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Global real analytic length parameters and angle parameters for Teichmüller spaces and the geometry of hyperbolic transformations

Yoshihide OKUMURA
奥村　善英（金沢大）

Abstract

We consider global real analytic parameters for Teichmüller spaces. First we show a parametrization of $T(2,0,0)$ by seven length parameters and describe this parameter space. Next we show a similar result about angle parameters. These parametrizations are obtained from the geometry of Möbius transformations. We define the one-half power of a Möbius transformation and consider the geometry of hyperbolic transformations related to these parametrizations.

1 Introduction

A Fuchsian group G acting on the unit disk $\mathbf{D}$ is of *type* $(g, n, m; \nu_1, \nu_2, ..., \nu_n)$, if the quotient space $\mathbf{D}/G$ is a Riemann surface of genus g with n branch points and punctures of orders $\nu_1, \nu_2, ..., \nu_n$ and m holes. This Riemann surface is also called of *type* $(g, n, m; \nu_1, \nu_2, ..., \nu_n)$. Teichmüller space $T(g, n, m; \nu_1, \nu_2, ..., \nu_n)$ is the set of equivalence classes of marked Fuchsian groups of type $(g, n, m; \nu_1, \nu_2, ..., \nu_n)$ and a global real analytic manifold of dimension $6g + 2n + 3m - 6$. We abbreviate $(g, n, m; \nu_1, \nu_2, ..., \nu_n)$ and $T(g, n, m; \nu_1, \nu_2, ..., \nu_n)$ to (g, n, m) and $T(g, n, m)$, respectively. There are various methods parametrizing $T(g, n, m)$. We will parametrize $T(g, n, m)$ by some lengths of closed geodesics and intersection angles between geodesics on a Riemann surface represented by a marked Fuchsian group. Such lengths and angles are called *length parameters* and *angle parameters*, respectively.

A hyperbolic Riemann surface R of type (g, n, m) is obtained by pasting sides of some geodesic polygon P in $\mathbf{D}$ which may have vertices on the circle at infinity and the boundary of P can ever contain arcs of the circle at infinity. The uniformization theorem implies that P is a fundamental domain of a Fuchsian group representing R. Since a side of P corresponds to a geodesic on R and P is determined by the lengths of the sides and the interior angles of P, R is parametrized real analytically by some lengths of geodesics on R and angle parameters of R. Constructing a special polygon, we can take such lengths from length parameters, that is, lengths of closed geodesics on R. Moreover, we can parametrize R by $3g + n + 2m - 3$ length

parameters and $3g+n+m-3$ angle parameters of R. By this plan, the following classical result is obtained.

Theorem 1.1 [5] *Teichmüller space $T(g,n,m)$ has global real analytic parameters consisting of $3g+n+2m-3$ length parameters and $3g+n+m-3$ angle parameters. These parameters correspond to some lengths of sides and interior angles of a geodesic polygon in $\mathbf{D}$ which is a fundamental domain of a marked Fuchsian group of type (g,n,m). The total number of these parameters is $\dim(T(g,n,m))$ and this parameter space is $\mathbf{R}_+^{3g+n+2m-3} \times (0,\pi)^{3g+n+m-3}$.*

In Section 2, we define the one-half power of a Möbius transformation, since this is useful for the geometry of Möbius transformations.

We consider a parametrization of $T(g,n,m)$ by only length parameters. It is well known that length parameters parametrize $T(g,n,m)$ global real analytically (see for example, [3], [7], [9], [11] and [15]). Wolpert [20] and [21] announced that in the case of $T(g,0,0)$, the minimal number of these parameters is greater than $\dim(T(g,0,0)) = 6g-6$. Recently, Schmutz [13] stated that this minimal number is $6g-5$. In the same time, the author also obtained this result and this parameter space independently. In Section 3, we show the result in the case of $T(2,0,0)$.

In the hyperbolic plane, a triangle is determined by three lengths of sides or three interior angles. Thus lengths and angles have same significance in the hyperbolic geometry. In Section 4, we show a parametrization of $T(2,0,0)$ by only angle parameters which is useful for some considerations.

These two parametrizations are obtained from the geometry of Möbius transformations. In Section 5, we show the geometry of hyperbolic transformations related to these parametrizations.

