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Remarks on Sakaguchi Functions

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1. Introduction

Let A denote the class of functions f which are analytic in the unit disc $D=\{z: |z|<1\}$, with

(1.1) $$f(0)=0 \text{ and } f'(0)=1.$$

We denote some subclasses of A as follows:

(1.2) $$S^{*}=\left\{f\in A: \operatorname{Re}\frac{zf'(z)}{f(z)}>0,\ z\in D\right\},$$

(1.3) $$S=\left\{f\in A: \operatorname{Re}\frac{zf'(z)}{f(z)-f(-z)}>0,\ z\in D\right\}$$

and

(1.4) $$R(\alpha)=\left\{f\in A: \operatorname{Re}\frac{f(z)}{z}>\alpha,\ z\in D\right\}$$

where $0\leqq\alpha<1$.

S^{*} is the usual class of starlike functions, and S is the class of Sakaguchi functions introduced by Sakaguchi in [2]. For relations between these two classes only the following result is known.

Theorem A (Sakaguchi [2]). $f(z) \in S$ if and only if

$$f(z) - f(-z) \over 2 \in S^*. \tag{1.5}$$

For $R(\alpha)$, Wu posed the following conjecture in [3].

Conjecture

If $f(z) \in S$, then $f(z) \in R\left(\frac{1}{2}\right)$.

And the present author in [4] showed by the counter-example

$$f(z) = z + \frac{3}{5} z^2 + \frac{1}{15} z^3 \tag{1.6}$$

that the conjecture is not true.

In this short paper we give two examples which show that

$$S^* \not\subset S \tag{1.7}$$

and

$$S \not\subset S^*. \tag{1.8}$$

2. Examples

Example 1. An example which shows the relation (1.7).

$$f(z) = z + \frac{4}{5} z^2 + \frac{1}{5} z^3. \tag{2.1}$$

If we let

$$\frac{z f'(z)}{f(z)} = \frac{5 + 8z + 3z^2}{5 + 4z + z^2} = \frac{u + i v}{s + i t}$$

and $z = r e^{i\theta}$, then we have

$$s = 5 + 4r\cos\theta + r^2\cos 2\theta, \qquad t = 4r\sin\theta + r^2\sin 2\theta$$

$$u=5+8r\cos\theta+3r^2\cos 2\theta, \quad v=8r\sin\theta+3r^2\sin 2\theta.$$

and it is evident that $\mathrm{Re}[zf'/f]>0$ if and only if

$$su+tv>0.$$

Since we easily deduce

$$\begin{aligned} su+tv &= 40r^2\cos^2\theta+20r(3+r^2)\cos\theta+25+12r^2+3r^4 \\ &\geqq 25-60r+52r^2-20r^3+3r^4 \quad (\cos\theta=-1) \\ &= (1-r)(25-35r+17r^2-3r^3), \end{aligned}$$

where

$$25-35r+17r^2-3r^3>0 \quad (0\leqq r<1).$$

we deduce $su+tv>0$, which shows that $f\in S^*$.

On the other hand, if we put $z_0=-\frac{3}{5}+\frac{4}{5}i$, then we have

$$\mathrm{Re}\,\frac{z_0 f'(z_0)}{f(z_0)-f(-z_0)}<0,$$

which shows that $f\notin S$.

Example 2. An example for the relation (1.8).

$$f(z)=\frac{1}{2}z+z^2+\log\frac{2+z}{2} \tag{2.2}$$

To show that the above function f belongs to S, we use the following theorem due to Miller and Mocanu.

Theorem B (Miller and Mocanu [1]). If $f(z)\in A$ satisfies

$$\left|\frac{zf''(z)}{f'(z)}\right|<2 \quad (z\in D) \tag{2.3}$$

then $f(z)$ belongs to S^*.

If we let

$$g(z)=\frac{f(z)-f(-z)}{2},$$

then we obtain

$$\left|\frac{zg''(z)}{g'(z)}\right|=\left|\frac{8z^2}{(4-z^2)(8-z^2)}\right|<\frac{8|z|^2}{(4-|z|^2)(8-|z|^2)}<\frac{8}{21}\quad (z\in D).$$

Hence from Theorem B, we deduce $g(z)\in S^*$. Therefore, by using Theorem A, we have $f(z)\in S$.

On the other hand, if we put $z_0=-5/8\in D$, then we have

$$\mathrm{Re}\,\frac{z_0 f'(z_0)}{f(z_0)}=-0.047\cdots\cdots<0,$$

which yields $f\notin S^*$.

References

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[3] Z.Wu, On classes of Sakaguchi functions and Hadamard products, Sci.Sinica 30(1987), 128-135.

[4] R.Yamakawa, Notes for Sakaguchi functions, (1991,preprint).