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On the Mehler-Fock Index Transform in L_p-Space

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Abstract

The present paper is devoted to study the transform by the index of the Legendre function which is known as the Mehler-Fock transform. Mapping properties of the Mehler-Fock transform in the weighted space $L_p(\omega(t); \mathbf{R}_+)$ are given as the inversion formula. The space of images is described.

1. Introduction

The present paper is devoted to discuss the Mehler-Fock transform, which is a transform by index of the Legendre function, namely the integral transform of the following type

$$MF[f](\tau) = \frac{\pi}{2}\int_0^\infty P_{-1/2+i\tau/2}(2y^2+1)f(y)dy \quad (\tau > 0), \tag{1.1}$$

where

$$P_\nu(z) = P_\nu^0(z) = {}_2F_1\left(-\nu, \nu+1; 1; \frac{1-z}{2}\right) \tag{1.2}$$

is the Legendre function of the first kind of the index ν represented by the Gauss hypergeometric function

$${}_2F_1(a,b,c,z) = \sum_{n=0}^\infty \frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!} \tag{1.3}$$

with $(a)_n = \Gamma(a+n)/\Gamma(a)$ $(n = 0, 1, 2, \cdots)$ being Pochhammer symbol [1].

To investigate this function we will use the integral representation of type [9, (2.16.21.1)]

$$\frac{\pi}{2\cosh(\pi\tau/2)}P_{-1/2+i\tau/2}(2x^2+1) = \int_0^\infty J_0(xy)K_{i\tau}(y)dy \quad (\tau, x > 0), \tag{1.4}$$

where $J_\nu(z)$ is the Bessel function and $K_\nu(z)$ is the Macdonald function [2].

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This transform was introduced in 1881 by Mehler [7] and it was investigated in more detail by Fock [3]. As it was shown by the first author the Mehler-Fock transform is very important as a basic one among the class of index transforms and it is related to the known Kontorovich-Lebedev transform [13]. In the literature there are not so many results concerning these transforms and such a theory is developing now. We note here the bibliography from [13] and the papers [12], [4], [5], [14].

The main purpose of this paper is to investigate the space of images by the Mehler-Fock transform (1.1) of the Banach spaces $L_p(\mathbf{R}_+)$. The results are known for the classical Fourier type transforms [10], [11] and such theorems are already received by first author for the Kontorovich-Lebedev transform [13].

Now we prepare several formulas which is useful in the following discussions. The Macdonald function has the expression [2]

$$K_{i\tau}(x) = \frac{1}{2}\int_{-\infty}^{\infty} e^{-x\cosh\beta} e^{i\tau\beta} d\beta \quad (x > 0). \tag{1.5}$$

By the analytic property of the integrand in (1.5) and by its asymptotic behavior at the contour we can shift it along the horizontal open infinite strip $(i\delta - \infty, i\delta + \infty)$ with $\delta \in [0, \pi/2)$ as

$$K_{i\tau}(x) = \frac{1}{2}\int_{i\delta-\infty}^{i\delta+\infty} e^{-x\cosh\beta} e^{i\tau\beta} d\beta \quad (x > 0). \tag{1.6}$$

If we substitute the formula (1.6) into the representation (1.4), then applying the Fubini theorem and evaluating the inside integral by the formula [9, (2.12.8.3)], we obtain

$$\frac{\pi}{\cosh(\pi\tau/2)} P_{-1/2+i\tau/2}(2x^2+1) = \int_{i\delta-\infty}^{i\delta+\infty} \frac{e^{i\tau\beta}}{\sqrt{x^2+\cosh^2\beta}}\, d\beta \quad (\tau, x > 0), \tag{1.7}$$

where we choose the main value of the square root in the integrand. Hence it is not difficult to see the following uniform estimate for the Legendre function:

$$\begin{aligned} &\frac{1}{\cosh(\pi\tau/2)} \left|P_{-1/2+i\tau/2}(2x^2+1)\right| \\ &\leqq e^{-\delta\tau}\left[C_1 P_{-1/2}(\cosh a) + C_2\frac{\sqrt{\cosh a}}{\sinh a} + C_3\frac{1}{\sqrt{\cosh a}}\right], \end{aligned} \tag{1.8}$$

where

$$\tau > 0, x > 0, \delta \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right), \cosh a = \frac{2(x^2+\sin^2\delta)+\cos 2\delta}{-\cos 2\delta}.$$

Here and in what follows C with suffix numbers denote positive constants. To show the estimate (1.8) we have from the representation (1.7) the relations

$$\frac{\pi}{\cosh(\pi\tau/2)} \left|P_{-1/2+i\tau/2}(2x^2+1)\right| \leqq e^{-\delta\tau}\int_{-\infty}^{\infty} \frac{1}{\sqrt{\left|x^2+\cosh^2(\beta+i\delta)\right|}}\, d\beta \tag{1.9}$$

$$\leqq 2e^{-\delta\tau}\int_0^\infty \frac{1}{\sqrt{\left|x^2+\sinh^2\delta+\cos 2\delta\cosh^2\beta\right|}}\,d\beta$$

$$\leqq \frac{\sqrt{2}e^{-\delta\tau}}{|\cos 2\delta|}\left(\int_0^a \frac{1}{\sqrt{\cosh a-\cosh\beta}}\,d\beta+\int_a^\infty \frac{1}{\sqrt{\cosh\beta-\cosh a}}\,d\beta\right)$$

$$= I(\tau, x),$$

where $\pi/4 < \delta < \pi/2$ and we put

$$\cosh a = \frac{2(x^2+\sin^2\delta)+\cos 2\delta}{-\cos 2\delta}.$$

Using the value of the integral [8, (2.4.6.1)] and choosing a constant $B > 1$, we have