2 Preliminaries

The group of Möbius transformations preserving $\mathbf{D}$, $M(\mathbf{D})$, is the group of isometries of $\mathbf{D}$ with respect to the Poincaré metric d. For distinct two points p_1 and p_2 in $\overline{\mathbf{D}}$, let $L(p_1,p_2)$ be the full geodesic through p_1 and p_2 with the direction from p_1 to p_2, where this direction is sometimes ignored. An elliptic element $A \in M(\mathbf{D})$ has the sole fixed point in $\mathbf{D}$. We denote it by $fp(A)$. A hyperbolic element $A \in M(\mathbf{D})$ has the *attracting fixed point*, $q(A)$, and the *repelling fixed point*, $p(A)$, which are characterized by $q(A) = \lim_{n\to\infty} A^n(z)$ and $p(A) = \lim_{n\to\infty} A^{-n}(z)$ for any $z \in \mathbf{D}$. The *axis* of A, $ax(A) = L(p(A),q(A))$, and the *translation length* of A, $tl(A) = \inf\{ d(z,A(z)) | z \in \mathbf{D}\}$, are characterized by

$$ax(A) = \{z \in \mathbf{D} | d(z,A(z)) = tl(A)\},$$

$$\cosh\frac{tl(A)}{2} = \frac{|trA|}{2}.$$

Let A be a hyperbolic element of a Fuchsian group G acting on $\mathbf{D}$. Then $ax(A)$ projects on a closed geodesic on $\mathbf{D}/G$ whose length is $tl(A)$ and corresponds to $|trA|$ real analytically. To define a marked Fuchsian group, we give the following:

Proposition 2.1 [5] *Let G be a Fuchsian group of type $(g, 0, m)$. Then G has a system of generators*

$$\Sigma = (A_1, B_1, ..., A_g, B_g, E_1, ..., E_m);$$

$$E_m E_{m-1} \cdots E_1 C_g C_{g-1} \cdots C_1 = identity,$$

where $A_j, B_j, C_j = [B_j, A_j] = B_j^{-1} A_j^{-1} B_j A_j$ $(j = 1, ..., g)$ and E_k $(k = 1, ..., m)$ are hyperbolic with axes illustrated as in Figure 2.1, and if $g = 0$ (resp. $m = 0$), then A_j, B_j and C_j (resp. E_k) are omitted.

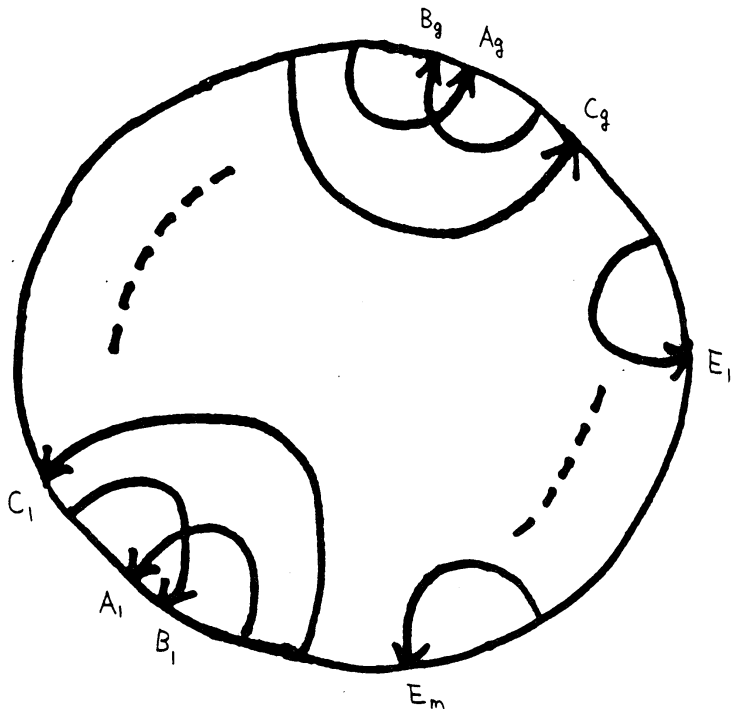

Figure 2.1.

