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A theory of tensor products for module categories for a vertex operator algebra, I

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Contents

Abstract

This is the first in a series of papers presenting a theory of tensor products for module categories for a vertex operator algebra. The theory applies in particular to any "rational" vertex operator algebra for which products of intertwining operators are known to be convergent, including the algebras associated with WZNW models, minimal models and the moonshine module. In this paper (Part I), we define the notions of $P(z)$- and $Q(z)$-tensor product, where $P(z)$ and $Q(z)$ are two special elements of a certain moduli space of spheres with punctures and local coordinates. For rational vertex operator algebras, we give a "tautological" construction of a $Q(z)$-tensor product, providing an existence proof. More generally, two constructions of a $Q(z)$-tensor product are given under a certain assumption, using certain results whose proofs are deferred to Part II of the series. The results entering into the second of these constructions and their natural generalizations constitute the foundation of the theory.

1 Introduction

In the representation theory of Lie algebras, we have the classical notion of tensor product of two modules, providing the conceptual foundation of the Clebsch-Gordan coefficients. The tensor product operation is an operation on the category of modules for a Lie algebra, giving a classical example of a tensor category satisfying an additional symmetry axiom. For quantum groups (Hopf algebras), the module categories are also tensor categories but in general do not satisfy the symmetry axiom, corresponding to the fact that the Hopf algebra need not be cocommutative. Instead, such tensor categories satisfy weaker conditions — braiding conditions. From the resulting braid group representations, one can construct knot and link invariants. See in particular [J], [K1], [K2], [Dr1], [Dr2], [RT].

Vertex operator algebras ([B1], [FLM2], [FHL]) are "complex analogues" of both Lie algebras and commutative associative algebras. For vertex operator algebras, we also have the notions of modules, intertwining operators among triples of modules and fusion rules (dimensions of spaces of intertwining operators) analogous to those for Lie algebras. We use the versions of these notions given in [FLM2] and [FHL], and recalled below. In the study of rational conformal field theories ([BPZ], [FS]), intertwining operators (or chiral vertex operators) are fundamental tools. Many important concepts and results, for example, representations of braid groups, the relationship between modular transformations and fusion rules, and duality relations, are obtained through the study of intertwining operators; see for example [KZ], [TK], [V] and [MS]. The fusion rules for a vertex operator algebra being the analogues of the Clebsch-Gordan coefficients for a Lie algebra, we have the natural question whether there exists a notion of tensor product for modules for a vertex operator algebra which would naturally provide a conceptual foundation for fusion rules and intertwining operators.

We noticed a few years ago that the Jacobi identity axiom (see [FLM2], [FHL]) for vertex operator algebras suggests an analogue of the coalgebra diagonal map for primitive elements of a Hopf algebra. In this paper we make this idea precise. Given two modules W_1 and W_2 for a vertex operator algebra V, when one tries to define a tensor product module, the first serious problem is that the tensor product vector space $W_1 \otimes W_2$ is not a V-module in any natural way (although it is a module for the vertex operator algebra $V \otimes V$), and so the underlying vector space of a tensor product module would not be expected to be the tensor product vector space. Another serious problem is that a vertex operator algebra is not a Hopf algebra in any natural sense. We need a new way to define and construct a tensor product module — both the underlying vector space and the action of the vertex operator algebra. As we shall see, the analogy between vertex operator algebras and Lie algebras, centered on the Jacobi identity axiom, provides an analogue of a Hopf algebra diagonal map for a construction of a tensor product module, under appropriate hypotheses. In addition, the analogy between vertex operator algebras and commutative associative algebras, via the geometric formulation of the notion of vertex operator algebra ([H1], [H2]), provides the geometric foundation for the construction.

One important class of examples of vertex operator algebras is constructed from certain modules for affine Lie algebras (see for example [FZ], [DL]). There are interesting relations between representations of affine Lie algebras and of quantum groups discussed in several of the works mentioned above, for example, and to understand these relations on a deeper level, one natural strategy is to compare categories of modules for affine Lie algebras with a fixed nonzero central charge and categories of modules for associated quantum groups. While the category of modules for a quantum group is a tensor category, a category of modules for an affine Lie algebras with fixed nonzero central charge does not close under the classical tensor product of Lie algebra modules. Thus an appropriate tensor product module, if it exists, could not be the ordinary one. Recently, Kazhdan and Lusztig ([KL1]–[KL5]) have found such a tensor product operation for certain module categories for an affine Lie algebra and have shown that these module categories can in fact be made into tensor categories. Moreover, they have shown that these tensor categories are equivalent to suitable categories of modules for corresponding quantum groups. More recently, Finkelberg [F] has extended their work to the important case of categories of certain positive integral level modules for an affine Lie algebra. The construction of Kazhdan-Lusztig was in fact motivated by conformal field theory, and we expected that their tensor product operation should come from more general and natural structures in conformal field theory.

In [HL1], partly motivated by the analogy between vertex operator algebras and Lie algebras and partly motivated by the announcement [KL1], we initiated a project whose goal is a theory of tensor products for modules for a vertex operator algebra. In this series of papers, we shall present this theory of tensor products for suitable module categories for a vertex operator algebra. Our methods are independent of the methods of [KL1]–[KL5] (even in the special case in which our vertex operator algebra is associated with an affine Lie algebra).

It should be emphasized that our theory applies (at present) to an arbitrary rational vertex operator algebra satisfying a certain convergence condition. It applies in particular to minimal models, WZNW models and the moonshine module for the Monster constructed in [FLM1], [FLM2]. (The rationality of the moonshine module vertex operator algebra has recently been proved by Dong [Do].) Many of the notions, constructions and techniques also apply to more general vertex operator algebras. We hope that this theory will provide not only a conceptual and unified way to study many different conformal-field-theoretic models but also insights regarding such phenomena as monstrous moonshine (see [CN], [B1], [FLM2], [L], [B2]).

We use the analogy between vertex operator algebras and Lie algebras as our main guide. In the theory of Lie algebras we have the following standard notion of intertwining map (of type $\binom{W_3}{W_1 W_2}$) among modules W_1, W_2, W_3 for a Lie algebra V, with corresponding actions π_1, π_2, π_3 of V: a linear map I from the tensor product vector space $W_1 \otimes W_2$ to W_3, satisfying the identity

$$\pi_3(v)I(w_{(1)} \otimes w_{(2)}) = I(\pi_1(v)w_{(1)} \otimes w_{(2)}) + I(w_{(1)} \otimes \pi_2(v)w_{(2)}) \tag{1.1}$$

for $v \in V$, $w_{(1)} \in W_1$, $w_{(2)} \in W_2$. This "Jacobi identity for intertwining maps" agrees with the Jacobi identity for V when all three modules are the adjoint module. Let us call a *product* of W_1 and W_2 a third module W_3 equipped with an intertwining map I of type $\binom{W_3}{W_1 W_2}$; we denote this by (W_3, I). Then a tensor product of W_1 and W_2 is a product $(W_1 \otimes W_2, \otimes)$ such that given any product (W_3, I), there exists a unique module map η from $W_1 \otimes W_2$ to W_3 such that

$$I = \eta \circ \otimes. \tag{1.2}$$

Thus any tensor product of two given modules has the following property: The intertwining maps from the tensor product vector space of the two modules to a third module correspond naturally to the module maps from the tensor product module to the third module. Moreover, this universal property characterizes the tensor product module up to unique isomorphism.