$$\begin{aligned}(1.10)\qquad I(\tau,x) &= \frac{\pi e^{-\delta\tau}}{|\cos 2\delta|}P_{-1/2}(\cosh a)\\ &+\frac{\sqrt{2}e^{-\delta\tau}}{|\cos 2\delta|}\left(\int_1^B \frac{\sqrt{\cosh a}}{\sqrt{u-1}\sqrt{u^2\cosh^2 a-1}}\,du\right.\\ &\left.+\frac{1}{\sqrt{\cosh a}}\int_B^\infty \frac{1}{\sqrt{u-1}\sqrt{u^2-1/\cosh^2 a}}\,du\right)\\ &\leqq e^{-\delta\tau}\left(\frac{\pi}{|\cos 2\delta|}P_{-1/2}(\cosh a)+C_4\frac{\sqrt{\cosh a}}{\sinh a}+C_5\frac{1}{\sqrt{\cosh a}}\right),\end{aligned}$$

and this lead us to the estimate (1.8). Then we conclude from the definition of the Legendre function and properties of the Gauss function (1.3) [1] that

$$(1.11)\qquad P_{-1/2}(\cosh a) = O(1)\quad (a\to 0+),$$

$$(1.12)\qquad P_{-1/2}(\cosh a) = O\left(\frac{1}{\sqrt{\cosh a}}\right) = O\left(P_{-1/2}(2x^2+1)\right) = O\left(\frac{1}{x}\right)\quad (x\to+\infty).$$

So the right side of (1.8) is $e^{-\delta\tau}O(P_{-1/2}(2x^2+1))$, where $\pi/4<\delta<\pi/2$ and $x>0$.

2. The Mehler-Fock Transform in L_p

Let us consider the Mehler-Fock transform (1.1) when the density $f(x)$ belongs to the space $L_p(\mathbf{R}_+)$ $(1\leqq p<\infty)$. As is evident from the Hölder inequality and from the asymptotic behavior of the Legendre function (1.11) and (1.12), the integral (1.1) converges absolutely for any $p\geq 1$. Let us consider the space of functions $g(\tau)$ represented by the Mehler-Fock transform (1.1) functions $f(y)$ belonging to $L_p(\mathbf{R}_+)$:

$$(2.1)\qquad MF(L_p) = \{MF[f](\tau) : f\in L_p(\mathbf{R}_+)\}\quad (p\geq 1).$$

We now show that the operator $MF[f]$ is a bounded mapping from $L_p(\mathbf{R}_+)$ $(1 \leq p < \infty)$ into $L_q(e^{-\alpha\tau}; \mathbf{R}_+)$ $(1 \leq q \leq \infty, \pi/4 < \alpha < \pi/2)$, where p and q have no dependence. Making use of the estimate (1.8) and the generalized Minkowski's inequality

$$\left(\int_0^\infty d\tau \left|\int_0^\infty F(\tau, y) dy\right|^p\right)^{1/p} \leqq \int_0^\infty dy \left(\int_0^\infty |F(\tau, y)|^p \, d\tau\right)^{1/p}, \tag{2.2}$$

we have

$$\begin{aligned} \|MF[f]\|_{L_q(e^{-\alpha\tau};\mathbf{R}_+)} &\leqq \frac{\pi}{2}\int_0^\infty |f(y)| \left(\int_0^\infty e^{-\alpha\tau} \left|P_{-1/2+i\tau/2}(2y^2+1)\right|^q d\tau\right)^{1/q} dy \\ &\leqq C_6 \int_0^\infty |f(y)| \, P_{-1/2}(2y^2+1) dy \left(\int_0^\infty e^{(\pi/2-\alpha/q-\delta)q\tau} d\tau\right)^{1/q} \\ &\leqq C_7 \left(\int_0^1 |f(y)| dy + \int_1^\infty |f(y)| \frac{dy}{y}\right) \\ &\leqq C_8 \|f\|_{L_p}, \end{aligned} \tag{2.3}$$

where $1 \leq p < \infty, \pi/2 - \alpha/q < \delta < \pi/2$. Here, in the last inequality we applied additionally the Hölder inequality.

In order to describe the space $MF(L_p)$, let us consider the operator

$$(I_\varepsilon g)(x) = \frac{x^{1-\varepsilon}}{\pi} \int_0^\infty \frac{\tau \sinh((\pi - \varepsilon)\tau)}{\cosh^2(\pi\tau/2)} P_{-1/2+i\tau/2}(2x^2+1) g(\tau) d\tau \quad (x > 0), \tag{2.4}$$

where $\varepsilon \in (0, 1)$.

Theorem 1. *For the Mehler-Fock transform $g(\tau) = MF[f](\tau)$ of $f(y) \in L_p(\mathbf{R}_+)$ $(1 \leq p < \infty)$, the operator (2.4) has the form*

$$(I_\varepsilon g)(x) = \int_0^\infty I(x, y, \varepsilon) f(y) dy \quad (x > 0), \tag{2.5}$$

where

$$I(x, y, \varepsilon) = \frac{2x^{1-\varepsilon} \sin \varepsilon}{\pi} \int_0^\infty \frac{u(x^2 + y^2u^2 + 1 + u^2 - 2u\cos\varepsilon)}{\left[(x^2 + y^2u^2 + 1 + u^2 - 2u\cos\varepsilon)^2 - 4x^2y^2u^2\right]^{3/2}} du. \tag{2.6}$$

Proof. Substituting the value of $g(\tau)$ as the Mehler-Fock transform (1.1), we obtain the iterated integral

$$\begin{aligned} (I_\varepsilon g)(x) = \frac{x^{1-\varepsilon}}{2} \int_0^\infty \frac{\tau \sinh((\pi - \varepsilon)\tau)}{\cosh^2(\pi\tau/2)} P_{-1/2+i\tau/2}(2x^2+1) \\ \times \int_0^\infty P_{-1/2+i\tau/2}(2y^2+1) f(y) dy \, d\tau, \end{aligned} \tag{2.7}$$

which is absolutely convergent for any $f(x) \in L_p(\mathbf{R}_+)$ $(1 \leqq p < \infty)$ by using the estimate (1.8). Now we need to treat the inner integral by index of the Legendre functions

$$(2.8)\qquad I(x,y,\varepsilon) \equiv \frac{x^{1-\varepsilon}}{2}\int_0^\infty \frac{\tau\sinh((\pi-\varepsilon)\tau)}{\cosh^2(\pi\tau/2)} P_{-1/2+i\tau/2}(2x^2+1)P_{-1/2+i\tau/2}(2y^2+1)d\tau$$

and we will prove that it coincides with (2.6). Invoking to the representation (1.4), we have

$$(2.9)\ I(x,y,\varepsilon) = \frac{2x^{1-\varepsilon}}{\pi^2}\int_0^\infty \tau\sinh((\pi-\varepsilon)\tau)d\tau \int_0^\infty J_0(xv)K_{i\tau}(v)dv \int_0^\infty J_0(yu)K_{i\tau}(u)du$$