A system Σ mentioned in Proposition 2.1 is a *canonical system of generators* of G. A pair of G and this system Σ, (G, Σ), is a *marked Fuchsian group* . Two marked Fuchsian groups (G_1, Σ_1) and (G_2, Σ_2) are *equivalent* if $G_2 = hG_1h^{-1}$ and $\Sigma_2 = h\Sigma_1 h^{-1}$ for some $h \in M(\mathrm{D})$. Teichmüller space $T(g, 0, m)$ is the set of equivalence classes of (G, Σ) of type $(g, 0, m)$. Similarly, $T(g, n, m)$, $n \neq 0$ are defined.

One of the matrix representations of a Möbius transformation A is denoted by $\tilde{A}$. For two Möbius transformations A and B, $tr[\tilde{B}, \tilde{A}]$ is invariant under the

choice of matrix representations. The following equations of commutator traces of $X, Y, Z = (YX)^{-1} \in SL(2,\mathbf{C})$ are useful: for $\varepsilon, \eta \in \{\pm 1\}$,

$$\begin{aligned} tr[X,Y] &= tr[X^{\varepsilon}, Y^{\eta}] = tr[Y^{\varepsilon}, X^{\eta}] \\ &= tr[Y^{\varepsilon}, Z^{\eta}] = tr[Z^{\varepsilon}, Y^{\eta}] \\ &= tr[Z^{\varepsilon}, X^{\eta}] = tr[X^{\varepsilon}, Z^{\eta}], \end{aligned}$$

Finally, we define the one-half power of $A \in M(\mathbf{D})$.

Definition and Proposition 2.2 *Let $A \in M(\mathbf{D})$ be hyperbolic or parabolic. Then there is a unique $X \in M(\mathbf{D})$ satisfying $X^2 = A$. X is called the one-half power of A and denoted by $A^{1/2}$. If $\tilde{A}$ is the matrix representation of A with negative trace (resp. positive trace), then the matrix representations of $A^{1/2}$ are*

$$\frac{\pm 1}{\sqrt{|trA|+2}}(\tilde{A} - I) \quad (resp. \ \frac{\pm 1}{\sqrt{|trA|+2}}(\tilde{A} + I)).$$

Thus

$$|trA^{1/2}| = \sqrt{|trA|+2}.$$

Since $(A^{1/2})^{-1} = (A^{-1})^{1/2}$, these are denoted by $A^{-1/2}$.

3 Global real analytic length parameters

Theorem 3.1 *$T(2,0,0)$ is parametrized global real analytically by seven length parameters which correspond to the absolute values of traces of the following hyperbolic elements of a marked Fuchsian group:*

$$A_1,\ B_1,\ B_1A_1,$$

$$A_2,\ B_2,\ B_2A_2A_1,\ B_2A_2B_1^{-1}.$$

Thus these length parameters are lengths of simple closed geodesics on a Riemann surface represented by a marked Fuchsian group. This parameter space is defined by

$$x_j, y_j, z_1, u, v > 2; \quad j = 1,2,$$

$$x_1^2 + y_1^2 + z_1^2 - x_1y_1z_1 = x_2^2 + y_2^2 + |trB_2A_2|^2 - x_2y_2|trB_2A_2| < 0,$$

$$\begin{aligned} |trB_2A_2| = \tfrac{1}{z_1^2-4}(z_1\sqrt{x_1y_1z_1 - (x_1^2+y_1^2+z_1^2)+4}\sqrt{uvz_1 - (u^2+v^2+z_1^2)+4} \\ +2(x_1u + y_1v) - z_1(y_1u + x_1v)) > 2, \end{aligned}$$

where $x_j = |trA_j|$, $y_j = |trB_j|$ $(j = 1,2)$, $z_1 = |trB_1A_1|$, $u = |trB_2A_2A_1|$ and $v = |trB_2A_2B_1^{-1}|$.

Remark 3.2 [12] In the case of $T(g,n,m), m \neq 0$, the minimal number of global real analytic length parameters is $\dim(T(g,n,m))$.