In this paper (Part I), we analogously define notions of $P(z)$-tensor product and $Q(z)$-tensor product of two modules for a vertex operator algebra, where z is a nonzero complex number and $P(z)$ and $Q(z)$ are two particular elements, depending on z, of a certain moduli space of spheres with punctures and local coordinates (see [H1], [H2], [H3] or [H4]). We give two constructions of a $Q(z)$-tensor product when the vertex operator algebra that we consider is such that its module category (or some fixed subcategory) is closed under a certain operation. This occurs in particular if the vertex operator algebra is "rational," that is, if the module category satisfies certain finiteness and semisimplicity conditions, and so the $Q(z)$-tensor product of two modules exists in this case. (For such algebras we also give a "tautological" construction which in fact provides an existence proof.) The construction of a $P(z)$-tensor product will be given in Part III using the results entering into the constructions of the $Q(z)$-tensor product in this paper (Part I). The first of our two constructions of a $Q(z)$-tensor product is straightforward and conceptually simple, but it is difficult to use. Our second, much more useful, construction presents the $Q(z)$-tensor product module of two modules W_1 and W_2 (when it exists) in terms of the subspace of the dual $(W_1 \otimes W_2)^*$ of the vector space tensor product consisting of the elements satisfying a certain set of conditions, the most important of which is what we call the "compatibility condition."

The dependence of the tensor product operation on the nonzero complex number z is a fundamental feature of our theory. In one of the papers in this series, we shall see that for every element of the moduli space mentioned above, we have a tensor product operation. The associativity, commutativity and coherence properties of this tensor product depend on the elements of this moduli space in a natural way. Such properties can be used to define a new concept — that of "vertex tensor category." In fact the notion of vertex tensor category depends only on the moduli space of spheres with punctures and local coordinates mentioned above. This moduli space is the $\mathbf{C}^{\times}$-rescalable associative partial operad used in the operadic formulation of vertex operator algebras ([HL2], [HL3], [H5]). Given a vertex tensor category, we can obtain a tensor category in a natural way. In the case that the vertex operator algebra is constructed from an affine Lie algebra, the tensor category obtained from the vertex tensor category of modules for this vertex operator algebra gives us a tensor category structure on the relevant category of modules for the corresponding affine Lie algebra. Thus this theory of tensor products for module categories for a vertex operator algebras is not only conceptually natural and satisfactory but is also powerful in the study of the conformal-field-theoretical properties of affine Lie algebras.

Our approach is based on the formal calculus developed in [FLM2], and also (in later papers in this series) on the geometric methods developed in [H1]. Our use of formal calculus (see [FLM2], [FHL]) is equivalent to the use of contour integral methods, but is much more natural for our formulations and arguments. For example, in Section 3, the space of rational functions whose action we must define is described conceptually by means of the formal δ-function.

Results in the present series of papers were announced in [HL1] and in talks presented by both authors at the June, 1992 AMS-IMS-SIAM Joint Summer Research Conference on Conformal Field Theory, Topological Field Theory and Quantum Groups at Mount Holyoke College.

Part I is organized as follows: Section 2 reviews basic concepts in the representation theory of vertex operator algebras. Section 3 discusses affinizations of vertex operator algebras, the opposite module structure on a module for a vertex operator algebra and a related $*$-operation. In [B1], Borcherds in fact placed a vertex algebra structure on a certain affinization of a vertex algebra (in his sense), while in this paper we are using more general affinizations of a vertex operator algebra, but in a simpler way. Section 4 gives the definitions of $P(z)$- and $Q(z)$-tensor product of two modules for a vertex operator algebra and establishes some straightforward consequences, including relations among intertwining operators, "intertwining maps" and tensor products, and the existence of a $Q(z)$-tensor product of the two modules for a rational vertex operator algebra. In this section, we formulate and use a result (Proposition 4.9) giving an isomorphism between certain spaces of intertwining operators and we defer its proof to Part II. Sections 5 and 6 present the first and second constructions of the $Q(z)$-tensor product of two modules, respectively. In the course of these constructions, we formulate and use three results, Proposition 5.2, Theorem 6.1 and Proposition 6.2, whose proofs will form the main content of Part II. Theorem 6.1 and Proposition

6.2 and their generalizations in fact constitute the foundation of our whole theory.

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2 Review of basic concepts

In this section, we review some basic definitions and concepts in the representation theory of vertex operator algebras. Except for Definition 2.11, everything in this section can be found in [FLM2] and [FHL].

In this section, all the variables x, $x_0, \ldots$ are independent commuting formal variables, and all expressions involving these variables are to be understood as formal Laurent series or, when explicitly so designated, as formal rational functions. (Later, we shall also use the symbols $z, z_0, \ldots$, which will denote complex numbers, *not* formal variables.) We use the formal expansion

$$\delta(x) = \sum_{n\in\mathbf{Z}} x^n. \tag{2.1}$$

This "formal δ-function" has the following simple and fundamental property: For any $f(x) \in \mathbf{C}[x, x^{-1}]$,

$$f(x)\delta(x) = f(1)\delta(x). \tag{2.2}$$

This property has many important variants. For example, for any

$$X(x_1, x_2) \in (\mathrm{End}\ W)[[x_1, x_1^{-1}, x_2, x_2^{-1}]]$$

(where W is a vector space) such that

$$\lim_{x_1\to x_2} X(x_1, x_2) = X(x_1, x_2)\Big|_{x_1=x_2} \tag{2.3}$$

exists, we have

$$X(x_1,x_2)\delta\left(\frac{x_1}{x_2}\right) = X(x_2,x_2)\delta\left(\frac{x_1}{x_2}\right). \tag{2.4}$$

The existence of the "algebraic limit" defined in (2.3) means that for an arbitrary vector $w \in W$, the coefficient of each power of x_2 in the formal expansion $X(x_1,x_2)w\Big|_{x_1=x_2}$ is a finite sum. We use the convention that negative powers of a binomial are to be expanded in nonnegative powers of the second summand. For example,

$$x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right) = \sum_{n\in\mathbf{Z}} \frac{(x_1-x_2)^n}{x_0^{n+1}} = \sum_{m\in\mathbf{N},\, n\in\mathbf{Z}} (-1)^m \binom{n}{m} x_0^{-n-1} x_1^{n-m} x_2^m. \tag{2.5}$$

We have the following identities:

$$x_1^{-1}\delta\left(\frac{x_2+x_0}{x_1}\right) = x_2^{-1}\left(\frac{x_1-x_0}{x_2}\right), \tag{2.6}$$

$$x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right) - x_0^{-1}\delta\left(\frac{x_2-x_1}{-x_0}\right) = x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right). \tag{2.7}$$

We shall use these properties and identities extensively later on without explicit comment. See [FLM2] and [FHL] for further discussion and many examples of their use.

