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ON REDUCIBLE FINITE SUBGROUPS OF MAPPING CLASS GROUPS OF SURFACES

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Introduction

Let Σ_g be the closed connected orientable surface of genus $g \geq 2$. By an automorphism of Σ_g, we mean an element of the mapping class group $\mathcal{M}_g$ which is the group of the isotopy classes of orientation preserving diffeomorphisms. We recall some definitions mainly from [T]. A periodic automorphism is the one which is of finite order in $\mathcal{M}_g$. A non empty 1-submanifold is said to be essential if it is compact, and its no two components are homotopic and no components are null-homotopic. A reducible automorphism is the one which fixes the isotopy class of some essential 1-submanifold of Σ_g.

In §1, we describe the relation between order and reducibility of periodic automorphisms. The result shows that the order of a periodic automorphism determine its reducibility unless g is even and the order is $2g+2$. This exception occurs because there is a periodic diffeomorphism $\Sigma_g \to \Sigma_g$ of order $4g+2$ with a fixed point for any $g \geq 1$. The proof is based on the geometric characterization of irreducible finite subgroup of Σ_g by Gilman, and cyclicity condition for 2-orbifolds by Harvey. Details of this section can be found in [Ka].

In §2, via Nielsen realization theorem [N, Ke], we consider decompositions of any finite subgroup of $\mathcal{M}_g$ along oriented essential 1-submanifolds, and describe the quotient orbifold types appearing in "*irreducible*" decompositions after capping off 2-disks to obtain closed orbifolds.

Notation. We denote by $\Sigma_\gamma(m_1, m_2, \cdots, m_n)$ the 2-dimensional orbifold whose underlying surface is Σ_γ and whose singular locus consists of n cone points with singular indices m_1, m_2, $\cdots$, m_n, respectively. We also write $S^2(m_1, \cdots, m_n)$ when $\gamma = 0$.

1. Reducibility and orders of periodic automorphisms

This section is devoted to prove the following.

Theorem 1.1. *Let $\mathfrak{f} \in \mathcal{M}_g$ be a periodic automorphism of order N. Then, the followings hold:*

(I) *if $\mathfrak{f}$ is irreducible, then $N \geq 2g + 1$,*

(II) *if $\mathfrak{f}$ is reducible, then $N \leq 2g + 2$ and $N \neq 2g + 1$;*
furthermore, if the genus g is odd, then $N \leq 2g$.

All the inequalities are best possible. That is to say, there certainly exists a periodic automorphism of Σ_g having as order the value of the right-hand term of each inequality, with required reducibility. On the other hand, Σ_g has always a periodic and irreducible automorphism of order $2g + 2$.

Proof of inequalities.

Given a periodic automorphism $\mathfrak{f} \in \mathcal{M}_g$ of order N, by Nielsen realization theorem, it can be represented by a periodic diffeomorphism $f\colon \Sigma_g \to \Sigma_g$ of the same order N. We denote by O_f the quotient orbifold of Σ_g by the cyclic action generated by f. Then $\mathfrak{f}$ is irreducible if and only if O_f is of the form $S^2(m_1, m_2, m_3)$ where m_1, m_2, $m_3 \geq 2$ for any (and then necessarily all) Nielsen realization f [Gi].

Then, the inequality of (i) directly follows from the Riemann-Hurwitz formula for the canonical projection $\pi\colon \Sigma_g \to O_f(= S^2(m_1, m_2, m_3))$ since each $m_i \leq N$.

To obtain the rest of the inequalities in (ii), instead of estimating order N while the genus g fixed, we obtain the minimum genus $g_{min}(N)$ of surfaces which admit a periodic and reducible automorphism of a fixed order N. Depending on the form of prime decomposition of N, it is described as follows:

Theorem 1.2. *Let N be an integer ≥ 2 with prime decomposition $p_1^{r_1} \cdots p_k^{r_k}$ where each p_i is prime, each $r_i \geq 1$, and $p_1 < p_2 < \cdots < p_k$. Then, the minimum genus*

$g_{min}(N)$ of surfaces which admit a periodic and reducible automorphism of order N is given by

$$\begin{aligned}
&(i) \quad g_{min}(N) = && \max\left\{2, (p_1-1)\frac{N}{p_1}\right\}, && \text{if } r_1>1 \text{ or } N \text{ is prime,}\\
&(ii) \quad g_{min}(N) = && N-\frac{1}{2}\left(\frac{N}{p_1}+\frac{N}{p_2}+\frac{N}{p_3}-1\right), && \text{if } N=p_1p_2p_3\\
&&&&& \text{and } p_3 \le \frac{p_1p_2-2p_1+1}{p_2-p_1},\\
&(iii) \quad g_{min}(N) = && (p_1-1)\left(\frac{N}{p_1}-1\right), && \text{otherwise.}
\end{aligned}$$

Now, we see that the rest of the inequalities follow from Theorem 1.2. Let N be the order of any periodic and reducible automorphism of Σ_g. Then, by definition, it holds that $g_{min}(N) \le g$. According to the form of the prime decomposition of N, replacing the left-hand side by the term given by Theorem 1.2, we obtain $N \le 2g+2$. Next, we can see that $g_{min}(2g+1) > g$ and therefore N cannot be $2g+1$.