4 Global real analytic angle parameters

Let $\Sigma_{(2,0,0)} = (A_1, B_1, A_2, B_2)$ be a canonical system of generators of type $(2,0,0)$. Let p_j be the intersection of $ax(A_j)$ and $ax(B_j)$ for $j = 1, 2$. By Poincaré's polygon theorem, the geodesic decagon P with vertices

$$A_1^{-1}B_1^{-1}(p_1),\ A_1^{-1}(p_1),\ p_1,\ B_1^{-1}(p_1),\ B_1^{-1}A_1^{-1}(p_1),$$
$$A_2^{-1}B_2^{-1}(p_2),\ A_2^{-1}(p_2),\ p_2,\ B_2^{-1}(p_2),\ B_2^{-1}A_2^{-1}(p_2),$$

is a fundamental domain of the Fuchsian group generated by $\Sigma_{(2,0,0)}$ (see Figure 4.1). The axes of A_j, B_j and B_jA_j determine a triangle T_j with vertices p_j, $A_j^{-1/2}(p_j)$ and $B_j^{1/2}(p_j)$ (see Lemma 5.2). Let $\theta(A_j), \theta(B_j)$ and $\theta(B_jA_j)$ be three interior angles of T_j. We can show that $ax(C_1)$ and the segment $[B_1^{-1}A_1^{-1}(p_1), A_2^{-1}B_2^{-1}(p_2)]$ intersect. Let μ be the intersection angle between them.

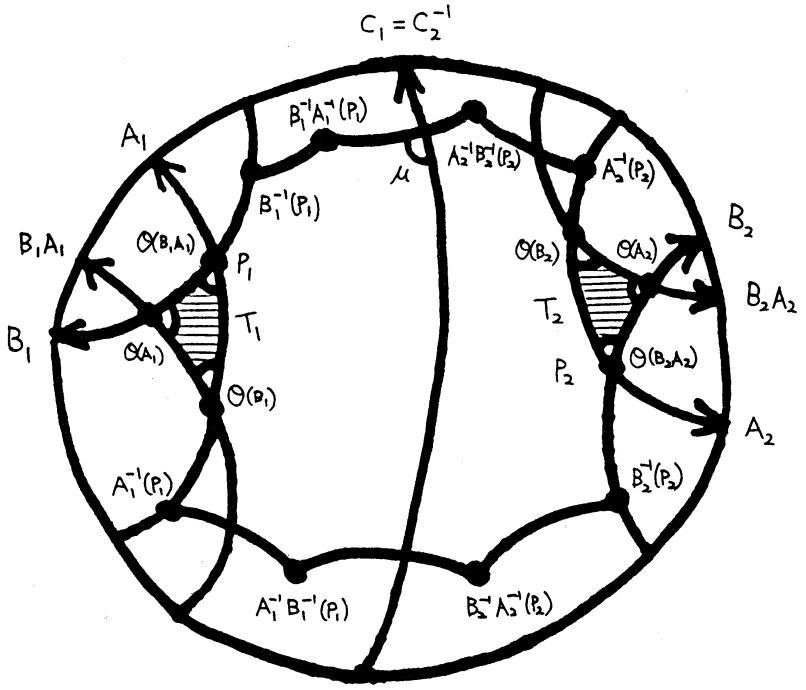

Figure 4.1.

Lemma 4.1 *Seven angles $\theta(A_j), \theta(B_j), \theta(B_jA_j)$ $(j = 1, 2)$ and μ determine $\Sigma_{(2,0,0)}$ global real analytically, up to conjugation by a Möbius transformation. This parameter space is defined by*

(4.1) $$\theta(A_j),\ \theta(B_j),\ \theta(B_jA_j),\ \mu \in (0, \pi),$$

$$\theta(A_j) + \theta(B_j) + \theta(B_jA_j) < \pi, \tag{4.2}$$

$$F(\theta(A_1), \theta(B_1), \theta(B_1A_1)) = F(\theta(A_2), \theta(B_2), \theta(B_2A_2)) > 1, \tag{4.3}$$

where $j = 1, 2$ *and*

$$F(x, y, z) = \frac{\cos^2 x + \cos^2 y + \cos^2 z + 2\cos x \cos y \cos z - 1}{\sin x \sin y \sin z}.$$

Remark 4.2 (4.1) and (4.2) imply that $F(\theta(A_j), \theta(B_j), \theta(B_jA_j)) > 0$.