We now quote the definition and basic "duality" properties of vertex operator algebras from [FLM2] or [FHL]:

Definition 2.1 A *vertex operator algebra* (over $\mathbf{C}$) is a $\mathbf{Z}$-graded vector space (graded by *weights*)

$$V = \coprod_{n\in\mathbf{Z}} V_{(n)}; \text{ for } v \in V_{(n)},\ n = \text{wt } v; \tag{2.8}$$

such that

$$\dim V_{(n)} < \infty \text{ for } n \in \mathbf{Z}, \tag{2.9}$$

$$V_{(n)} = 0 \text{ for } n \text{ sufficiently small}, \tag{2.10}$$

equipped with a linear map $V \otimes V \to V[[x,x^{-1}]]$, or equivalently,

$$\begin{aligned} V &\to (\text{End } V)[[x,x^{-1}]] \\ v &\mapsto Y(v,x) = \sum_{n\in\mathbf{Z}} v_n x^{-n-1} \text{ (where } v_n \in \text{End } V), \end{aligned} \tag{2.11}$$

$Y(v,x)$ denoting the *vertex operator associated with* v, and equipped also with two distinguished homogeneous vectors $\mathbf{1} \in V_{(0)}$ (the *vacuum*) and $\omega \in V_{(2)}$. The following conditions are assumed for $u, v \in V$: the *lower truncation condition* holds:

$$u_n v = 0 \text{ for } n \text{ sufficiently large} \tag{2.12}$$

(or equivalently, $Y(u,x)v \in V((x))$);

$$Y(\mathbf{1},x) = 1 \text{ (1 on the right being the identity operator);} \tag{2.13}$$

the *creation property* holds:

$$((Y(v,x)\mathbf{1} \in V[[x]] \text{ and } \lim_{x\to 0} Y(v,x)\mathbf{1} = v \tag{2.14}$$

(that is, $Y(v,x)\mathbf{1}$ involves only nonnegative integral powers of x and the constant term is v); the *Jacobi identity* (the main axiom) holds:

$$x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right)Y(u,x_1)Y(v,x_2) - x_0^{-1}\delta\left(\frac{x_2-x_1}{-x_0}\right)Y(v,x_2)Y(u,x_1)$$
$$= x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right)Y(Y(u,x_0)v,x_2) \tag{2.15}$$

(note that when each expression in (2.15) is applied to any element of V, the coefficient of each monomial in the formal variables is a finite sum; on the right-hand side, the notation $Y(\cdot, x_2)$ is understood to be extended in the obvious way to $V[[x_0, x_0^{-1}]]$); the Virasoro algebra relations hold:

$$[L(m), L(n)] = (m-n)L(m+n) + \frac{1}{12}(m^3-m)\delta_{n+m,0}c \tag{2.16}$$

for $m, n \in \mathbf{Z}$, where

$$L(n) = \omega_{n+1} \text{ for } n \in \mathbf{Z}, \text{ i.e., } Y(\omega,x) = \sum_{n\in\mathbf{Z}} L(n)x^{-n-2} \tag{2.17}$$

and

$$c \in \mathbf{C}; \tag{2.18}$$

$$L(0)v = nv = (\text{wt } v)v \text{ for } n \in \mathbf{Z} \text{ and } v \in V_{(n)}; \tag{2.19}$$

$$\frac{d}{dx}Y(v,x) = Y(L(-1)v,x) \tag{2.20}$$

(the $L(-1)$*-derivative property*).

The vertex operator algebra just defined is denoted by $(V, Y, \mathbf{1}, \omega)$ (or simply by V). The complex number c is called the *central charge* or *rank* of V. Homomorphisms of vertex operator algebras are defined in the obvious way.

Remark 2.2 The axioms above imply that if $v \in V$ is homogeneous and $n \in \mathbf{Z}$,

$$\mathrm{wt}\ v_n = \mathrm{wt}\ v - n - 1 \tag{2.21}$$

as an operator. We shall also use the fact that in the presence of other axioms, the Virasoro algebra commutator relations (2.16) are equivalent to the relation

$$Y(\omega, x)\omega = \frac{1}{2}c\mathbf{1}x^{-4} + 2\omega x^{-2} + L(-1)\omega x^{-1} + v \tag{2.22}$$

where $v \in V[[x]]$.

Vertex operator algebras have important "rationality," "commutativity" and "associativity" properties, collectively called "duality" properties. These properties can in fact be used as axioms replacing the Jacobi identity in the definition of vertex operator algebra, as we now recall.

In the propositions below, $\mathbf{C}[x_1, x_2]_S$ is the ring of rational functions obtained by inverting (localizing with respect to) the products of (zero or more) elements of the set S of nonzero homogeneous linear polynomials in x_1 and x_2. Also, ι_{12} (which might also be written as $\iota_{x_1x_2}$) is the operation of expanding an element of $\mathbf{C}[x_1, x_2]_S$, that is, a polynomial in x_1 and x_2 divided by a product of homogeneous linear polynomials in x_1 and x_2, as a formal series containing at most finitely many negative powers of x_2 (using binomial expansions for negative powers of linear polynomials involving both x_1 and x_2); similarly for ι_{21} and so on. (The distinction between rational functions and formal Laurent series is crucial.)

For any $\mathbf{Z}$-graded, or more generally, $\mathbf{C}$-graded, vector space $W = \coprod W_{(n)}$, we use the notation

$$W' = \coprod W^*_{(n)} \tag{2.23}$$

for its graded dual.

Proposition 2.3 (a) (rationality of products) *For v, v_1, $v_2 \in V$ and $v' \in V'$, the formal series $\langle v', Y(v_1, x_1)Y(v_2, x_2)v\rangle$, which involves only finitely* *many negative powers of* x_2 *and only finitely many positive powers of* x_1, *lies in the image of the map* ι_{12}:

$$\langle v', Y(v_1,x_1)Y(v_2,x_2)v\rangle = \iota_{12} f(x_1,x_2), \tag{2.24}$$

where the (uniquely determined) element $f \in \mathbf{C}[x_1,x_2]_S$ *is of the form*

$$f(x_1,x_2) = \frac{g(x_1,x_2)}{x_1^r x_2^s (x_1-x_2)^t} \tag{2.25}$$

for some $g \in \mathbf{C}[x_1,x_2]$ *and* $r,s,t \in \mathbf{Z}$.

(b) (commutativity) *We also have*

$$\langle v', Y(v_2,x_2)Y(v_1,x_1)v\rangle = \iota_{21} f(x_1,x_2). \qquad \square \tag{2.26}$$

Proposition 2.4 (a) (rationality of iterates) *For* v, v_1, $v_2 \in V$ *and* $v' \in V'$, *the formal series* $\langle v', Y(Y(v_1,x_0)v_2,x_2)v\rangle$, *which involves only finitely many negative powers of* x_0 *and only finitely many positive powers of* x_2, *lies in the image of the map* ι_{20}:

$$\langle v', Y(Y(v_1,x_0)v_2,x_2)v\rangle = \iota_{20} h(x_0,x_2), \tag{2.27}$$

where the (uniquely determined) element $h \in \mathbf{C}[x_0,x_2]_S$ *is of the form*

$$h(x_0,x_2) = \frac{k(x_0,x_2)}{x_0^r x_2^s (x_0+x_2)^t} \tag{2.28}$$

for some $k \in \mathbf{C}[x_0,x_2]$ *and* $r,s,t \in \mathbf{Z}$.

