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Certain subclasses of starlike functions of order α

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1. Introduction

Let A denote the class of functions of the form

$$f(z)=z+\sum_{n=2}^{\infty}a_{n}z^{n}$$

which are analytic in the unit disk $U=\{z:|z|<1\}$. Let S denote the subclass of A consisting of functions which are univalent in the unit disk U.

A function $f(z)$ in A is said to be starlike of order α if it satisfies

$$\mathrm{Re}\frac{zf'(z)}{f(z)}>\alpha$$

for some $\alpha(0\leq\alpha<1)$ and for all z in U.

We denote by $S^{*}(\alpha)$ the subclass of S consisting of all starlike functions of order α in the unit disk U.

Let $S_{1}(\alpha,a), S_{2}(\alpha,a)$ and $S_{3}(\alpha,a)$ denote the subclasses of S consisting of functions which satisfy

$$\left|\frac{zf'(z)}{f(z)}-a\right|<a-\alpha \quad (0\leq\alpha<1),$$

for $1\leq a\leq 2, a>2$ and $\frac{1+\alpha}{2}<a<1$, respectively.

It is clear that $S_{i}(\alpha,a)\subset S^{*}(\alpha)$ and $S_{i}(\alpha,a)\subseteq S_{i}(\alpha,b)(a\leq b)$ for $i=1,2,3$.

In [1. Theorem 1], Silverman have showed the sufficient condition for a function in S belongs to $S^{*}(\alpha)$. In this paper, we shall reconsider the sufficient condition by using the subclasses $S_{i}(\alpha,a)(i=1,2,3)$ of $S^{*}(\alpha)$ defined above. Further, we determine the distortion theorems for certain subclasses $\tilde{S}_{i}(\alpha,a)$ of $S_{i}(\alpha,a)(i=1,2,3)$.

2. Coefficient inequality

We shall now prove the following theorems in a same way of Theorem 1 of Silverman [1].

Theorem 1 *Let $f(z)\in S$, $0\leq\alpha<1$ and $1\leq a\leq 2$. If*

$$\sum_{n=2}^{\infty}(n-\alpha)|a_{n}|\leq 1-\alpha,$$

then $f(z)\in S_{1}(\alpha,a)$.

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Proof. We have

$$\left|\frac{zf'(z)}{f(z)} - a\right| = \left|\frac{1 - a + \sum_{n=2}^{\infty}(n-a)a_n z^{n-1}}{1 + \sum_{n=2}^{\infty} a_n z^{n-1}}\right|$$

$$= \left|\frac{a - 1 - \sum_{n=2}^{\infty}(n-a)a_n z^{n-1}}{1 + \sum_{n=2}^{\infty} a_n z^{n-1}}\right|$$

$$\leq \frac{a - 1 + \sum_{n=2}^{\infty}(n-a)|a_n||z|^{n-1}}{1 - \sum_{n=2}^{\infty}|a_n||z|^{n-1}}$$

$$\leq \frac{a - 1 + \sum_{n=2}^{\infty}(n-a)|a_n|}{1 - \sum_{n=2}^{\infty}|a_n|}.$$

This last expression is bounded above by $a - \alpha$ if

$$a - 1 + \sum_{n=2}^{\infty}(n-a)|a_n| \leq (a-\alpha)\left(1 - \sum_{n=2}^{\infty}|a_n|\right). \tag{1}$$

Since the hypothessis of Theorem 1 is equivalent to the coefficient inequality (1), Theorem 1 is proved.

In the proof of Theorem 1 of Silverman [1], only the case of a=1 of the Theorem 1 above was considered.

Theorem 2 *Let $f(z) \in S$, $0 \leq \alpha < 1$ and $a > 2$. If*

$$\sum_{n=2}^{j}(2a - n - \alpha)|a_n| + \sum_{n=j+1}^{\infty}(n-\alpha)|a_n| \leq 1 - \alpha$$

for $j < a \leq j+1$, then $f(z) \in S_2(\alpha, a)$.