Let R be a Riemann surface represented by $\Sigma_{(2,0,0)}$. Let (a_1, b_1, a_2, b_2) be a canonical homotopy basis of the fundamental group of R corresponding to $\Sigma_{(2,0,0)}$. We put same labels on a closed curve on R and the geodesic freely homotopic to it. Then $\theta(A_j)$, $\theta(B_j)$ and $\theta(B_jA_j)$ are three interior angles of a triangle determined by a_j, b_j and a_jb_j. Let q_j be the intersection of a_j and b_j. The intersection angle of $a_1b_1a_1^{-1}b_1^{-1}$ and the segment $[q_1, q_2]$ is μ (see Figure 4.2).

Theorem 4.3 $T(2, 0, 0)$ *is parametrized global real analytically by the above seven angle parameters. This parameter space is defined by* (4.1), (4.2) *and* (4.3).

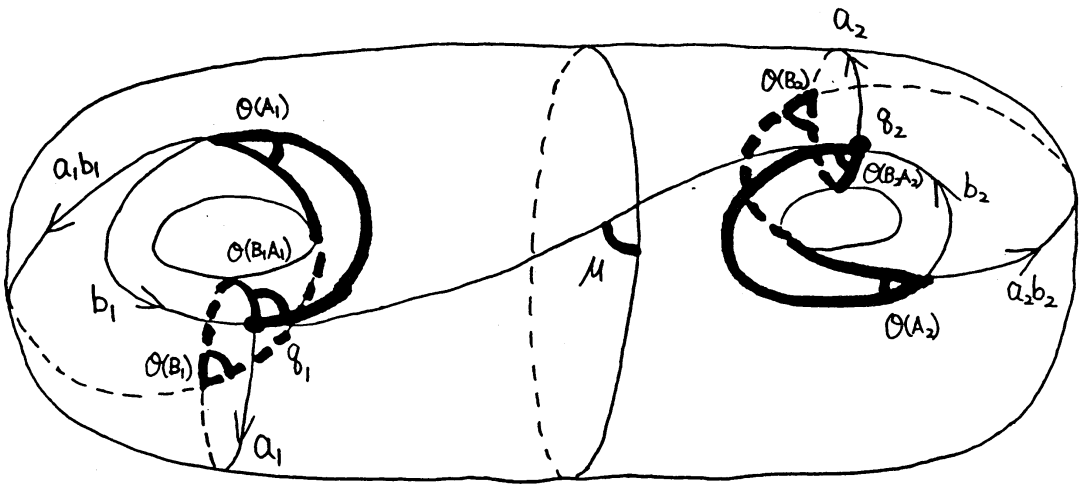

Figure 4.2.

5 The geometry of hyperbolic transformations

First we show the position of the axes of two hyperbolic transformations.

Lemma 5.1 *Let $A, B \in M(\mathrm{D})$ be hyperbolic. Then $ax(A)$ and $ax(B)$ intersect if and only if $tr[\tilde{A}, \tilde{B}] < 2$.*

Lemma 5.2 *Let $A, B \in M(\mathrm{D})$ be hyperbolic elements with intersecting axes. Then eight elements $A^{\varepsilon}B^{\eta}, B^{\varepsilon}A^{\eta}$; $\varepsilon, \eta \in \{\pm 1\}$ are hyperbolic. Let p be the intersection of $ax(A)$ and $ax(B)$. Then*

$$\begin{aligned} ax(BA) &= L(A^{-1/2}(p), B^{1/2}(p)), \\ ax(B^{-1}A) &= L(A^{-1/2}(p), B^{-1/2}(p)), \\ d(A^{-1/2}(p), B^{1/2}(p)) &= \frac{tl(BA)}{2}, \\ d(A^{-1/2}(p), B^{-1/2}(p)) &= \frac{tl(B^{-1}A)}{2}. \end{aligned}$$

Especially, $ax(A)$, $ax(B)$ and $ax(BA)$ determine a triangle with vertices $p, A^{-1/2}(p)$ and $B^{1/2}(p)$ (see Figure 5.1).

