(A+1)]\delta(n,k)}{(B-A)} a_{k,1} \leq 1 \tag{8.2}$$

and

$$\sum_{k=2}^{\infty} \frac{[(1+D)k-(C+1)]\delta(n,k)}{(D-C)} a_{k,2} \leq 1 \cdot \tag{8.3}$$

We want to find the largest $\beta = \beta(n,A,B,C,D)$ such that

$$\sum_{k=2}^{\infty} \frac{[2k-(\beta+1)]\delta(n,k)}{(1-\beta)} a_{k,1}\, a_{k,2} \leq 1. \tag{8.4}$$

From (8.2) and (8.3) by means of Cauchy-Shwarz inequality we obtain

$$\sum_{k=2}^{\infty} \sqrt{\frac{[(1+B)k-(A+1)][(1+D)k-(C+1)]}{(B-A)(D-C)}}\,\delta(n,k)\sqrt{a_{k,1}a_{k,2}} \leq 1. \tag{8.5}$$

Hence (8.4) will be satisfied if

$$\sqrt{a_{k,1}\, a_{k,2}} \le \frac{(1-\beta)}{[2k-(\beta+1)]}\sqrt{\frac{[(1+B)k-(A+1)][(1+D)k-(C+1)]}{(B-A)(D-C)}}$$

$$(k \ge 2). \qquad (8.6)$$

From (8.5) it follows that

$$\sqrt{a_{k,1}\, a_{k,2}} \le \frac{1}{\delta(n,k)}\sqrt{\frac{(B-A)(D-C)}{[(1+B)k-(A+1)][(1+D)k-(C+1)]}} \quad (k \ge 2).$$

$$(8.7)$$

Therefore (8.4) will be satisfied if

$$\frac{1}{\delta(n,k)}\sqrt{\frac{(B-A)(D-C)}{[(1+B)k-(A+1)][(1+D)k-(C+1)]}}$$

$$\le \frac{(1-\beta)}{[2k-(\beta+1)]}\sqrt{\frac{[(1+B)k-(A+1)][(1+D)k-(C+1)]}{(B-A)(D-C)}} \quad (k \ge 2) \quad (8.8)$$

that is, that

$$\beta \le 1 - \frac{2(k-1)(B-A)(D-C)}{[(1+B)k-(A+1)][(1+D)k-(C+1)]\delta(n,k)-(B-A)(D-C)}.$$

$$(8.9)$$

The right hand side of (8.9) is an increasing function of k

$(k \geq 2)$. Therefore, setting $k = 2$ in (8.9) we get

$$\beta \leq 1 - \frac{2(B-A)(D-C)}{[2B-A+1][2D-C+1](n+1)-(B-A)(D-C)}. \tag{8.10}$$

The result is sharp, with equality when

$$f_1(z) = z - \frac{B-A}{[2B-A+1](n+1)} z^2 \tag{8.11}$$

and

$$f_2(z) = z - \frac{D-C}{[2D-C+1](n+1)} z^2. \tag{8.12}$$

Theorem 15. Let the functions $f_i(z)$ $(i = 1,2)$ defined by (5.1) be in the class $R_n^*(A,B)$. Then we have the function $h(z)$ defined by

$$h(z) = z - \sum_{k=2}^{\infty} \sqrt{a_{k,1}\, a_{k,2}}\; z^k \tag{8.13}$$

belongs to the class $R_n^*(A,B)$. The result is sharp.

Proof. Since $f_i(z)$ $(i = 1,2)$ belongs to the class $R_n^*(A,B)$, we have

$$\sum_{k=2}^{\infty} \frac{[(1+B)k-(A+1)]\delta(n,k)}{(B-A)} a_{k,1} \le 1 \tag{8.14}$$

and

$$\sum_{k=2}^{\infty} \frac{[(1+B)k-(A+1)]\delta(n,k)}{(B-A)} a_{k,2} \le 1. \tag{8.15}$$

From (8.14) and (8.15) we get by means of Cauchy-Schwarz inequality

$$\sum_{k=2}^{\infty} \frac{[(1+B)k-(A+1)]\delta(n,k)}{(B-A)} \sqrt{a_{k,1}\, a_{k,2}} \le 1 \tag{8.16}$$

by Theorem 1, it follows that $h(z) \in R_n^*(A,B)$. Finally, the result is sharp for the functions

$$f_i(z) = z - \frac{B-A}{[2B-A+1](n+1)} z^2 \quad (i = 1,2). \tag{8.17}$$

Theorem 16. Let $f_1(z) \in R_{n_1}^*(A,B)$ and $f_2(z) \in R_{n_2}^*(A,B)$. Then the modified Hadamard product $f_1 * f_2(z) \in R_{n_1}^*(A,B) \cap R_{n_2}^*(A,B)$.

PROOF. Since $f_2(z) \in R^*_{n_2}(A,B)$ we have from (4.4)

$$a_{k,2} \le \frac{(B-A)}{[2B-A+1](n_2+1)}. \tag{8.18}$$

From Theorem 1, since $f_1(z) \in R^*_{n_1}(A,B)$, we have

$$\sum_{k=2}^{\infty} \frac{[(1+B)k-(A+1)]\delta(n_1,k)}{(B-A)} a_{k,1} \le 1. \tag{8.19}$$

Now, from (8.18) and (8.19),

$$\sum_{k=2}^{\infty} \frac{[(1+B)k-(A+1)]\delta(n_1,k)}{(B-A)} a_{k,1}\, a_{k,2}$$

$$\le \frac{(B-A)}{[2B-A+1](n_2+1)} \sum_{k=2}^{\infty} \frac{[(1+B)k-(A+1)]\delta(n_1,k)}{(B-A)} a_{k,1}$$

$$\le \frac{(B-A)}{[2B-A+1](n_2+1)} \le (B-A). \tag{8.20}$$

Hence $f_1 * f_2(z) \in R^*_{n_1}(A,B)$. Interchanging n_1 and n_2 by each other in the above, we get $f_1 * f_2(z) \in R^*_{n_2}(A,B)$. Hence the theorem.

References

[1] H. S. Al-Amiri, On Ruscheweyh derivatives, Ann. Polon. Math. 38 (1980), 87-94.

[2] V.P. Gupta and P.K. Jain, Certain classes of univalent functions with negative coefficients, Bull. Austral. Math. Soc. 14 (1976), 409-416.

[3] S. Owa, On new classes of univalent functions with negative coefficients, Bull. Korean Math. Soc. 22 (1985), no. 1, 43-52.

[4] St. Ruscheweyh, New criteria for univalent functions, Proc. Amer. Math. Soc. 49 (1975), 109-115.

[5] H. Silverman, Univalent functions with negative coefficients, Proc. Amer Math. Soc. 51 (1975), 109-116.

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