数理解析研究所講究録
第 882 巻 1994 年 121-124

A Note on the Arithmeticity of the Figure-Eight Knot Orbifolds

渡辺 綾子 (晃華学園)
AYAKO WATANABE

1. INTRODUCTION

Let K be the figure-eight knot, (K, n) the orbifold with underlying space $\mathbf{S}^3$, singular set K and isotropy group cyclic of order n.

Proposition 1 (Thurston [5], Hilden-Lozano-Montesinos [1]). *If $n > 3$, (K, n) is hyperbolic. Furthermore, (K, n) is arithmetic exactly for $n = 4, 5, 6, 8, 12$.*

In this paper, our aim is to describe concretely the arithmeticity of (K, n) for $n = 4, 5, 6, 8, 12$.

2. PRELIMINARIES

We can take a Kleinian model of (K, n) as follows ([1]):

$$\Gamma_n = \langle A, B \mid A^{-1}BAB^{-1}ABA^{-1}B^{-1}AB^{-1} = I, A^n = B^n = -I \rangle,$$

$$A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \; B = \begin{pmatrix} \mu & 1/\alpha \\ \alpha & \mu(\alpha - \mu) - 1 \end{pmatrix},$$

where

$$\alpha = 2\cos\frac{\pi}{n}$$

$$\beta = \frac{1}{2}(1 + \alpha^2 + \sqrt{(\alpha^2 - 1)(\alpha^2 - 5)})$$

$$\lambda = \frac{1}{2}(\alpha + \sqrt{\alpha^2 - 4})$$

$$\mu = \frac{\lambda\beta - \alpha}{\lambda^2 - 1}.$$

Definition. Let Γ be a non-elementary Kleinian group and $\Gamma^{(2)}$ the subgroup generated by the squares of the elements of Γ. The invariant trace field of Γ is the field $\mathbb{Q}(tr\Gamma^{(2)})$, and denoted by $k\Gamma$. The invariant quaternion algebra is given by

$$\left\{ \sum a_i \gamma_i (\text{finite sum}) \mid a_i \in k\Gamma, \gamma_i \in \Gamma^{(2)} \right\},$$

and denoted by $A\Gamma$.

In fact, we see that $A\Gamma$ is a quaternion algebra over $k\Gamma$ from:

Lemma 2. *Let Γ be a Kleinian group of finite covolume. Then $A\Gamma$ is quaternion algebra over $k\Gamma$ if*

(1) *$k\Gamma$ is a number field with one complex place, and*
(2) *$tr\Gamma^{(2)}$ consists of algebraic integers.*

Furthermore, if we define

$$R_{k\Gamma} = \{a \in k\Gamma \mid a \text{ is an algebraic integer}\} \quad \text{and}$$

$$O\Gamma = \{\sum b_i\gamma_i (\text{finite sum}) \mid b_i \in R_{k\Gamma},\ \gamma_i \in \Gamma^{(2)}\},$$

then $O\Gamma$ is an order of $A\Gamma$.

The following lemma shows that if Γ is arithmetic, it is sufficient to take $k\Gamma$ and $A\Gamma$ as its algebraic tools (see [2] [6]).

Lemma 3. *Suppose that Γ is an arithmetic Kleinian group. Then*

$$\Gamma^{(2)} \subset P(O^1\Gamma)$$

where $P : SL(2,\mathbb{C}) \longrightarrow PSL(2,\mathbb{C})$ and $O^1\Gamma = \{x \in O\Gamma \mid \text{the norm of } x \text{ is } 1\}$.

We shall calculate $k\Gamma_n$ for $n = 4, 5, 6, 8, 12$. Since $k\Gamma_n = \mathbb{Q}(\alpha^2, \beta)$ by [1], we see that

$$k\Gamma_4 = \mathbb{Q}(\sqrt{-3}) \qquad (\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}});$$

$$k\Gamma_5 = \mathbb{Q}(\sqrt{\frac{-1-3\sqrt{5}}{2}}) \qquad (\cos\frac{\pi}{5} = \frac{1+\sqrt{5}}{4});$$

$$k\Gamma_6 = \mathbb{Q}(\sqrt{-1}) \qquad (\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2});$$

$$k\Gamma_8 = \mathbb{Q}(\sqrt{-1-2\sqrt{2}}) \qquad (\cos\frac{\pi}{8} = \frac{\sqrt{2+\sqrt{2}}}{2});$$

$$k\Gamma_{12} = \mathbb{Q}(\sqrt{-2\sqrt{3}}) \qquad (\cos\frac{\pi}{12} = \frac{\sqrt{2+\sqrt{3}}}{2}).$$

All of them are the extension fields over $\mathbb{Q}$ of degree 2.
On the other hand, by [1],

$$A\Gamma_n = \Big(\frac{\frac{1}{4}(\lambda^2-\lambda^{-2})^2,\ \alpha^2\{\mu(\alpha-\mu)-1\}}{k\Gamma_n}\Big),$$

where

$$1 = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \; i = \begin{pmatrix} \frac{1}{2}(\lambda^2 - \lambda^{-2}) & 0 \\ 0 & -\frac{1}{2}(\lambda^2 - \lambda^{-2}) \end{pmatrix}$$

and

$$j = \begin{pmatrix} 0 & 1 \\ \alpha^2\{\mu(\alpha - \mu) - 1\} & 0 \end{pmatrix}.$$

In fact,

$$i^2 = \frac{1}{4}(\lambda^2 - \lambda^{-1})^2, \; j^2 = \alpha^2\{\mu(\alpha - \mu) - 1\}, \; ij = -ji.$$

3. Main Theorem and its Proof

In the beginning of this section, we mention our main theorem.