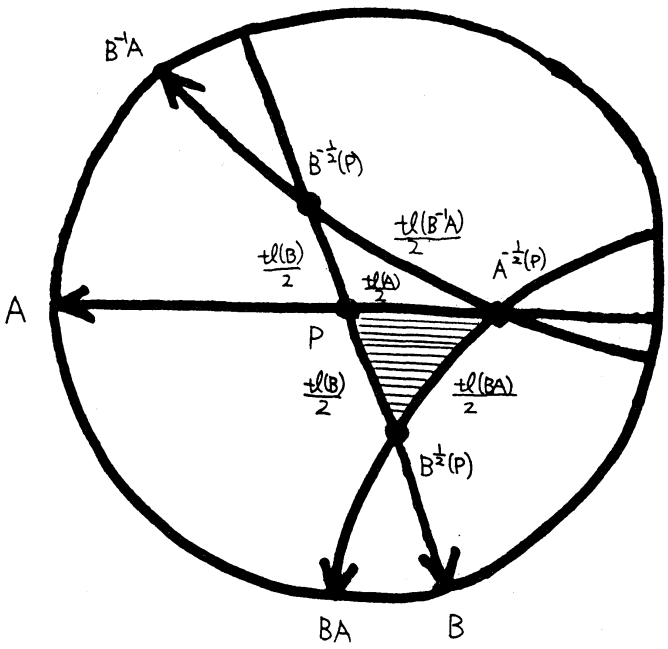

Figure 5.1. In the case that $p(A)$, $q(B)$, $q(A)$ and $p(B)$ are arranged clockwise in this order on the circle at infinity.

Let $A, B \in M(\mathbf{D})$ be hyperbolic. If BA is not hyperbolic, then $ax(A)$ and $ax(B)$ do not intersect, by Lemma 5.2. If BA is hyperbolic, then $ax(A), ax(B)$ and $ax(BA)$ are characterized as follows:

Lemma 5.3 *Let $A, B, BA \in M(\mathbf{D})$ be hyperbolic.*

(i) $ax(A)$, $ax(B)$ and $ax(BA)$ are disjoint, if and only if some two axes are disjoint.

(ii) $ax(A)$, $ax(B)$ and $ax(BA)$ are not disjoint, if and only if these three axes do not intersect at one point and any two axes intersect each other. Thus there are three cases of the positions of axes illustrated in Figure 5.2. These cases are characterized by $tr\tilde{A}$, $tr\tilde{B}$ and $tr\tilde{B}\tilde{A}$ as follows:

$$\begin{aligned}
(a) &\Leftrightarrow tr\tilde{A}tr\tilde{B}tr\tilde{B}\tilde{A} < 0,\ tr[\tilde{B},\tilde{A}] > 18,\\
(b) &\Leftrightarrow tr\tilde{A}tr\tilde{B}tr\tilde{B}\tilde{A} > 0,\ tr[\tilde{B},\tilde{A}] > 2,\\
(c) &\Leftrightarrow tr\tilde{A}tr\tilde{B}tr\tilde{B}\tilde{A} > 0,\ tr[\tilde{B},\tilde{A}] < 2.
\end{aligned}$$

Remark 5.4 $tr[\tilde{B},\tilde{A}] = tr^2\tilde{A} + tr^2\tilde{B} + tr^2\tilde{B}\tilde{A} - tr\tilde{A}tr\tilde{B}tr\tilde{B}\tilde{A} - 2$ and $tr\tilde{A}tr\tilde{B}tr\tilde{B}\tilde{A}$ are invariant under the choice of matrix representations.

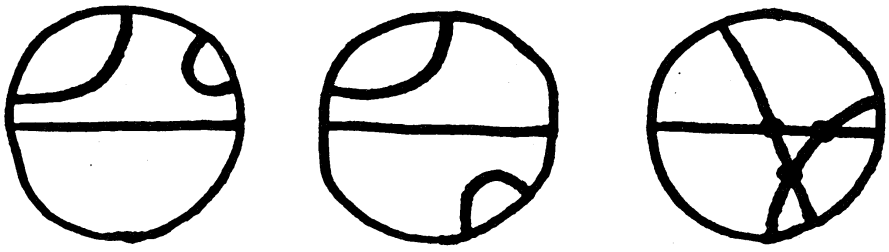

Figure 5.2.

Remark 5.5 Similarly, for any non trivial elements $A, B, BA \in M(\mathbf{D})$, the position of their fixed points and the *direction* of their actions are characterized by such three traces.