(b) *The formal series* $\langle v', Y(v_1,x_0+x_2)Y(v_2,x_2)v\rangle$, *which involves only finitely many negative powers of* x_2 *and only finitely many positive powers of* x_0, *lies in the image of* ι_{02}, *and in fact*

$$\langle v', Y(v_1,x_0+x_2)Y(v_2,x_2)v\rangle = \iota_{02} h(x_0,x_2). \qquad \square \tag{2.29}$$

Proposition 2.5 (associativity) *We have the following equality of rational functions:*

$$\iota_{12}^{-1}\langle v', Y(v_1,x_1)Y(v_2,x_2)v\rangle = \left(\iota_{20}^{-1}\langle v', Y(Y(v_1,x_0)v_2,x_2)v\rangle\right)\Big|_{x_0=x_1-x_2}. \qquad \square \tag{2.30}$$

Proposition 2.6 *In the presence of the other axioms, the Jacobi identity follows from the rationality of products and iterates, and commutativity and associativity. In particular, in the definition of vertex operator algebra, the Jacobi identity may be replaced by these properties.* □

We have the following notions of module and of intertwining operator for vertex operator algebras:

Definition 2.7 Given a vertex operator algebra $(V, Y, \mathbf{1}, \omega)$, a *module for V* (or *V-module* or *representation space*) is a **C**-graded vector space (graded by *weights*)

$$W = \coprod_{n \in \mathbf{C}} W_{(n)}; \text{ for } w \in W_{(n)}, \ n = \text{wt } w; \tag{2.31}$$

such that

$$\dim W_{(n)} < \infty \text{ for } n \in \mathbf{C}, \tag{2.32}$$

$$W_{(n)} = 0 \text{ for } n \text{ whose real part is sufficiently small,} \tag{2.33}$$

equipped with a linear map $V \otimes W \to W[[x, x^{-1}]]$, or equivalently,

$$\begin{aligned} V &\to (\text{End } W)[[x, x^{-1}]] \\ v &\mapsto Y(v, x) = \sum_{n \in \mathbf{Z}} v_n x^{-n-1} \quad (\text{where } v_n \in \text{End } W) \end{aligned} \tag{2.34}$$

(note that the sum is over **Z**, not **C**), $Y(v, x)$ denoting the *vertex operator associated with v*, such that "all the defining properties of a vertex operator algebra that make sense hold." That is, for $u, v \in V$ and $w \in W$,

$$v_n w = 0 \text{ for } n \text{ sufficiently large} \tag{2.35}$$

(the lower truncation condition);

$$Y(\mathbf{1}, z) = 1; \tag{2.36}$$

$$\begin{aligned} x_0^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right) Y(u, x_1)Y(v, x_2) - x_0^{-1}\delta\left(\frac{x_2 - x_1}{-x_0}\right) Y(v, x_2)Y(u, x_1) \\ = x_2^{-1}\delta\left(\frac{x_1 - x_0}{x_2}\right) Y(Y(u, x_0)v, x_2) \end{aligned} \tag{2.37}$$

(the Jacobi identity for operators on W); note that on the right-hand side, $Y(u, x_0)$ is the operator associated with V; the Virasoro algebra relations hold on W with scalar c equal to the central charge of V:

$$[L(m), L(n)] = (m-n)L(m+n) + \frac{1}{12}(m^3 - m)\delta_{m+n,0}c \tag{2.38}$$

for $m, n \in \mathbf{Z}$, where

$$L(n) = \omega_{n+1} \text{ for } n \in \mathbf{Z}, \text{ i.e., } Y(\omega, z) = \sum_{n\in\mathbf{Z}} L(n)x^{-n-2}; \tag{2.39}$$

$$L(0)w = nw = (\text{wt } w)w \text{ for } n \in \mathbf{C} \text{ and } w \in W_{(n)}; \tag{2.40}$$

$$\frac{d}{dx}Y(v, x) = Y(L(-1)v, x), \tag{2.41}$$

where $L(-1)$ is the operator on V.

This completes the definition of module. We may denote the module just defined by (W, Y) (or simply by W). If necessary, we shall use Y_W or similar notation to indicate that the vertex operators concerned act on W. Homomorphisms (or maps) of V-modules are defined in the obvious way. For V-modules W_1 and W_2, we shall denote the space of module maps from W_1 to W_2 by $\mathrm{Hom}_V(W_1, W_2)$.

Remark 2.8 Formula (2.21) holds for modules. Also note that the Virasoro algebra commutator relations (2.38) are in fact consequences of the other axioms, in view of (2.22).

For any vector space W and any formal variable x, we use the notation

$$W\{x\} = \left\{\sum_{n\in\mathbf{C}} a_n x^n \middle| a_n \in W,\ n \in \mathbf{C}\right\}. \tag{2.42}$$

In particular, we shall allow complex powers of our commuting formal variables.

Definition 2.9 Let V be a vertex operator algebra and let (W_1, Y_1), (W_2, Y_2) and (W_3, Y_3) be three V-modules (not necessarily distinct, and possibly equal to V). An *intertwining operator of type* $\binom{W_3}{W_1\,W_2}$ is a linear map $W_1 \otimes W_2 \to W_3\{x\}$, or equivalently,

$$\begin{array}{rcl} W_1 & \to & (\mathrm{Hom}(W_2, W_3))\{x\} \\ w & \mapsto & \mathcal{Y}(w, x) = \sum_{n \in \mathbb{C}} w_n x^{-n-1} \quad (\text{where } w_n \in \mathrm{Hom}(W_2, W_3)) \end{array} \tag{2.43}$$

such that "all the defining properties of a module action that make sense hold." That is, for $v \in V$, $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$, we have the lower truncation condition

$$(w_{(1)})_n w_{(2)} = 0 \text{ for } n \text{ whose real part is sufficiently large;} \tag{2.44}$$

the following Jacobi identity holds for the operators $Y_1(v, \cdot)$, $Y_2(v, \cdot)$, $Y_3(v, \cdot)$ and $\mathcal{Y}(\cdot, x_2)$ acting on the element $w_{(2)}$:

$$\begin{aligned} x_0^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right) & Y_3(v, x_1)\mathcal{Y}(w_{(1)}, x_2)w_{(2)} \\ & - x_0^{-1}\delta\left(\frac{x_2 - x_1}{-x_0}\right)\mathcal{Y}(w_{(1)}, x_2)Y_2(v, x_1)w_{(2)} \\ & = x_2^{-1}\delta\left(\frac{x_1 - x_0}{x_2}\right)\mathcal{Y}(Y_1(v, x_0)w_{(1)}, x_2)w_{(2)} \end{aligned} \tag{2.45}$$

(note that the first term on the left-hand side is algebraically meaningful because of the condition (2.44), and the other terms are meaningful by the usual properties of modules; also note that this Jacobi identity involves integral powers of x_0 and x_1 and complex powers of x_2);

$$\frac{d}{dx}\mathcal{Y}(w_{(1)}, x) = \mathcal{Y}(L(-1)w_{(1)}, x), \tag{2.46}$$

where $L(-1)$ is the operator acting on W_1.

The intertwining operators of the same type $\binom{W_3}{W_1\,W_2}$ form a vector space, which we denote by $\mathcal{V}^{W_3}_{W_1 W_2}$. The dimension of this vector space is called the *fusion rule* for W_1, W_2 and W_3 and is denoted by $N^{W_3}_{W_1 W_2}$ ($\leq \infty$). Formula (2.21) holds for intertwining operators, with v_n replaced by w_n ($n \in \mathbb{C}$).

There are also duality properties for modules and intertwining operators. See [FHL] and [DL] for details.