Thus the main problem comes to change the order of integration in (2.7) and to apply the Fubini theorem. From the estimate for the Macdonald function [13]

$$(2.10)\qquad |K_{i\tau}(x)| \leqq C_9\frac{\tau+1}{\tau}e^{-\delta\tau - x\cos\delta} \quad (\tau, x > 0),$$

where $0 < \delta < \pi/2$, and the uniform bondedness of the Bessel function $J_0(xy)$ for positive variables x, y, we have the estimate

$$(2.11)\qquad |I(x,y,\varepsilon)| \leqq C_{10}x^{1-\varepsilon}\left[\int_0^1 (\tau+1)^2 e^{-2\delta\tau}d\tau + \int_1^\infty \frac{(\tau+1)^2}{\tau}e^{-(2\delta+\varepsilon-\pi)\tau}d\tau\right]$$
$$\times \int_0^\infty e^{-v\cos\delta}dv \int_0^\infty e^{-u\cos\delta}du < +\infty,$$

where we choose the parameter δ in (2.10) as $(\pi-\varepsilon)/2 < \delta < \pi/2$. Now we first treat the inner integral by τ

$$(2.12)\qquad I_1(u,v,\varepsilon) \equiv \frac{2}{\pi^2}\int_0^\infty \tau\sinh((\pi-\varepsilon)\tau)K_{i\tau}(v)K_{i\tau}(u)d\tau,$$

using the formula [9, (2.16.52.6)], we have

$$(2.13)\qquad \int_0^\infty \cosh((\pi-\varepsilon)\tau)K_{i\tau}(v)K_{i\tau}(u)d\tau = \frac{\pi}{2}K_0\left(\sqrt{u^2+v^2-2uv\cos\varepsilon}\right).$$

Then by differentiating the integral (2.13) by parameter ε we find

$$(2.14)\qquad I_1(u,v,\varepsilon) = -\frac{1}{\pi}\frac{\partial}{\partial\varepsilon}K_0\left(\sqrt{u^2+v^2-2uv\cos\varepsilon}\right).$$

We substitute (2.14) into (2.9) and we obtain the double integral

$$(2.15)\qquad I(x,y,\varepsilon) = -\frac{x^{1-\varepsilon}}{\pi}\frac{\partial}{\partial\varepsilon}\int_0^\infty\int_0^\infty J_0(xv)J_0(yu)K_0\left(\sqrt{u^2+v^2-2uv\cos\varepsilon}\right)du\,dv,$$

where of course we need to justify the validity of differentiability by ε under the sign of this double integral. Differentiating (2.14), we get

$$(2.16)\qquad I_1(u,v,\varepsilon) = \frac{\sin\varepsilon}{\pi}\frac{uvK_1\left(\sqrt{u^2+v^2-2uv\cos\varepsilon}\right)}{\sqrt{u^2+v^2-2uv\cos\varepsilon}},$$

and by the polar coordinates $v = r\cos\varphi, u = r\sin\varphi$ $(r > 0, 0 \leqq \varphi \leqq \pi/2)$ the integral (2.15) can be written in the form

$$I(x,y,\varepsilon) = \frac{x^{1-\varepsilon}\sin\varepsilon}{2\pi}\int_0^{\pi/2}\frac{\sin 2\varphi}{\sqrt{1-\sin 2\varphi\cos\varepsilon}}d\varphi$$
$$\times\int_0^\infty r^2 J_0(xr\cos\varphi)J_0(yr\sin\varphi)K_1\left(r\sqrt{1-\sin 2\varphi\cos\varepsilon}\right)dr.$$

Hence its uniform convergence for $\varepsilon \geqq \varepsilon_0 > 0$ follows from the estimate

$$(2.17)\quad |I(x,y,\varepsilon)| \leqq C_{11}y^{-1/2}x^{1/2-\varepsilon}\sin\varepsilon\int_0^{\pi/2}\frac{d\varphi}{(1-\sin 2\varphi\cos\varepsilon)^{3/2}}\int_0^\infty tK_1(t)dt$$
$$\leqq C_{12}\frac{y^{-1/2}x^{1/2-\varepsilon}}{\sin\varepsilon}\int_{-\infty}^\infty\frac{\sqrt{1+(1+|t|)^2}}{(t^2+1)^{3/2}}dt = O\left(\frac{1}{\varepsilon}\right)\quad(0<\varepsilon<1),$$

because

$$(2.18)\qquad |J_0(x)| \leqq C_{13}x^{-1/2}\quad(x>0)$$

and the Macdonald function $K_1(t)$ has the asymptotic at zero and infinity as $K_1(t) = O(t^{-1})$ $(t\to 0), K_1(t) = O(e^{-t}/\sqrt{t})$ $(t\to\infty)$ [2]. Thus we obtain the following representation

$$(2.19)\qquad I(x,y,\varepsilon) = -\frac{x^{1-\varepsilon}}{\pi}\frac{\partial}{\partial\varepsilon}\int_0^{\pi/2}d\varphi\int_0^\infty rJ_0(xr\cos\varphi)J_0(yr\sin\varphi)$$
$$\times K_0\left(r\sqrt{1-\sin 2\varphi\cos\varepsilon}\right)dr.$$

But the integral in r is evaluated by the formula [9, (2.16.37.2)] and the representation (2.19) takes the form

$$(2.20)\qquad I(x,y,\varepsilon) = -\frac{x^{1-\varepsilon}}{\pi}\frac{\partial}{\partial\varepsilon}\int_0^{\pi/2}\left[(x\cos\varphi-y\sin\varphi)^2+1-\sin 2\varphi\cos\varepsilon\right]^{-1/2}$$
$$\times\left[(x\cos\varphi+y\sin\varphi)^2+1-\sin 2\varphi\cos\varepsilon\right]^{-1/2}d\varphi.$$

Let us change the variable $\tan\varphi = u$ in the integral (2.20), and simple transformations carry out the differentiation by ε. Hence we obtain the formula (2.6) and Theorem 1 is proved.