Proof. We have

$$\left|\frac{zf'(z)}{f(z)} - a\right| = \left|\frac{a - 1 + \sum_{n=2}^{j}(a-n)a_n z^{n-1} - \sum_{n=j+1}^{\infty}(n-a)a_n z^{n-1}}{1 + \sum_{n=2}^{\infty} a_n z^{n-1}}\right|$$

$$\leq \frac{a-1+\sum_{n=2}^{j}(a-n)|a_n||z|^{n-1}+\sum_{n=j+1}^{\infty}(n-a)|a_n||z|^{n-1}}{1-\sum_{n=2}^{j}|a_n||z|^{n-1}-\sum_{n=j+1}^{\infty}|a_n||z|^{n-1}}$$

$$\leq \frac{a-1+\sum_{n=2}^{j}(a-n)|a_n|+\sum_{n=j+1}^{\infty}(n-a)|a_n|}{1-\sum_{n=2}^{j}|a_n|-\sum_{n=j+1}^{\infty}|a_n|}.$$

This last expression is bounded above by $a-\alpha$ if

$$a-1+\sum_{n=2}^{j}(a-\alpha)|a_n|+\sum_{n=j+1}^{\infty}(n-\alpha)|a_n| \leq (a-\alpha)\left(1-\sum_{n=2}^{j}|a_n|-\sum_{n=j+1}^{\infty}|a_n|\right). \tag{2}$$

(2) is equivalent to the hypothesis of Theorem 2. This completes the proof.

We can prove the following theorem by using the same way of Theorem 1 and Theorem 2.

Theorem 3 *Let $f(z) \in S$, $0 \leq \alpha < 1$ and $\frac{1+\alpha}{2} < a < 1$. If*

$$\sum_{n=2}^{\infty}(n-\alpha)|a_n| \leq 2a-1-\alpha,$$

then $f(z) \in S_3(a,\alpha)$.

3. Distortion theorems

We denote by $\widetilde{S}_1(\alpha,a)$ the subclass of $S(\alpha,a)$ $(0 \leq \alpha < 1, 1 \leq a \leq 2)$ consisting of functions which satisfy

$$\sum_{n=2}^{\infty}(n-\alpha)|a_n| \leq 1-\alpha. \tag{3}$$

Since a function

$$f(z) = z + \frac{1-\alpha}{n-\alpha}z^n \tag{4}$$

in S belongs to $S(\alpha,a)$ and satisfies the coefficient inequality (3), this function (4) belongs to $\widetilde{S}_1(\alpha,a)$. Therefore, the subclass $\widetilde{S}_1(\alpha,a)$ is not empty.

Theorem 4 *If $f(z) \in \widetilde{S}_1(\alpha,a)$ $(0 \leq \alpha < 1, 1 \leq a \leq 2)$, then*

$$|z| - \frac{1-\alpha}{2-\alpha}|z|^2 \leq |f(z)| \leq |z| + \frac{1-\alpha}{2-\alpha}|z|^2.$$

Equality holds for the function

$$f(z) = z + \frac{1-\alpha}{2-\alpha}z^2.$$

Proof. By the assumption, note that

$$(2-\alpha)\sum_{n=2}^{\infty}|a_n| \le \sum_{n=2}^{\infty}(n-\alpha)|a_n| \le 1-\alpha,$$

that is, that

$$\sum_{n=2}^{\infty}|a_n| \le \frac{1-\alpha}{2-\alpha}.$$

Thus, we have

$$\begin{aligned} |f(z)| &\le |z| + \sum_{n=2}^{\infty}|a_n||z|^n \\ &\le |z| + |z|^2\sum_{n=2}^{\infty}|a_n| \\ &\le |z| + |z|^2\frac{1-\alpha}{2-\alpha}, \end{aligned}$$

and

$$\begin{aligned} |f(z)| &\ge |z| - \sum_{n=2}^{\infty}|a_n||z|^n \\ &\ge |z| - |z|^2\sum_{n=2}^{\infty}|a_n| \\ &\ge |z| - |z|^2\frac{1-\alpha}{2-\alpha}. \end{aligned}$$

We denote by $\widetilde{S}_2(\alpha, a)$ the subclass of $S(\alpha, a)$ $(0 \le \alpha < 1, a > 2)$ consisting of functions which satisfy

$$\sum_{n=2}^{j}(2a-n-\alpha)|a_n| + \sum_{n=j+1}^{\infty}(n-\alpha)|a_n| \le 1-\alpha,$$

where $j < a \le j+1$. Just as in the case of $\widetilde{S}_1(\alpha, a)$, by taking a function

$$f(z) = z + \frac{1-\alpha}{2(2a-k-\alpha)}z^k + \frac{1-\alpha}{2(l-\alpha)}z^l,$$

where $2 \le k \le j$ and $j+1 \le l < \infty$, we see that $\widetilde{S}_2(\alpha, a)$ is not empty.