Let (W, Y), with

$$W = \coprod_{n\in\mathbf{C}} W_{(n)}, \tag{2.47}$$

be a module for a vertex operator algebra $(V, Y, \mathbf{1}, \omega)$, and consider its graded dual space W' (recall (2.23)). We define the *contragredient vertex operators* (called "adjoint vertex operators" in [FHL]) $Y'(v, x)$ $(v \in V)$ by means of the linear map

$$\begin{aligned} V &\to (\mathrm{End}\, W')[[x, x^{-1}]] \\ v &\mapsto Y'(v, x) = \sum_{n\in\mathbf{Z}} v'_n x^{-n-1} \quad (\text{where } v'_n \in \mathrm{End}\, W'), \end{aligned} \tag{2.48}$$

determined by the condition

$$\langle Y'(v, x)w', w\rangle = \langle w', Y(e^{xL(1)}(-x^{-2})^{L(0)}v, x^{-1})w\rangle \tag{2.49}$$

for $v \in V$, $w' \in W'$, $w \in W$. The operator $(-x^{-2})^{L(0)}$ has the obvious meaning; it acts on a vector of weight $n \in \mathbf{Z}$ as multiplication by $(-x^{-2})^n$. Also note that $e^{xL(1)}(-x^{-2})^{L(0)}v$ involves only finitely many (integral) powers of x, that the right-hand side of (2.49) is a Laurent polynomial in x, and that the components v'_n of the formal Laurent series $Y'(v, x)$ indeed preserve W'.

We give the space W' a $\mathbf{C}$-grading by setting

$$W'_{(n)} = W^*_{(n)} \quad \text{for } n \in \mathbf{C} \tag{2.50}$$

(cf. (2.23)). The following theorem defines the *V-module W' contragredient to W* (see [FHL], Theorem 5.2.1 and Proposition 5.3.1):

Theorem 2.10 *The pair (W', Y') carries the structure of a V-module and* $(W'', Y'') = (W, Y)$. □

Given a module map $\eta : W_1 \to W_2$, there is a unique module map $\eta' : W'_2 \to W'_1$, the *adjoint* map, such that

$$\langle \eta'(w'_{(2)}), w_{(1)}\rangle = \langle w'_{(2)}, \eta(w_{(1)})\rangle \tag{2.51}$$

for $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$. (Here the pairings $\langle \cdot, \cdot \rangle$ on the two sides refer to two different modules.) Note that

$$\eta'' = \eta. \tag{2.52}$$

In the construction of the tensor product module of two modules for a vertex operator algebra, we shall need the following generalization of the notion of module recalled above:

Definition 2.11 A *generalized V-module* is a $\mathbf{C}$-graded vector space W equipped with a linear map of the form (2.34) satisfying all the axioms for a V-module except that the homogeneous subspaces need not be finite-dimensional and that they need not be zero even for n whose real part is sufficiently small; that is, we omit (2.32) and (2.33) from the definition.

3 Affinizations of vertex operator algebras and the $*$-operation

In order to use the Jacobi identity to construct a tensor product of modules for a vertex operator algebra, we shall study various "affinizations" of a vertex operator algebra with respect to certain algebras and vector spaces of formal Laurent series and formal rational functions.

Let V be a vertex operator algebra and W a V-module. We adjoin the formal variable t to our list of commuting formal variables. This variable will play a special role. Consider the vector spaces

$$V[t,t^{-1}] = V \otimes \mathbf{C}[t,t^{-1}] \subset V \otimes \mathbf{C}((t)) \subset V \otimes \mathbf{C}[[t,t^{-1}]] \subset V[[t,t^{-1}]]$$

(note carefully the distinction between the last two, since V is typically infinite-dimensional) and $W \otimes \mathbf{C}\{t\} \subset W\{t\}$ (recall (2.42)). The linear map

$$\begin{array}{rcl} \tau_W : V[t,t^{-1}] & \to & \mathrm{End}\, W \\ v \otimes t^n & \mapsto & v_n \end{array} \tag{3.1}$$

($v \in V$, $n \in \mathbf{Z}$) extends canonically to

$$\begin{array}{rrcl} \tau_W : & V \otimes \mathbf{C}((t)) & \to & \mathrm{End}\, W \\ & v \otimes \sum_{n>N} a_n t^n & \mapsto & \sum_{n>N} a_n v_n . \end{array} \tag{3.2}$$

(but not to $V((t))$), in view of (2.21). It further extends canonically to

$$\tau_W : (V \otimes \mathbf{C}((t)))[[x,x^{-1}]] \to (\mathrm{End}\, W)[[x,x^{-1}]], \tag{3.3}$$

where of course $(V \otimes \mathbf{C}((t)))[[x,x^{-1}]]$ can be viewed as the subspace of $V[[t,t^{-1},x,x^{-1}]]$ such that the coefficient of each power of x lies in $V \otimes \mathbf{C}((t))$.

Let $v \in V$ and define the "generic vertex operator"

$$Y_t(v,x) = \sum_{n\in\mathbf{Z}} (v\otimes t^n)x^{-n-1} \in (V\otimes \mathbf{C}[t,t^{-1}])[[x,x^{-1}]]. \tag{3.4}$$

Then

$$\begin{aligned} Y_t(v,x) &= v\otimes x^{-1}\delta\left(\frac{t}{x}\right) \\ &= v\otimes t^{-1}\delta\left(\frac{x}{t}\right) \\ &\in V\otimes \mathbf{C}[[t,t^{-1},x,x^{-1}]] \\ &(\subset V[[t,t^{-1},x,x^{-1}]]) \end{aligned} \tag{3.5}$$

and the linear map

$$\begin{aligned} V &\to V\otimes \mathbf{C}[[t,t^{-1},x,x^{-1}]] \\ v &\mapsto Y_t(v,x) \end{aligned} \tag{3.6}$$

is simply the map given by tensoring by the "universal element" $x^{-1}\delta\left(\frac{t}{x}\right)$. We have

$$\tau_W(Y_t(v,x)) = Y_W(v,x). \tag{3.7}$$

For all $f(x) \in \mathbf{C}[[x,x^{-1}]]$, $f(x)Y_t(v,x)$ is defined and

$$f(x)Y_t(v,x) = f(t)Y_t(v,x). \tag{3.8}$$

In case $f(x) \in \mathbf{C}((x))$, then $\tau_W(f(x)Y_t(v,x))$ is also defined, and

$$f(x)Y_W(v,x) = f(x)\tau_W(Y_t(v,x)) = \tau_W(f(x)Y_t(v,x)) = \tau_W(f(t)Y_t(v,x)). \tag{3.9}$$

The expansion coefficients, in powers of x, of $Y_t(v,x)$ span $v\otimes\mathbf{C}[t,t^{-1}]$, the x-expansion coefficients of $Y_W(v,x)$ span $\tau_W(v\otimes\mathbf{C}[t,t^{-1}])$ and the x-expansion coefficients of $f(x)Y_t(v,x)$ span $v\otimes f(t)\mathbf{C}[t,t^{-1}]$. In case $f(x)\in\mathbf{C}((x))$, the x-expansion coefficients of $f(x)Y_W(v,x)$ span $\tau_W(v\otimes f(t)\mathbf{C}[t,t^{-1}])$.