The inversion formula for the Mehler-Fock transform will be established by the following:

Theorem 2. *Let* $g(\tau) = MF[f](\tau), f(y) \in L_p(\mathbf{R}_+)$ $(1<p<\infty)$. *Then*

$$(2.21)\qquad f(x) = (Ig)(x),$$

where

$$(2.22)\qquad (Ig)(x) = \underset{\varepsilon\to 0+}{\text{l.i.m.}}\,(I_\varepsilon g)(x) \quad (x>0)$$

and $(I_\varepsilon g)(x)$ *is defined in* (2.4). *Here the limit in* (2.22) *is meant in the norm of* $L_p(\mathbf{R}_+)$, *and it exists almost everywhere on* $\mathbf{R}_+$.

Proof. The inequality (1.8) implies the uniform estimate for the function $I(x,y,\varepsilon)$ in (2.8) by $x,y>0$ and $\varepsilon\in(0,1)$

$$(2.23)\qquad |I(x,y,\varepsilon)| \leqq C_{14}x^{1-\varepsilon}P_{-1/2}(2x^2+1)P_{-1/2}(2y^2+1).$$

After the replacement $y = x(1+t\varepsilon)$ in the integral (2.5) we obtain

$$(2.24)\qquad (I_\varepsilon g)(x) = x\varepsilon\int_{-1/\varepsilon}^{\infty} I(x,x(1+t\varepsilon),\varepsilon)f(x(1+t\varepsilon))dt.$$

Now we have to estimate more carefully the right part of the inequality (2.23). From the Mellin-Barnes representation [6, 11.5(1)]

$$(2.25)\qquad P_{-1/2}(2x^2+1) = \frac{1}{2\pi^2 i}\int_{\mu-i\infty}^{\mu+i\infty}\frac{\Gamma(s)\Gamma(1/2-s)\Gamma(1/2-s)}{\Gamma(1-s)}x^{-2s}ds \quad \left(0<\mu<\frac{1}{2}\right)$$

we get $P_{-1/2}(2x^2+1)\leqq C_{15}x^{-2\mu}$ uniformly for all $x>0$, because the integral (2.25) is absolutely convergent due to the asymptotic of the gamma-function as the absolute value of the argument diverges. With the aid of the generalized Minkowski inequality (2.2) we estimate the $L_p(\mathbf{R}_+)$-norm of the operator (2.24)

$$(2.26)\qquad \begin{aligned} \|(I_\varepsilon g)\|_{L_p} &\leqq \int_{-1/\varepsilon}^{\infty}\|f(x(1+t\varepsilon))x\varepsilon I(x,x(1+t\varepsilon),\varepsilon)\|_{L_p}dt \\ &= \int_{-1/\varepsilon}^{0}\|f(x(1+t\varepsilon))x\varepsilon I(x,x(1+t\varepsilon),\varepsilon)\|_{L_p}dt \\ &\quad + \int_{0}^{\infty}\|f(x(1+t\varepsilon))x\varepsilon I(x,x(1+t\varepsilon),\varepsilon)\|_{L_p}dt \\ &\equiv I_1+I_2. \end{aligned}$$

To estimate I_1 and I_2 we use (2.23) and the representation (2.25). Indeed,

$$(2.27)\qquad \begin{aligned} |xI(x,x(1+t\varepsilon),\varepsilon)| &\leqq C_{16}x^{2-\varepsilon}P_{-1/2}\left(2x^2+1\right)P_{-1/2}\left(2x^2(1+t\varepsilon)^2+1\right) \\ &\leqq C_{17}x^{2(1-\mu_1-\mu_2)-\varepsilon}(1+t\varepsilon)^{-2\mu_2}, \end{aligned}$$

where there exist such parameters μ_1,μ_2 for the representation (2.25) that $0<\mu_1,\mu_2<1/2$ and $1-\varepsilon/2<\mu_1+\mu_2$. Hence I_1 can be estimated as

$$(2.28)\qquad I_1 \leqq \int_{-1/\varepsilon}^{0}\|f(x(1+t\varepsilon))x\varepsilon I(x,x(1+t\varepsilon),\varepsilon)\|_{L_p(0,1)}dt$$

$$+\int_{-1/\varepsilon}^{0}\|f(x(1+t\varepsilon))x\varepsilon I(x,x(1+t\varepsilon),\varepsilon)\|_{L_p(1,\infty)}dt$$

$$\leqq C_{18}\left\|f(x)x^{1-2\mu_1-\varepsilon}\right\|_{L_p(0,1)}\int_{-1/\varepsilon}^{0}(1+t\varepsilon)^{2\mu_1+\varepsilon-1/p-1}dt$$

$$+C_{19}\|f\|_{L_p(1,\infty)}\int_{-1/\varepsilon}^{0}(1+t\varepsilon)^{-2\mu_2-1/p}dt$$

$$\leqq C_{20}\|f\|_{L_p(\mathbf{R}_+)},$$

if we choose in (2.25) $\mu_1,\mu_2\in(0,1/2)$ and $\mu_1>1/(2p)-\varepsilon/2, \mu_2<(1-1/p)/2, \mu_1+\mu_2>1-\varepsilon/2$. Similarly, we have

$$I_2 \leqq C_{21}\left\|f(x)x^{2-2\mu_1-\varepsilon}\right\|_{L_p(0,1)}\int_1^{\infty}u^{2\mu_1+\varepsilon-1/p-2}du \tag{2.29}$$

$$+C_{22}\left\|f(x)x^{2(1-\mu_1-\mu_2)-\varepsilon}\right\|_{L_p(1,\infty)}\int_1^{\infty}u^{2\mu_1+\varepsilon-1/p-2}du$$

$$\leqq C_{23}\|f\|_{L_p(\mathbf{R}_+)}.$$

Finally from the estimates (2.28) and (2.29) we get the inequality of the operator (2.8):

$$\|(I_\varepsilon g)\|_{L_p} \leqq C_{24}\|f\|_{L_p}. \tag{2.30}$$

Let us now proceed to estimate the norm of difference $\|(I_\varepsilon g)-f\|_{L_p}$ and to show that it is tends to zero when $\varepsilon\to 0+$.