Suppose now g is odd. Then we can also see $g_{min}(2g+2) > g$, which implies that N cannot be $2g+2$, and therefore $N \le 2g$.

A sketchy proof of Theorem 1.2 is given in the end of this section.

Examples.

Now, we describe examples of periodic automorphisms which should assure the best possibility of each inequality. It is known that an orbifold $\Sigma_\gamma(m_1, m_2, \cdots, m_n)$ is an N-cyclic quotient of some compact surface if and only if it satisfies the following conditions [H]:

(i) $lcm(m_1, \cdots, \hat{m}_i, \cdots, m_n) = lcm(m_1, \cdots, m_n)$ where m_i denotes the omission of m_i. $(i = 1, 2, \cdots, n)$;

(ii) $lcm(m_1, \cdots, m_n)$ divides N, and if $\gamma = 0$, $lcm(m_1, \cdots, m_n) = N$;

(iii) $n \neq 1$;

(iv) if $lcm(m_1, \cdots, m_n)$ is even, then the number of m_i's divisible by the maximum power of 2 dividing $lcm(m_1, \cdots, m_n)$ is even.

We call such an orbifold N-cyclic. Note that the genus of N-cyclically covering surface of a given N-cyclic orbifold is determined *uniquely* by the Riemann-Hurwitz formula. Now, it is easy to see that the following three orbifolds give examples of periodic and reducible automorphisms of Σ_g which show that equality holds for each inequality of Theorem 1.1, respectively: $S^2(2g+1, 2g+1, 2g+1)$; $S^2(2, 2, g+1, g+1)$ (g: even); $S^2(2, 2, 2g, 2g)$.

Also, the orbifold $S^2(g+1, 2g+2, 2g+2)$ gives an example of periodic and irreducible automorphism of Σ_g of order $2g+2$. This complete the proof of Theorem 1.1.

Proof of Theorem 1.2.

For an N-cyclic orbifold $\Sigma_\gamma(m_1, \cdots, m_n)$, the genus of the N-cyclic covering surface g is given by

$$(*) \qquad g = 1 + N(\gamma - 1) + \frac{1}{2}N\sum_{i=1}^{n}(1 - \frac{1}{m_i})$$

Therefore, $g_{min}(N)$ is the minimum value of (*) where $\Sigma_\gamma(m_1, \cdots, m_n)$ varies all the orbifolds which are not of the type $S^2(m_1, m_2, m_3)$, satisfying Harvey's cyclicity conditions (i)-(iv).

So far as $\gamma = 0$ and $n = 4$, the minimum of (*) corresponds to the maximum of $1/m_1 + 1/m_2 + 1/m_3 + 1/m_4$ where $lcm(m_2, m_3, m_4) = lcm(m_1, m_3, m_4) = lcm(m_1, m_2, m_4) = lcm(m_1, m_2, m_3) = N$. By dividing into several subcases carefully, the calculation of this maximum is reduced to the calculation of the maximum of $1/x + 1/y + 1/z$ where $lcm(x, y) = lcm(y, z) = lcm(z, x) =$ given positive integer. The latter maximum was given by Harvey [H]. The result of calculation gives the value expected for $g_{min}(N)$.

If $\gamma \neq 0$ or $n \neq 4$, it can be checked that the value of (*) does not exceed the minimum for the case $\gamma = 0$ and $n = 4$ so far as γ and m_i's satisfy (i)-(iv). Therefore, $g_{min}(N)$ is not less than the expected value.

The following three N-cyclic orbifolds realize the minimum genus according to the form of prime decomposition of N: $S^2(p_1, p_1, N, N)$; $S^2(p_1, p_2, p_3)$ $(N = p_1p_2p_3)$; $S^2(p_1, p_1, N/p_1, N/p_1)$ $(r_1 = 1,\ k \geq 2)$. This completes the proof of Theorem 1.2.