Using this viewpoint, we shall examine each of the three terms in the Jacobi identity (2.45) in the definition of intertwining operator. First we consider the formal Laurent series in x_0, x_1, x_2 and t given by

$$\begin{aligned} x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right)Y_t(v,x_0) &= x_1^{-1}\delta\left(\frac{x_2+x_0}{x_1}\right)Y_t(v,x_0) \\ &= v\otimes x_1^{-1}\delta\left(\frac{x_2+t}{x_1}\right)x_0^{-1}\delta\left(\frac{t}{x_0}\right) \end{aligned} \tag{3.10}$$

(cf. the right-hand side of (2.45)). The expansion coefficients in powers of x_0, x_1 and x_2 of (3.10) span just the space $v \otimes \mathbf{C}[t,t^{-1}]$. However, the expansion coefficients in x_0 and x_1 only (but not in x_2) of

$$\begin{aligned} x_1^{-1}\delta\left(\frac{x_2+x_0}{x_1}\right) & Y_t(v,x_0) = \\ &= v \otimes x_1^{-1}\delta\left(\frac{x_2+t}{x_1}\right) x_0^{-1}\delta\left(\frac{t}{x_0}\right) \\ &= v \otimes \left(\sum_{m\in\mathbf{Z}} (x_2+t)^m x_1^{-m-1}\right)\left(\sum_{n\in\mathbf{Z}} t^n x_0^{-n-1}\right) \end{aligned} \tag{3.11}$$

span

$$v \otimes \iota_{x_2,t}\mathbf{C}[t,t^{-1},x_2+t,(x_2+t)^{-1}] \subset v \otimes \mathbf{C}[x_2,x_2^{-1}]((t)),$$

where $\iota_{x_2,t}$ is the operation of expanding a formal rational function in the indicated algebra as a formal Laurent series involving only finitely many negative powers of t (cf. the notation ι_{12}, etc., above). We shall use similar ι-notations below. Specifically, the coefficient of $x_0^{-n-1}x_1^{-m-1}$ $(m,n \in \mathbf{Z})$ in (3.11) is $v \otimes (x_2+t)^m t^n$.

We may specialize $x_2 \mapsto z \in \mathbf{C}^\times$, and (3.11) becomes

$$\begin{aligned} z^{-1}\delta\left(\frac{x_1-x_0}{z}\right) & Y_t(v,x_0) = \\ &= x_1^{-1}\delta\left(\frac{z+x_0}{x_1}\right) Y_t(v,x_0) \\ &= v \otimes x_1^{-1}\delta\left(\frac{z+t}{x_1}\right) x_0^{-1}\delta\left(\frac{t}{x_0}\right) \\ &= v \otimes \left(\sum_{m\in\mathbf{Z}} (z+t)^m x_1^{-m-1}\right)\left(\sum_{n\in\mathbf{Z}} t^n x_0^{-n-1}\right). \end{aligned} \tag{3.12}$$

The coefficient of $x_0^{-n-1}x_1^{-m-1}$ $(m,n \in \mathbf{Z})$ in (3.12) is $v \otimes (z+t)^m t^n \in V \otimes \mathbf{C}((t))$, and these coefficients span

$$v \otimes \mathbf{C}[t,t^{-1},(z+t)^{-1}] \subset v \otimes \mathbf{C}((t)). \tag{3.13}$$

Our tensor product construction will be based on a certain action of the space $V \otimes \mathbf{C}[t,t^{-1},(z+t)^{-1}]$, and the description of this space as the span of the coefficients of the expression (3.12) (as $v \in V$ varies) will be very useful.

Now consider

$$\begin{aligned}
&x_0^{-1}\delta\left(\frac{-x_2+x_1}{x_0}\right)Y_t(v,x_1) = \\
&\quad = v\otimes x_0^{-1}\delta\left(\frac{-x_2+t}{x_0}\right)x_1^{-1}\delta\left(\frac{t}{x_1}\right) \\
&\quad = v\otimes\Big(\sum_{n\in\mathbf{Z}}(-x_2+t)^n x_0^{-n-1}\Big)\Big(\sum_{m\in\mathbf{Z}}t^m x_1^{-m-1}\Big)
\end{aligned} \tag{3.14}$$

(cf. the second term on the left-hand side of (2.45). The expansion coefficients in powers of x_0 and x_1 (but not x_2) span

$$v\otimes\iota_{x_2,t}\mathbf{C}[t,t^{-1},-x_2+t,(-x_2+t)^{-1}],$$

and in fact the coefficient of $x_0^{-n-1}x_1^{-m-1}$ $(m,n\in\mathbf{Z})$ in (3.14) is $v\otimes(-x_2+t)^n t^m$. Again specializing $x_2\mapsto z\in\mathbf{C}^\times$, we obtain

$$\begin{aligned}
&x_0^{-1}\delta\left(\frac{-z+x_1}{x_0}\right)Y_t(v,x_1) = \\
&\quad = v\otimes x_0^{-1}\delta\left(\frac{-z+t}{x_0}\right)x_1^{-1}\delta\left(\frac{t}{x_1}\right) \\
&\quad = v\otimes\Big(\sum_{n\in\mathbf{Z}}(-z+t)^n x_0^{-n-1}\Big)\Big(\sum_{m\in\mathbf{Z}}t^m x_1^{-m-1}\Big).
\end{aligned} \tag{3.15}$$

The coefficient of $x_0^{-n-1}x_1^{-m-1}$ $(m,n\in\mathbf{Z})$ in (3.15) is $v\otimes(-z+t)^n t^m$, and these coefficients span

$$v\otimes\mathbf{C}[t,t^{-1},(-z+t)^{-1}]\subset v\otimes\mathbf{C}((t)). \tag{3.16}$$

Finally, consider

$$\begin{aligned}
&x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right)Y_t(v,x_1) = \\
&\quad = v\otimes x_0^{-1}\delta\left(\frac{t-x_2}{x_0}\right)x_1^{-1}\delta\left(\frac{t}{x_1}\right).
\end{aligned} \tag{3.17}$$

The coefficient of $x_0^{-n-1}x_1^{-m-1}$ $(m,n\in\mathbf{Z})$ is $v\otimes(t-x_2)^n t^m$, and these expansion cofficients span

$$v\otimes\iota_{t,x_2}\mathbf{C}[t,t^{-1},t-x_2,(t-x_2)^{-1}].$$

If we again specialize $x_2 \mapsto z$, we get

$$x_0^{-1}\delta\left(\frac{x_1 - z}{x_0}\right) Y_t(v, x_1) = v \otimes x_0^{-1}\delta\left(\frac{t - z}{x_0}\right) x_1^{-1}\delta\left(\frac{t}{x_1}\right), \tag{3.18}$$

whose coefficient of $x_0^{-n-1}x_1^{-m-1}$ is $v \otimes (t-z)^n t^m$. These coefficients span

$$v \otimes \mathbf{C}[t, t^{-1}, (t-z)^{-1}] \subset v \otimes \mathbf{C}((t^{-1})) \tag{3.19}$$

(cf. (3.13), (3.16)).

Later we shall evaluate the identity (2.45) on the elements of the contragredient module W_3'. This will allow us to convert the expansion (3.19) into an expansion in positive powers of t. It will be useful to examine the notion of contragredient vertex operator ((2.48), (2.49)) more closely. For a V-module W, we define the *opposite vertex operator* associated to $v \in V$ by

$$Y_W^*(v, x) = Y_W(e^{xL(1)}(-x^{-2})^{L(0)}v, x^{-1}) \tag{3.20}$$

and we define its components by:

$$Y_W^*(v, x) = \sum_{n \in \mathbf{Z}} v_n^* x^{-n-1}. \tag{3.21}$$

Then $v_n^* \in \operatorname{End} W$ and $v \mapsto Y_W^*(v, x)$ is a linear map $V \to (\operatorname{End} W)[[x, x^{-1}]]$ such that $V \otimes W \to W((x^{-1}))$ $(v \otimes w \mapsto Y_W^*(v, x)w)$. Note that the contragredient vertex operators are the adjoints of the opposite vertex operators:

$$\langle w', Y_W^*(v, x)w\rangle = \langle Y'(v, x)w', w\rangle \tag{3.22}$$

and that if v is homogeneous, the weight of the operator v_n^* is $n + 1 - \operatorname{wt} v$, by (2.21). The proof of Theorem 5.2.1 in [FHL], which asserts that (W', Y') is a V-module, in fact proves the following *opposite Jacobi identity* for Y_W^*:

$$\begin{aligned} x_0^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right) & Y_W^*(v_2, x_2) Y_W^*(v_1, x_1) \\ & - x_0^{-1}\delta\left(\frac{x_2 - x_1}{-x_0}\right) Y_W^*(v_1, x_1) Y_W^*(v_2, x_2) \\ = x_2^{-1}\delta\left(\frac{x_1 - x_0}{x_2}\right) & Y_W^*(Y(v_1, x_0)v_2, x_2) \end{aligned} \tag{3.23}$$

(which of course also follows from the assertion that (W', Y') is a V-module). The pair (W, Y^*) should be thought of as a "right module" for V.