We first prove the relation

$$\lim_{\varepsilon\to 0+}\varepsilon I(x,x(1+t\varepsilon),\varepsilon)=\frac{1}{\pi}\frac{\sqrt{x^2+1}}{x^2t^2+x^2+1}\quad (x>0, t\in\mathbf{R}) \tag{2.31}$$

by virtue of the representation (2.6). Indeed, substituting the replacement $y=x(1+t\varepsilon)$ in (2.6) and changing the variable $u=1+v\varepsilon$ we obtain

$$I(x,x(1+t\varepsilon),\varepsilon)=\frac{2x^{1-\varepsilon}\varepsilon^2\sin\varepsilon}{\pi} \tag{2.32}$$

$$\times\int_{-1/\varepsilon}^{\infty}(1+v\varepsilon)\left[x^2\left(1+(1+t\varepsilon)^2(1+v\varepsilon)^2\right)+\varepsilon^2v^2+4\sin^2\left(\frac{\varepsilon}{2}\right)\right]$$

$$\times\varepsilon^{-3}\left[(x^2+1)v^2+2x^2tv+x^2t^2+1+o(\varepsilon^2)\right]^{-3/2}$$

$$\times\left[x^2(1+(1+\varepsilon t)(1+\varepsilon v))^2+\varepsilon^2v^2+4\sin^2\left(\frac{\varepsilon}{2}\right)\right]^{-3/2}dv.$$

In view of (2.17) $\varepsilon I(x,x(1+t\varepsilon),\varepsilon)$ converges uniformly in ε, and we have

$$\lim_{\varepsilon\to 0+}\varepsilon I(x,x(1+t\varepsilon),\varepsilon)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{((x^2+1)v^2+2x^2tv+x^2t^2+1)^{3/2}}dv \tag{2.33}$$

$$=\frac{1}{\pi}\frac{\sqrt{x^2+1}}{x^2t^2+x^2+1},$$

where the formula [9, (2.2.9.22)] is applied, and we obtain (2.31). Observing that for all $x > 0$ the equality

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{x\sqrt{x^2+1}}{x^2t^2+x^2+1}\,dt = 1$$

is true and the application of the generalized Minkowski inequality (2.2) yields the desired estimate of the difference $\|(I_\varepsilon g) - f\|_{L_p}$. In fact, using the representation (2.24), we obtain

$$(2.34)\qquad \|(I_\varepsilon g) - f\|_{L_p} = \left\| \int_{-\infty}^{\infty} H(t+1/\varepsilon)x\varepsilon I(x, x(1+t\varepsilon), \varepsilon) f(x(1+t\varepsilon))dt \right.$$

$$\left. -\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{x\sqrt{x^2+1}f(x)}{x^2t^2+x^2+1}\,dt \right\|_{L_p}$$

$$\leq \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{dt}{t^2+1} \left\| H\left(\frac{\sqrt{x^2+1}}{x}t + \frac{1}{\varepsilon}\right) \pi\sqrt{x^2+1}(t^2+1)\varepsilon \right.$$

$$\left. \times I\left(x, x\left(1+\frac{\sqrt{x^2+1}}{x}t\varepsilon\right), \varepsilon\right) f\left(x\left(1+\frac{\sqrt{x^2+1}}{x}t\varepsilon\right)\right) - f(x) \right\|_{L_p},$$

where $H(x)$ is the Heaviside function: $H(x) = 1$ $(x \geqq 0), H(x) = 0$ $(x < 0)$. Thus the right side tends to zero when $\varepsilon \to 0+$ due to the Lebesgue theorem and the relation (2.31). So we established (2.22) and inversion formula (2.21) for L_p-functions. The existence of the limit almost everywhere on $\mathbf{R}_+$ follows from the radial property of the Poisson kernel $P(t) = 1/(\pi(t^2+1)) = P(|t|) \in L_1(\mathbf{R}_+)$ and Theorem 2 is proved.

From the estimate (2.30) the inequality

$$(2.35)\qquad \|(I_\varepsilon g)\|_{L_p} \leqq C_{25}\|(Ig)\|_{L_p}$$

holds in view of (2.21). Theorem 2 shows that $MF[f](\tau) \equiv 0$ for $f(y) \in L_p(\mathbf{R}_+)$ $(1 < p < \infty)$, iff $f(y) \equiv 0$. So, in the space $MF(L_p)$ we can introduce a norm by the equality

$$(2.36)\qquad \|MF[f]\|_{MF(L_p)} = \|f\|_{L_p}\,.$$

It is easily find that the space $MF(L_p)$ is a Banach space with the norm (2.36) and is isometric to L_p.

3. The Description of the Space $MF(L_p)$

The main result of this section is to describe the space $MF(L_p)$ defined in (2.1) in term of the operator I_ε defined in (2.4).