Theorem 4. *For the Kleinian model Γ_n of (K, n), the following are satisfied:*

(1) *If $n = 4, 6, 8, 12$, then*

$$\Gamma_n \cap P(O^1\Gamma_n) = \Gamma_n^{(2)} \quad \textit{and}$$

$$[\Gamma_n : \Gamma_n \cap P(O^1\Gamma_n)] = 2.$$

(2) *For $n = 5$,*

$$\Gamma_5 \cap P(O^1\Gamma_5) = \Gamma_5, \; \textit{that is,}$$

$$[\Gamma_5 : \Gamma_5 \cap P(O^1\Gamma_5)] = 1.$$

To prove this theorem, we need the next lemma.

Lemma 5. *Let Γ be a finitely generated group, m the number of the generators of Γ. Then*

$$[\Gamma : \Gamma^{(2)}] \leq 2^m.$$

Proof. See [6].

Proof of Theorem 4. Lemma 3 and Lemma 5 show that

$$[\Gamma_n : \Gamma_n \cap P(O^1\Gamma_n)] \leq [\Gamma_n : \Gamma_n^{(2)}] \leq 4.$$

Furthermore by the relation

$$A^{-1}BAB^{-1}ABA^{-1}B^{-1}AB^{-1} = I,$$

we see that $AB \in \Gamma_n^{(2)}$. Hence $[\Gamma_n : \Gamma_n \cap P(O^1\Gamma_n)] \leq 2$, and it is sufficient to consider A (or B). We set $A = a_0 \cdot 1 + a_1 i + a_2 j + a_3 ij$. In this case, solving linear equations, we see that

$$a_0 = \alpha/2, \; a_1 = 1/\alpha, \; a_2 = a_3 = 0.$$

And since $\lambda^2 - \lambda^{-2} = \alpha\sqrt{\alpha^2 - 4}$, the norm of A equals to 1.

Now, we shall classify into two cases. In case $n = 4, 6, 8, 12$: Since $\alpha \notin k\Gamma_n$, we see $A \notin A\Gamma_n$. Hence $A \notin O\Gamma_n$. In case $n = 5$: By $A^5 = -I$, we have $-A = A^{-4} \in \Gamma_5^{(2)}$. On the other hand, since $\Gamma_5^{(2)} \subset \Gamma_5 \cap P(O^1\Gamma_5)$,

$$A = \sum(-b_i)\gamma_i$$

where $-b_i \in R_{k\Gamma_5}$, $\gamma_i \in \Gamma_5^{(2)}$ and the norm of A equals to 1. Therefore $A \in O^1\Gamma_5$.

The proof of Theorem 4 is now completed.

4. The Difficulty about the Complement

For $S^3 - K$, there is Riley's model Γ as its Kleinian model ([4]), so we know it is arithmetic. But it is difficult to calculate its arithmeticity same as the case of (K, n). The difficulty comes from the lack of definite information about the order $O\Gamma$, but by relations in the fundermental group of $S^3 - K$ and experimental calculation in [6], we shall except the next problem.

Problem 6. *For the Kleinian model Γ of $S^3 - K$, is it satisfied that $\Gamma \cap P(O^1\Gamma) = \Gamma$? In other words,*

$$[\Gamma : \Gamma \cap P(O^1\Gamma)] = 1?$$

In future, our subject is to investigate geometrical properties of arithmetic hyperbolic 3-manifolds.

References

1. H. M. Hilden, M. T. Lozano and J. M. Montesinos, *Arithmeticity of the figure-eight knot orbifolds,* in TOPOLOGY'90, Proceeding of the Research Semester in Low dimensional Topology at Ohio State University, De Gryter Verlag, 1992, pp 169–183.
2. C. Maclachlan and A. W. Reid, *Commensurability classes of arithmetic Kleinian groups and their Fuchsian subgroups*, Math. Proc. Camb. Phil. Soc. **102** (1987), 251–257.
3. A. W. Reid, *A non-Haken hyperbolic 3-manifold covered by a surface bundle,* preprint.
4. R. Riley, *A quadratic parabolic groups*, Math. Proc. Camb. Phil. Soc. **77** (1975), 281–288.
5. W. P. Thurston, *The geometry and topology of 3-manifolds,* Mineographed lecture notes at Princeton University, 1978/79.
6. A. Watanabe, Master thesis, Dept. of Math., Tokyo Inst. Tech. (1994).

Kōka Gakuen High School, 5-28-1 Sazu Machi, Chofu City, Tokyo 182, Japan