Finally, we show the following result.

Theorem 5.6 *Let $A, B \in M(\mathbf{D})$ be hyperbolic elements with intersecting axes. Let p be the intersection of these axes.*

(i) The axes of $A^{\varepsilon}B^{\eta}$, $B^{\varepsilon}A^{\eta}$; $\varepsilon, \eta \in \{\pm 1\}$ determine the parallelogram with vertices $A^{-1/2}(p), B^{1/2}(p), A^{1/2}(p)$ and $B^{-1/2}(p)$.

Let $C = [B, A]$ be hyperbolic. Let $p(A)$, $q(B)$, $q(A)$ and $p(B)$ be arranged clockwise in this order on the circle at infinity.

(ii) $(A, B^{-1}A^{-1}B, C^{-1})$, $(BA, B^{-1}A^{-1}, C^{-1})$ and $(A^{-1}BA, B^{-1}, C^{-1})$ are canonical systems of generators of type $(0, 0, 3)$.

(iii) Let $R \in M(\mathbf{D})$ be elliptic of order 2 with fixed point p. Then we have

$$\begin{aligned} C^{1/2} &= RBA, \\ A^{-1} &= RAR, \\ B^{-1} &= RBR, \\ A &= [R, A^{1/2}], \\ B &= [R, B^{1/2}], \\ \tilde{R} &= \frac{\pm 1}{det(\tilde{B}\tilde{A} - \tilde{A}\tilde{B})^{1/2}}(\tilde{B}\tilde{A} - \tilde{A}\tilde{B}). \end{aligned}$$

(iv) $C^{-1/2}A$, $C^{-1/2}B^{-1}$ and $C^{-1/2}BA$ are elliptic of order 2 satisfying

$$\begin{aligned} fp(C^{-1/2}A) &= (ABA)^{-1/2}(p) = (BA)^{-1/2}A^{-1/2}(p), \\ fp(C^{-1/2}B^{-1}) &= A^{-1/2}(p), \\ ax(ABA) &= L(fp(C^{-1/2}A), p). \end{aligned}$$

(v) Let $A_{1/2}$ (resp. $A_{-1/2}$) be elliptic of order 2 with fixed point $A^{1/2}(p)$ (resp. $A^{-1/2}(p)$), namely, $A_{1/2} = A^{1/2}RA^{-1/2}$ and $A_{-1/2} = A^{-1/2}RA^{1/2}$. Similarly, $B_{1/2}$ and $B_{-1/2}$ are defined. Then we have

$$\begin{aligned} A &= RA_{-1/2} = A_{1/2}R, \\ B &= RB_{-1/2} = B_{1/2}R. \end{aligned}$$

Thus

$$\begin{aligned} BA &= B_{1/2}A_{-1/2}, \\ AB &= A_{1/2}B_{-1/2}, \\ C &= B_{-1/2}A_{1/2}B_{1/2}A_{-1/2}. \end{aligned}$$

Especially, C is determined by four elliptic transformations of order 2 whose fixed points are four vertices of the parallelogram (see Figure 5.3).

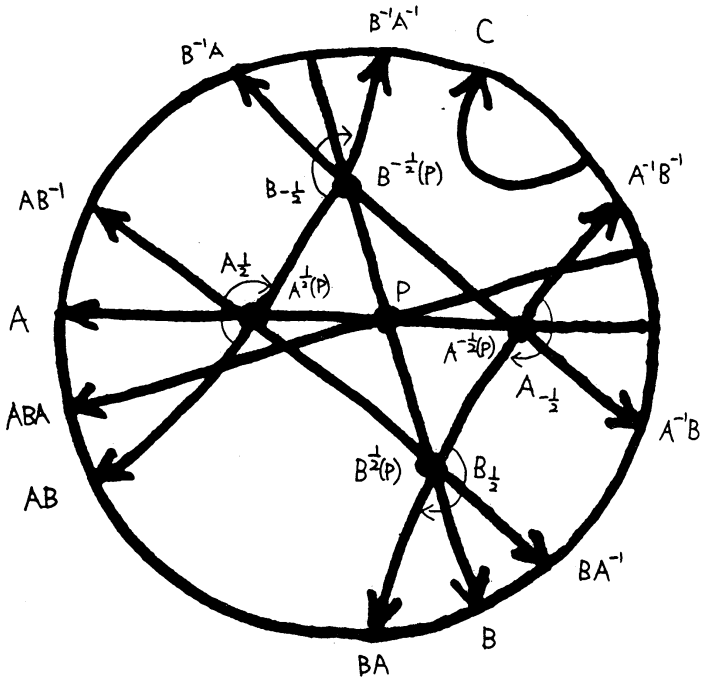

Figure 5.3.