Theorem 3. *In order to* $g(\tau) \in MF(L_p)$ $(1 < p < \infty)$, *it is necessary and sufficient that the following conditions hold:*

$$\text{(3.1)} \qquad \underset{\varepsilon \to 0+}{\text{l.i.m.}} (I_\varepsilon g) \in L_p(\mathbf{R}_+),$$

and $g(\tau) \in L_r(e^{-\alpha\tau}; \mathbf{R}_+)$ $(\pi/4 < \alpha < \pi/2, 1 \leqq r \leqq \infty)$ *at the necessity part and* $g(\tau) \in L_r(\mathbf{R}_+)$ $(1 \leqq r \leqq \infty)$ *at the sufficiency part.*

Proof. The neccessity of the conditions follows from Theorem 2 and the estimate (2.3). Let us prove the sufficiency. Let $g(\tau) \in L_r(\mathbf{R}_+)$ and the condition (3.1) holds. We show that in this case there is a function $f \in L_p$ such that the equality

$$\text{(3.2)} \qquad g = MF[f](\tau)$$

takes place. From the inequality (2.35), we conclude that the operator $(I_\varepsilon g)$ belongs to $\in L_p(\mathbf{R}_+)$ for each $\varepsilon \in (0,1)$ and we can evaluate the composition

$$\text{(3.3)} \qquad MF[I_\varepsilon g](\tau) = \frac{\pi}{2} \int_0^\infty P_{-1/2+i\tau/2}(2y^2+1)\,(I_\varepsilon g)\,(y)dy.$$

At least for a smooth function with a compact support on $\mathbf{R}_+$ by substituting (2.4) in equality (3.3) and after changing the order of integrations by the Fubini theorem we have the relation

$$\text{(3.4)} \qquad MF[I_\varepsilon g](\tau) \equiv g_\varepsilon(\tau) = \int_0^\infty M(\nu,\tau,\varepsilon) g(\nu) d\nu,$$

where

$$\text{(3.5)} \quad M(\nu,\tau,\varepsilon) = \frac{\nu \sinh((\pi-\varepsilon)\nu)}{\cosh^2(\pi\nu/2)} \int_0^\infty y^{1-\varepsilon} P_{-1/2+i\tau/2}(2y^2+1) P_{-1/2+i\nu/2}(2y^2+1) dy.$$

Let us treat the integral (3.5). We first observe that the uniform inequality

$$\text{(3.6)} \qquad |M(\nu,\tau,\varepsilon)| \leqq C_{26} \nu e^{(\pi/2-\delta_1)\tau + (\pi/2-\varepsilon-\delta_2)\nu}$$

$$\left(0 < \varepsilon_0 < \varepsilon < 1, \delta_1 \in \left(\frac{\pi}{2} - \alpha, \frac{\pi}{2}\right), \delta_2 \in \left(\frac{\pi}{2} - \varepsilon, \frac{\pi}{2}\right)\right)$$

is true from the estimate (1.8). Hence we see that from the generalized Minkowski inequality (2.2) the operator at the right side of (3.4) is bounded on the space $L_r(e^{-\alpha\tau}; \mathbf{R}_+)$. Now let us represent the kernel $M(\nu,\tau,\varepsilon)$. For this we substitute integral (1.4) into (3.5) and we get the equality

$$\text{(3.7)} \quad M(\nu,\tau,\varepsilon) = \frac{2\nu \sinh((\pi-\varepsilon)\nu)}{\pi^2} \int_0^\infty y^{1-\varepsilon} \int_0^\infty J_0(yu) K_{i\tau}(u) du \int_0^\infty J_0(yv) K_{i\nu}(v) dv\, dy.$$

Changing the order of integration in (3.7), we see that the inside integrals by u and v are absolutely convergent and uniformly in y in $[0,\lambda]$ for some positive λ dueing to the estimate (2.10) and the inequality (2.18). Thus the integral (3.7) can be written in the form

$$\text{(3.8)} \qquad M(\nu,\tau,\varepsilon) = \frac{2\nu \sinh((\pi-\varepsilon)\nu)}{\pi^2}$$

$$\times \lim_{\lambda \to +\infty} \int_0^\infty \int_0^\infty K_{i\tau}(u) K_{i\nu}(v) du\, dv \int_0^\lambda y^{1-\varepsilon} J_0(yu) J_0(yv) dy.$$

The polar coordinates give

$$(3.9)\qquad M(\nu,\tau,\varepsilon)=\frac{2\nu\sinh((\pi-\varepsilon)\nu)}{\pi^2}$$

$$\times\lim_{\lambda\to+\infty}\int_0^{\pi/2}d\varphi\int_0^{\infty}K_{i\tau}(r\cos\varphi)K_{i\nu}(r\sin\varphi)r^{\varepsilon-1}dr$$

$$\times\int_0^{r\lambda}y^{1-\varepsilon}J_0(y\cos\varphi)J_0(y\sin\varphi)dy.$$

Let us treat the last integral by y. We have

$$(3.10)\quad \int_0^{r\lambda}y^{1-\varepsilon}J_0(y\cos\varphi)J_0(y\sin\varphi)dy$$

$$=\int_0^{1}y^{1-\varepsilon}J_0(y\cos\varphi)J_0(y\sin\varphi)dy+\int_1^{r\lambda}y^{1-\varepsilon}J_0(y\cos\varphi)J_0(y\sin\varphi)dy.$$

By the mean value theorem the second integral in (3.10) is equal to

$$(3.11)\qquad \int_1^{r\lambda_1}yJ_0(y\cos\varphi)J_0(y\sin\varphi)dy\quad(\lambda_1<\lambda).$$

Making use of the formula [9, (1.8.3.10)] we obtain

$$(3.12)\quad \int_1^{r\lambda_1}yJ_0(y\cos\varphi)J_0(y\sin\varphi)dy$$

$$=\frac{r\lambda_1\left[\cos\varphi J_1(r\lambda_1\cos\varphi)J_0(r\lambda_1\sin\varphi)-\sin\varphi J_1(r\lambda_1\sin\varphi)J_0(r\lambda_1\cos\varphi)\right]}{\cos^2\varphi-\sin^2\varphi}$$

$$-\frac{\cos\varphi J_1(\cos\varphi)J_0(\sin\varphi)-\sin\varphi J_1(\sin\varphi)J_0(\cos\varphi)}{\cos^2\varphi-\sin^2\varphi}.$$