2. Irreducible decomposition

Let $\overrightarrow{\mathcal{E}}$ be the set of the isotopy classes of oriented essential 1-submanifolds of Σ_g. Transformation of 1-submanifolds by diffeomorphisms naturally induces an action of $\mathcal{M}_g$ on $\overrightarrow{\mathcal{E}}$. Let $\mathfrak{G}$ be a finite subgroup of $\mathcal{M}_g$. We denote by $\overrightarrow{\mathcal{E}\mathfrak{G}}$ the subset of $\overrightarrow{\mathcal{E}}$ consisting of the elements fixed by every $g \in \mathfrak{G}$. If $G \subset \mathrm{Diff}^+ \Sigma_g$ is any Niesen realization of $\mathfrak{G}$, it is easy to see that any $\overrightarrow{e} \in \overrightarrow{\mathcal{E}\mathfrak{G}}$ has a representative $\overrightarrow{E} \subset \Sigma_g$ such that $G(\overrightarrow{E}) = \overrightarrow{E}$. Then, the action of G on Σ_g decomposes into the pair of:

(1) the permutation of the connected components of $\Sigma_g \setminus \overrightarrow{E}$;

(2) actions on each connected component of $\Sigma_g \setminus \overrightarrow{E}$ of its stabilizer.

Note that any $\overrightarrow{e} \in \overrightarrow{\mathcal{E}\mathfrak{G}}$ is contained in a maximal element of $\overrightarrow{\mathcal{E}\mathfrak{G}}$ according to the inclusion order since the number of the connected components of an essential 1-submanifold is at most $3g-3$. Among the decompositions as above, it might be natural to call a decomposition corresponding to a maximal element of $\overrightarrow{\mathcal{E}\mathfrak{G}}$ an *irreducible decomposition* of G.

In this section, we describe the orbifolds appearing as the quotient of connected component of $\Sigma_g \setminus \overrightarrow{E}$ by its stabilizer after capping off 2-disks to the boundary of the component.

Now, we set the notation. We fix G and $\overrightarrow{E}$ as above. We denote by S_i a connected component of $\Sigma_g \setminus \overrightarrow{E}$. We take a completion M_i' of S_i as follows. Let $\tilde{S}_i$ be the universal covering of S_i embedded in $\tilde{\Sigma}_g$ via a lift of the inclusion $S_i \to \Sigma_g$. Then $\pi_1(S_i)$ acts on the closure $\bar{\tilde{S}}_i$. We set M_i' as the quotient $\bar{\tilde{S}}_i/\pi_1(S_i)$. Next, for each boundary component of M_i', we cap off 2-disk identifying it with the cone of the boundary component, and obtain a closed surface M_i. Then, the stabilizer G_i of S_i naturally acts on M_i. We denote the quotient orbifold M_i/G_i by O_i.

Theorem 2.1. *Let $\overrightarrow{E} \subset \Sigma_g$ be an oriented essential 1-submanifold which is invariant under the G-action. If its representing class $[\overrightarrow{E}]$ is maximal in $\overrightarrow{\mathcal{E}\mathfrak{G}}$, then the corresponding quotient orbifold $O_i = M_i/G_i$ for any connected component S_i of $\Sigma_g \setminus \overrightarrow{E}$ is described as follows:*

(i) If G_i is a trivial group, then O_i is isomorphic to the 2-sphere S^2.

(ii) If G_i is not trivial, then the orbifold isomorphism class of O_i is one of the followings according to the genus g_i of M_i.

(a) $g_i \geq 2$: $S^2(2,2,2,2,2)$, $S^2(2,2,2,m)$ $(m \geq 3)$, $S^2(m_1, m_2, m_3)$ $(m_1,\ m_2,\ m_3 \geq 2$, *and* $\frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3} < 1)$;

(b) $g_i = 1$: $S^2(2,2,2,2)$, $S^2(3,3,3)$, $S^2(2,4,4)$, $S^2(2,3,6)$;

(c) $g_i = 0$: $S^2(2,3,3)$, $S^2(2,3,4)$, $S^2(2,3,5)$, $S^2(2,2,m)$, $S^2(m,m)$ $(m \geq 2)$.

Moreover, any orbifold type above certainly appears in some irreducible decomposition for some $g \geq 2$.

The theorem follows from the next two lemmas.

Lemma 2.2. *There exists an oriented essential 1-submanifold $\overrightarrow{E}_0$ of M_i invariant under the G_i-action so that $\overrightarrow{E}_0 \subset \mathring{M}'_i$.*

Lemma 2.3. *Let $\overrightarrow{E}_0 \subset S_i$ be another G_i-invariant oriented essential 1-submanifold of Σ_g. Suppose that $\overrightarrow{E}_0 \cup \overrightarrow{E}$ also form an essential 1-submanifold of Σ_g. Then, $G(\overrightarrow{E}_0) \cup \overrightarrow{E}$ is a G-invariant oriented essential 1-submanifold of Σ_g.*

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