References

[1] W. Abikoff, The Real Analytic Theory of Teichmüller Space, Lecture Notes in Mathematics 820, Springer-Verlag, Berlin, Heidelberg, New York, 1980.

[2] W. J. Harvey, Spaces of discrete groups, in Discrete Groups and Automorphic Functions, (W. J. Harvey ed.), Academic Press, London, New York, San Francisco, 1977, 295-348.

[3] Y. Imayoshi and M. Taniguchi, An Introduction to Teichmüller Spaces, Springer-Verlag, Tokyo, Berlin, Heidelberg, New York, Paris, Hong Kong, Barcelona, 1992.

[4] L. Keen, Canonical polygons for finitely generated Fuchsian Groups, Acta Math., 115(1966), 1-16.

[5] L. Keen, Intrinsic moduli on Riemann surfaces, Ann. of Math., 84(1966), 404-420.

[6] L. Keen, On infinitely generated Fuchsian groups, J. Indian Math. Soc., 35(1971), 67-85.

[7] L. Keen, On Fricke moduli, in Advances in the Theory of Riemann Surfaces, (L. V. Ahlfors ed. et al.), Ann. Math. Studies 66, Princeton Univ. Press, Princeton, N. J., 1971, 205-224.

[8] L. Keen, A correction to "On Fricke moduli", Proc. Amer. Math. Soc., 40(1973), 60-62.

[9] L. Keen, A rough fundamental domain for Teichmüller spaces, Bull. Amer. Math. Soc., 83(1977), 1199-1226.

[10] S. P. Kerckhoff, The Nielsen realization problem, Ann. of Math., 117(1983), 235-265.

[11] Y. Okumura, On the global real analytic coordinates for Teichmüller spaces, J. Math. Soc. Japan, 42(1990), 91-101.

[12] Y. Okumura, Global real analytic coordinates for Teichmüller Spaces, Doctor's Thesis, Kanazawa Univ., 1989.

[13] P. Schmutz, Die Parametrisierung des Teichmüllerraumes durch geodätische Längenfunktionen, Comment. Math. Helv., 68(1993), 278-288.

[14] M. Seppälä and T. Sorvali, On geometric parametrization of Teichmüller spaces, Ann. Acad. Sci. Fenn. Ser. A I Math., 10(1985), 515-526.

[15] M. Seppälä and T. Sorvali, Parametrization of Möbius groups acting in a disk, Comment. Math. Helv., 61(1986), 149-160.

[16] M. Seppälä and T. Sorvali, Parametrization of Teichmüller spaces by geodesic length functions, in Holomorphic Functions and Moduli II (D. Drasin ed. et al.), Mathematical Sciences Research Institute Publications 11, Springer-Verlag, New York, Berlin, Heidelberg, London, Paris, Tokyo, 1988, 267-284.

[17] M. Seppälä and T. Sorvali, Geometry of Riemann Surfaces and Teichmüller Spaces, Mathematics Studies 169, North-Holland, Amsterdam, London, New York, Tokyo, 1992.

[18] T. Sorvali, Parametrization of free Möbius groups, Ann. Acad. Sci. Fenn. Ser. A I Math., 579, 1974, 1-12.

[19] S. A. Wolpert, The Fenchel-Nielsen deformation, Ann. of Math. 115(1982), 501-528.

[20] S. A. Wolpert, On the symplectic geometry of deformations of a hyperbolic surface, Ann. of Math., 117(1983), 207-234.

[21] S. A. Wolpert, Geodesic length functions and the Nielsen problem, J. Differential Geom., 25(1987), 275-296.

Faculty of Technology, Kanazawa University, Kanazawa 920, Japan
E-mail address: okumura@funah01.eg.t.kanazawa-u.ac.jp