We shall interpret the operator Y_W^* by means of a $*$-operation on $V \otimes \mathbf{C}[[t, t^{-1}]]$. This operation will be an involution. We proceed as follows: First we generalize Y^* in the following way: Given any vector space U and any linear map

$$\begin{aligned} Z(\cdot, x) : V &\to U[[x, x^{-1}]] \ \left(= \prod_{n \in \mathbf{Z}} U \otimes x^n\right) \\ v &\mapsto Z(v, x) \end{aligned} \tag{3.24}$$

from V into $U[[x, x^{-1}]]$ (i.e., given any family of linear maps from V into the spaces $U \otimes x^n$), we define $Z^*(\cdot, x) : V \to U[[x, x^{-1}]]$ by

$$Z^*(v, x) = Z(e^{xL(1)}(-x^{-2})^{L(0)}v, x^{-1}), \tag{3.25}$$

where we use the obvious linear map $Z(\cdot, x^{-1}) : V \to U[[x, x^{-1}]]$, and where we extend $Z(\cdot, x^{-1})$ canonically to a linear map $Z(\cdot, x^{-1}) : V[x, x^{-1}] \to U[[x, x^{-1}]]$. Then by formula (5.3.1) in [FHL] (the proof of Proposition 5.3.1), we have

$$\begin{aligned} Z^{**}(v, x) &= Z^*(e^{xL(1)}(-x^{-2})^{L(0)}v, x^{-1}) \\ &= Z(e^{x^{-1}L(1)}(-x^2)^{L(0)}e^{xL(1)}(-x^{-2})^{L(0)}v, x) \\ &= Z(v, x). \end{aligned} \tag{3.26}$$

That is,

$$Z^{**}(\cdot, x) = Z(\cdot, x). \tag{3.27}$$

Moreover, if $Z(v, x) \in U((x))$, then $Z^*(v, x) \in U((x^{-1}))$ and vice versa.

Now we expand $Z(v, x)$ and $Z^*(v, x)$ in components. Write

$$Z(v, z) = \sum_{n \in \mathbf{Z}} v_{(n)} x^{-n-1}, \tag{3.28}$$

where for all $n \in \mathbf{Z}$,

$$\begin{aligned} V &\to U \\ v &\mapsto v_{(n)} \end{aligned} \tag{3.29}$$

is a linear map depending on $Z(\cdot,x)$ (and in fact, as $Z(\cdot,x)$ varies, these linear maps are arbitrary). Also write

$$Z^*(v,x)=\sum_{n\in\mathbf{Z}} v^*_{(n)}x^{-n-1} \tag{3.30}$$

where

$$\begin{aligned} V &\to U \\ v &\mapsto v^*_{(n)} \end{aligned} \tag{3.31}$$

is a linear map depending on $Z(\cdot,x)$. We shall compute $v^*_{(n)}$. First note that

$$\sum_{n\in\mathbf{Z}} v^*_{(n)}x^{-n-1}=\sum_{n\in\mathbf{Z}}(e^{zL(1)}(-x^{-2})^{L(0)}v)_{(n)}x^{n+1}. \tag{3.32}$$

For convenience, suppose that $v\in V_{(h)}$, for $h\in\mathbf{Z}$. Then the right-hand side of (3.32) is equal to

$$\begin{aligned} &(-1)^h\sum_{n\in\mathbf{Z}}(e^{zL(1)}v)_{(-n)}x^{-n+1-2h} \\ &=(-1)^h\sum_{n\in\mathbf{Z}}\sum_{m\in\mathbf{N}}\frac{1}{m!}(L(1)^m v)_{(-n)}x^{m-n+1-2h} \\ &=(-1)^h\sum_{m\in\mathbf{N}}\frac{1}{m!}\sum_{n\in\mathbf{Z}}(L(1)^m v)_{(-n-m-2+2h)}x^{-n-1}, \end{aligned} \tag{3.33}$$

that is,

$$v^*_{(n)}=(-1)^h\sum_{m\in\mathbf{N}}\frac{1}{m!}(L(1)^m v)_{(-n-m-2+2h)}. \tag{3.34}$$

For $v\in V$ not necessarily homogeneous, $v^*_{(n)}$ is given by the appropriate sum of such expressions.

Now consider the special case where $U=V\otimes\mathbf{C}[t,t^{-1}]$ and where $Z(\cdot,x)$ is the "generic" linear map

$$\begin{aligned} Y_t(\cdot,x):V &\to (V\otimes\mathbf{C}[t,t^{-1}])[[x,x^{-1}]] \\ v &\mapsto Y_t(v,x)=\sum_{n\in\mathbf{Z}}(v\otimes t^n)x^{-n-1} \end{aligned} \tag{3.35}$$

(recall (3.4)), i.e.,

$$v_{(n)}=v\otimes t^n. \tag{3.36}$$

Then for $v \in V_{(h)}$,

$$v^*_{(n)} = (-1)^h \sum_{m \in \mathbf{N}} \frac{1}{m!}((L(1))^m v) \otimes t^{-n-m-2+2h} \tag{3.37}$$

in this case.

This motivates defining a $*$-operation on $V \otimes \mathbf{C}[t, t^{-1}]$ as follows: For any $n, h \in \mathbf{Z}$ and $v \in V_{(h)}$, define

$$(v \otimes t^n)^* = (-1)^h \sum_{m \in \mathbf{N}} \frac{1}{m!}(L(1)^m v) \otimes t^{-n-m-2+2h} \in V \otimes \mathbf{C}[t, t^{-1}], \tag{3.38}$$

and extend by linearity to $V \otimes \mathbf{C}[t, t^{-1}]$. That is, $(v \otimes t^n)^* = v^*_{(n)}$ for the special case $Z(\cdot, x) = Y_t(\cdot, x)$ discussed above. (Note that for general Z, we cannot expect to be able to define an analogous $*$-operation on U.) Also consider the map

$$\begin{array}{rcl} Y_t^*(\cdot, x) = (Y_t(\cdot, x))^* : V & \to & (V \otimes \mathbf{C}[t, t^{-1}])[[x, x^{-1}]] \\ v & \mapsto & Y_t^*(v, x) = \sum_{n \in \mathbf{Z}} (v \otimes t^n)^* x^{-n-1} \end{array} \tag{3.39}$$

Then for general $Z(\cdot, x)$ as above, we can define a linear map

$$\begin{array}{rcl} \varepsilon_Z : V \otimes \mathbf{C}[t, t^{-1}] & \to & U \\ v \otimes t^n & \mapsto & v_{(n)} \end{array} \tag{3.40}$$