To passing to the limit $\lambda_1\to\infty$ under the integral sign of (3.9) it is sufficient to consider the contribution of the first term of (3.12). That is we estimate the integral

$$(3.13)\quad \int_0^{\pi/2}d\varphi\int_0^{\infty}K_{i\tau}(r\cos\varphi)K_{i\nu}(r\sin\varphi)r^{\varepsilon-1}$$

$$\times\frac{r\lambda_1\left[\cos\varphi J_1(r\lambda_1\cos\varphi)J_0(r\lambda_1\sin\varphi)-\sin\varphi J_1(r\lambda_1\sin\varphi)J_0(r\lambda_1\cos\varphi)\right]}{\cos^2\varphi-\sin^2\varphi}dr.$$

Dividing the outside integral by φ into three parts by taking a fixed number $\xi\in(0,\pi/4)$, we obtain

$$(3.14)\quad \left[\int_0^{\pi/4-\xi}+\int_{\pi/4-\xi}^{\pi/4+\xi}+\int_{\pi/4+\xi}^{\pi/2}\right]d\varphi\int_0^{\infty}K_{i\tau}(r\cos\varphi)K_{i\nu}(r\sin\varphi)r^{\varepsilon-1}$$

$$\times\frac{r\lambda_1\left[\cos\varphi J_1(r\lambda_1\cos\varphi)J_0(r\lambda_1\sin\varphi)-\sin\varphi J_1(r\lambda_1\sin\varphi)J_0(r\lambda_1\cos\varphi)\right]}{\cos^2\varphi-\sin^2\varphi}dr$$

$$\equiv I_1+I_2+I_3.$$

Let us estimate the integral I_1. For this we need to use the formula [9, (2.16.33.1)]

$$\int_0^\infty K_{i\tau}(r\cos\varphi)K_{i\nu}(r\sin\varphi)r^{\varepsilon-1}dr \tag{3.15}$$

$$= 2^{\varepsilon-3}\frac{\sin^{-\varepsilon}\varphi\cot^{-i\tau}\varphi}{\Gamma(\varepsilon)}\left|\Gamma\left(\frac{\varepsilon+i(\tau+\nu)}{2}\right)\Gamma\left(\frac{\varepsilon+i(\tau-\nu)}{2}\right)\right|^2$$

$$\times{}_2F_1\left(\frac{\varepsilon+i(\tau+\nu)}{2},\frac{\varepsilon+i(\tau-\nu)}{2};\varepsilon;1-\cot^2\varphi\right)\quad(\varepsilon>0).$$

Accounting the simple inequality for the Macdonald function $|K_{i\tau}(x)|\leqq K_0(x)$ being deduced from the representation (1.6), asymptotic of the Gauss function at infinity [6] and the inequality (2.18), we obtain that

$$|I_1|\leqq C_{28}\int_0^{\pi/4-\xi}\frac{d\varphi}{\sqrt{\sin\varphi}}\int_0^\infty K_0(r\cos\varphi)K_0(r\sin\varphi)r^{\varepsilon-1}dr \tag{3.16}$$

$$\leqq C_{29}\int_0^{\pi/4-\xi}{}_2F_1\left(\frac{\varepsilon}{2},\frac{\varepsilon}{2};\varepsilon;1-\cot^2\varphi\right)\frac{1}{\sin^{\varepsilon+1/2}\varphi}\,d\varphi$$

$$\leqq C_{30}\int_0^{\pi/4-\xi}\frac{d\varphi}{\sqrt{\sin\varphi}}<\infty,$$

where C_{28} does not depend on λ_1. Similarly we can estimate the integral I_3. Concerning the integral I_2 its estimation can be accounted by the behavior of the integrand in the neighborhood of the point $\varphi=\pi/4$ as

$$|I_2|\leqq C_{31}\int_{\pi/4-\xi}^{\pi/4-\xi}\frac{d\varphi}{\sin\varphi+\cos\varphi}<\infty.$$

Thus we established the possibility pass to limit under the sign of the iterated integral (3.9). Using formula [9, (2.12.31.1)]

$$\int_0^\infty y^{1-\varepsilon}J_0(yr\cos\varphi)J_0(yr\sin\varphi)dy=\frac{2^{1-\varepsilon}r^{\varepsilon-2}}{(\cos\varphi+\sin\varphi)^{2-\varepsilon}}\frac{\Gamma(1-\varepsilon/2)}{\Gamma(\varepsilon/2)} \tag{3.17}$$

$$\times{}_2F_1\left(1-\frac{\varepsilon}{2},\frac{1}{2};1;\frac{2\sin2\varphi}{(\sin\varphi+\cos\varphi)^2}\right)$$

and the integral (3.15), we have the representation

$$M(\nu,\tau,\varepsilon)=\frac{\nu\sinh((\pi-\varepsilon)\nu)\Gamma(1-\varepsilon/2)}{2\pi^2\Gamma(\varepsilon)\Gamma(\varepsilon/2)}\left|\Gamma\left(\frac{\varepsilon+i(\tau+\nu)}{2}\right)\Gamma\left(\frac{\varepsilon+i(\tau-\nu)}{2}\right)\right|^2 \tag{3.18}$$

$$\times\int_0^{\pi/2}\sin^{-\varepsilon}\varphi\cot^{-i\tau}\varphi\,{}_2F_1\left(\frac{\varepsilon+i(\tau+\nu)}{2},\frac{\varepsilon+i(\tau-\nu)}{2};\varepsilon;1-\cot^2\varphi\right)$$

$$\times{}_2F_1\left(1-\frac{\varepsilon}{2},\frac{1}{2};1;\frac{2\sin2\varphi}{(\sin\varphi+\cos\varphi)^2}\right)\frac{d\varphi}{(\cos\varphi+\sin\varphi)^{2-\varepsilon}}.$$