Theorem 5 *If $f(z) \in \widetilde{S}_2(\alpha, a)$ $(0 \le \alpha < 1, a > 2)$, then*

$$|z| - \frac{1-\alpha}{2a-j-\alpha}|z|^2 \le |f(z)| \le |z| + \frac{1-\alpha}{2a-j-\alpha}|z|^2, \tag{5}$$

for $a - \frac{1}{2} \le j < a$

and

$$|z| - \frac{1-\alpha}{j+1-\alpha}|z|^2 \le |f(z)| \le |z| + \frac{1-\alpha}{j+1-\alpha}|z|^2, \tag{6}$$

for $a - 1 \le j < a - \frac{1}{2}$.
These results are sharp.

Proof. Since $2a - n - \alpha$ and $n - \alpha$ are decreasing and increasing for n, respectively, by the assumption we note that

$$(2a - j - \alpha)\sum_{n=2}^{j}|a_n| + (j+1-\alpha)\sum_{n=j+1}^{\infty}|a_n| \le 1-\alpha.$$

Then, we have

$$\sum_{n=2}^{\infty}|a_n| \le \frac{1-\alpha}{2a-j-\alpha} \qquad (a - \frac{1}{2} \le j < a), \tag{7}$$

and

$$\sum_{n=2}^{\infty}|a_n| \le \frac{1-\alpha}{j+1-\alpha} \qquad (a - 1 \le j < a - \frac{1}{2}). \tag{8}$$

By using (7), we have

$$\begin{aligned} |f(z)| &\le |z| + \sum_{n=2}^{\infty}|a_n||z|^n \\ &\le |z| + |z|^2\sum_{n=2}^{\infty}|a_n| \\ &\le |z| + |z|^2\frac{1-\alpha}{2a-j-\alpha}, \end{aligned}$$

and

$$\begin{aligned} |f(z)| &\ge |z| - \sum_{n=2}^{\infty}|a_n||z|^n \\ &\ge |z| - |z|^2\sum_{n=2}^{\infty}|a_n| \\ &\ge |z| - |z|^2\frac{1-\alpha}{2a-j-\alpha}. \end{aligned}$$

Equality of (5) holds for the function

$$f(z) = z + \frac{1-\alpha}{2a-j-\alpha}z^2.$$

Using coefficient inequality (8) we can obtain the latter of Theorem 1 similarly.
Equality of (6) holds for the function

$$f(z) = z + \frac{1-\alpha}{j+1-\alpha} z^2.$$

Let $\widetilde{S}_3(\alpha, a)$ be the subclass of $S(\alpha, a)$ $(0 \leq \alpha < 1, \frac{1+\alpha}{2} < a < 1)$ consisting of functions which satisfy

$$\sum_{n=2}^{\infty} (n-\alpha)|a_n| \leq 2a - 1 - \alpha.$$

Then, the subclass $\widetilde{S}_3(\alpha, a)$ is not empty, because there exists a function

$$f(z) = z + \frac{2a-1-\alpha}{n-\alpha} z^n$$

in $\widetilde{S}_3(\alpha, a)$.

We can prove the following theorem in the same way of Theorem 4 and 5.

Theorem 6 *If* $f(z) \in \widetilde{S}_3(\alpha, a)$ $(0 \leq \alpha < 1, \frac{1+\alpha}{2} < a < 1)$, *then*

$$|z| - \frac{2a-1-\alpha}{2-\alpha}|z|^2 \leq |f(z)| \leq |z| + \frac{2a-1-\alpha}{2-\alpha}|z|^2.$$

Equality holds for the function

$$f(z) = z + \frac{2a-1-\alpha}{2-\alpha} z^2.$$

References

[1] H.Silverman, Univalent functions with negative coefficients, Proc. Amer. Math. Soc., 51 (1975), 109-116.

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