Let us substitute this value into the composition (3.4) preliminary changing the variable $\tan\varphi = u$ in the integral (3.18). Hence we get that the composition (3.4) is equal to

$$(3.19)\qquad g_\varepsilon(\tau) = \frac{\Gamma(1-\varepsilon/2)}{2\pi^2\Gamma(\varepsilon)\Gamma(\varepsilon/2)}\int_0^\infty \left|\Gamma\left(\frac{\varepsilon+i(\tau+\nu)}{2}\right)\Gamma\left(\frac{\varepsilon+i(\tau-\nu)}{2}\right)\right|^2 \times\nu\sinh((\pi-\varepsilon)\nu)M_1(\nu,\tau,\varepsilon)g(\nu)d\nu,$$

where

$$(3.20)\qquad M_1(\nu,\tau,\varepsilon) = \int_0^\infty {}_2F_1\left(\frac{\varepsilon+i(\tau+\nu)}{2},\frac{\varepsilon+i(\tau-\nu)}{2};\varepsilon;1-\frac{1}{u^2}\right) \times {}_2F_1\left(1-\frac{\varepsilon}{2},\frac{1}{2};1;\frac{4u}{(u+1)^2}\right)\frac{u^{i\tau-\varepsilon}}{(u+1)^{2-\varepsilon}}du.$$

Let us return to the composition (3.4). After the substitution $\nu = \tau + \varepsilon t$ we obtain

$$(3.21)\qquad MF[I_\varepsilon g](\tau) = \varepsilon\int_{-\infty}^\infty H(\tau+\varepsilon t)M(\tau+\varepsilon t,\tau,\varepsilon)g(\tau+\varepsilon t)dt,$$

where $H(x)$ is the Heaviside function. As we know from the estimate (3.6) the kernel of (3.21) is bounded function of three variables and moreover let us prove that the following limit relation is true

$$(3.22)\qquad \lim_{\varepsilon\to 0+}\varepsilon M(\tau+\varepsilon t,\tau,\varepsilon) = \frac{1}{\pi}\frac{1}{t^2+1}\quad (\tau>0, t\in\mathbf{R}).$$

Indeed using the self-transformation formula for the Gauss hypergeometric function [6]

$$(3.23)\qquad {}_2F_1(a,b;c;z) = (1-z)^{c-a-b}{}_2F_1(c-a,c-b,c;z)$$

we have the representation for the integral (3.20) as

$$(3.24)\qquad M_1(\tau+\varepsilon t,\tau,\varepsilon) = \int_0^\infty {}_2F_1\left(-i\tau+\frac{\varepsilon(1-it)}{2},\frac{\varepsilon(1+it)}{2};\varepsilon;1-\frac{1}{u^2}\right) \times {}_2F_1\left(\frac{\varepsilon}{2},\frac{1}{2};1;\frac{4u}{(u+1)^2}\right)\frac{|u-1|^{\varepsilon-1}u^{3i\tau-\varepsilon}}{u+1}du.$$

For some fixed number $\mu > 0$ we divide the integral (3.24) into three parts

$$(3.25)\qquad M_1(\tau+\varepsilon t,\tau,\varepsilon) = \left[\int_0^{1-\mu}+\int_{1-\mu}^{1+\mu}+\int_{1+\mu}^\infty\right] \times {}_2F_1\left(-i\tau+\frac{\varepsilon(1-it)}{2},\frac{\varepsilon(1+it)}{2};\varepsilon;1-\frac{1}{u^2}\right) \times {}_2F_1\left(\frac{\varepsilon}{2},\frac{1}{2};1;\frac{4u}{(u+1)^2}\right)\frac{|u-1|^{\varepsilon-1}u^{3i\tau-\varepsilon}}{u+1}du \equiv I_1+I_2+I_3.$$

It is easily seen that the integrals εI_1 and εI_3, tend to zero as $\varepsilon \to 0+$ because these integrals are absolutely and uniformly convergent in $\varepsilon \in [0,1]$. Concerning the middle integral we use the mean value theorem and get the relation

$$\varepsilon I_2 = {}_2F_1\left(-i\tau + \frac{\varepsilon(1-it)}{2}, \frac{\varepsilon(1+it)}{2}; \varepsilon; 1-\mu_1^{-2}\right) \tag{3.26}$$

$$\times {}_2F_1\left(\varepsilon/2, 1/2; 1; \frac{4\mu_1}{(\mu_1+1)^2}\right) \frac{\mu_1^{3i\tau-\varepsilon}}{\mu_1+1}\, \varepsilon \int_{1-\mu}^{1+\mu} |u-1|^{\varepsilon-1} du,$$

where $\mu_1 \in (1-\mu, 1+\mu)$. So putting now $\mu = \varepsilon$ after the evaluation of the integral, we get $\lim\limits_{\varepsilon\to 0+} \varepsilon I_2 = 1$. Hence the relation (3.22) can be deduced by using the supplement formula for gamma-function $\Gamma(z+1) = z\Gamma(z)$.

As in Theorem 2, we obtain the following estimates for norms of the function $g_\varepsilon(\tau)$ in the space $L_r(e^{-\alpha\tau}; \mathbf{R}_+) \subset L_r(\mathbf{R}_+)$

$$\|g_\varepsilon(\tau) - g(\tau)\|_{L_r(e^{-\alpha\tau};\mathbf{R}_+)} \tag{3.27}$$

$$\leqq \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{1}{t^2+1} \left\|g(\tau+\varepsilon t)\varepsilon\pi(t^2+1)M(\tau+\varepsilon t, \tau, \varepsilon) - g(\tau)\right\|_{L_r(\mathbf{R}_+)} dt \to 0 \quad (\varepsilon \to 0+).$$

But, on the other hand, since the operator $MF[f](\tau)$ is bounded on $L_q(e^{-\alpha\tau}; \mathbf{R}_+)$ $(1 \le q < \infty)$, there exists the limit in L_q-norm

$$\underset{\varepsilon\to 0+}{\text{l.i.m.}}\, MF[I_\varepsilon g](\tau) = MF\left[\underset{\varepsilon\to 0+}{\text{l.i.m.}}\,(I_\varepsilon g)\right](\tau) = MF[f](\tau), \tag{3.28}$$

where $f = Ig \in L_p$. Since the operator $MF[I_\varepsilon g]$ converges in the norm of $L_r(e^{-\alpha\tau}; \mathbf{R}_+)$ too, then the limit function must coincide almost everywhere on $\mathbf{R}_+$. Thus, from the equality (3.28) we obtain (3.2) and Theorem 3 is completely proved.

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