("evaluation with respect to Z"), i.e.,

$$\varepsilon_Z : Y_t(v, x) \mapsto Z(v, x), \tag{3.41}$$

and a linear map

$$\begin{array}{rcl} \varepsilon_Z^* : V \otimes \mathbf{C}[t, t^{-1}] & \to & U \\ v \otimes t^n & \mapsto & v^*_{(n)}, \end{array} \tag{3.42}$$

i.e.,

$$\varepsilon_Z^* : Y_t(v, x) \mapsto Z^*(v, x). \tag{3.43}$$

Then

$$\varepsilon_Z^* = \varepsilon_Z \circ *, \tag{3.44}$$

that is,

$$\varepsilon_Z(Y_t^*(v,x)) = Z^*(v,x), \tag{3.45}$$

or equivalently, the diagram

$$\begin{array}{ccc} Y_t(v,x) & \stackrel{\varepsilon_Z}{\longmapsto} & Z(v,x) \\ *\downarrow & & \downarrow (Z(\cdot,x) \mapsto Z^*(\cdot,x)) \\ Y_t^*(v,x) & \stackrel{\varepsilon_Z}{\longmapsto} & Z^*(v,x) \end{array} \tag{3.46}$$

commutes. Note that the components $v^*_{(n)}$ of $Z^*(v,x)$ depend on all the components $v_{(n)}$ of $Z(v,z)$ (for arbitrary v), whereas the component $(v\otimes t^n)^*$ of $Y_t^*(v,z)$ can be defined generically and abstractly; $(v\otimes t^n)^*$ depends linearly on $v \in V$ alone.

Since in general $Z^{**}(v,x) = Z(v,x)$, we know that

$$Y_t^{**}(v,x) = Y_t(v,x) \tag{3.47}$$

as a special case, and in particular (and equivalently),

$$(v\otimes t^n)^{**} = v \otimes t^n \tag{3.48}$$

for all $v \in V$ and $n \in \mathbf{Z}$. Thus $*$ is an involution of $V \otimes \mathbf{C}[t,t^{-1}]$.

Furthermore, the involution $*$ of $V \otimes \mathbf{C}[t,t^{-1}]$ extends canonically to a linear map

$$V \otimes \mathbf{C}[[t,t^{-1}]] \xrightarrow{*} V \otimes \mathbf{C}[[t,t^{-1}]].$$

In fact, consider the restriction of $*$ to $V = V \otimes t^0$:

$$\begin{array}{rcl} V & \xrightarrow{*} & V \otimes \mathbf{C}[t,t^{-1}] \\ v & \mapsto & v^* = (-1)^h \displaystyle\sum_{m\in\mathbf{N}} \frac{1}{m!}(L(1)^m v) \otimes t^{-m-2+2h}, \end{array} \tag{3.49}$$

extended by linearity from $V_{(h)}$ to V. Then for $v \in V$ and $n \in \mathbf{Z}$,

$$(v\otimes t^n)^* = v^* t^{-n}, \tag{3.50}$$

and it is clear that $*$ extends to $V \otimes \mathbf{C}[[t,t^{-1}]]$: For $f(t) \in \mathbf{C}[[t,t^{-1}]]$,

$$(v \otimes f(t))^* = v^* f(t^{-1}). \tag{3.51}$$

To see that $*$ is an involution of this larger space, first note that

$$v^{**} = v \tag{3.52}$$

(although $v^* \notin V$ in general). (This could of course alternatively be proved by direct calculation using formula (3.38).) Also, for $g(t) \in \mathbf{C}[t, t^{-1}]$ and $f(t) \in \mathbf{C}[[t, t^{-1}]]$,

$$(v \otimes g(t) f(t))^* = v^* g(t^{-1}) f(t^{-1}) = (v \otimes g(t))^* f(t^{-1}). \tag{3.53}$$

Thus for all $x \in V \otimes \mathbf{C}[t, t^{-1}]$ and $f(t) \in \mathbf{C}[[t, t^{-1}]]$,

$$(x f(t))^* = x^* f(t^{-1}). \tag{3.54}$$

It follows that

$$\begin{aligned} (v \otimes f(t))^{**} &= (v^* f(t^{-1}))^* \\ &= v^{**} f(t) \\ &= v f(t) \\ &= v \otimes f(t), \end{aligned} \tag{3.55}$$

and we have shown that $*$ is an involution of $V \otimes \mathbf{C}[[t, t^{-1}]]$. We have

$$* : V \otimes \mathbf{C}((t)) \leftrightarrow V \otimes \mathbf{C}((t^{-1})). \tag{3.56}$$

Note that

$$\begin{aligned} Y_t^*(v, x) &= \sum_{n \in \mathbf{Z}} (v \otimes t^n)^* x^{-n-1} \\ &= v^* \sum_{n \in \mathbf{Z}} t^{-n} x^{-n-1} \\ &= v^* x^{-1} \delta(tx) \\ &= v^* t \delta(tx) \\ &\in V \otimes \mathbf{C}[[t, t^{-1}, x, x^{-1}]]. \end{aligned} \tag{3.57}$$

Thus the map $v \mapsto Y_t^*(v, x)$ is the linear map given by multiplying v^* by the "universal element" $t\delta(tx)$ (cf. the comment following (3.6)). For all $f(x) \in \mathbf{C}[[x, x^{-1}]]$, $f(x) Y_t^*(v, x)$ is defined and

$$\begin{aligned} f(x) Y_t^*(v, x) &= f(t^{-1}) Y_t^*(v, x) \\ &= v^* f(t^{-1}) t \delta(tx). \end{aligned} \tag{3.58}$$

Now we return to the starting point — the original special case: $U =$ End W and $Z(\cdot, z) = Y_W(\cdot, z) : V \to (\text{End } W)[[x, x^{-1}]]$. The corresponding map

$$\begin{aligned} \varepsilon_Z = \varepsilon_{Y_W} : V[t,t^{-1}] &\to \text{End } W \\ v \otimes t^n &\mapsto v_{(n)} \end{aligned} \tag{3.59}$$

(recall (3.40)) is just the map $\tau_W : v \otimes t^n \mapsto v_n$ (recall (3.1)), i.e., $v_{(n)} = v_n$ in this case. Recall that this map extends canonically to $V \otimes \mathbf{C}((t))$. The map ε_Z^* is $\tau_W \circ * : V \otimes \mathbf{C}[t, t^{-1}] \to \text{End } W$ and this map extends canonically to $V \otimes \mathbf{C}((t^{-1}))$. In addition to (3.7), we have

$$\tau_W(Y_t^*(v, z)) = Y_W^*(v, z) \tag{3.60}$$

($v_{(n)}^* = v_n^*$ in this case; recall (3.21)). In case $f(x) \in \mathbf{C}((x^{-1}))$,

$$f(x)Y_W^*(v, x) = \tau_W(f(x)Y_t^*(v, x))$$

is defined and is equal to $\tau_W(f(t^{-1})Y_t^*(v, z))$ (which is also defined).

The x-expansion coefficients of $f(x)Y_t^*(v, x)$, for $f(x) \in \mathbf{C}[[x, x^{-1}]]$, span

$$\begin{aligned} v^* f(t^{-1})\mathbf{C}[t, t^{-1}] &= (v\mathbf{C}[t, t^{-1}])^* f(t^{-1}) \\ &= (vf(t)\mathbf{C}[t, t^{-1}])^* \end{aligned} \tag{3.61}$$

The x-expansion coefficients of $Y_W^*(v, x)$ span

$$\begin{aligned} \tau_W(v^*\mathbf{C}[t, t^{-1}]) &= \tau_W((v \otimes \mathbf{C}[t, t^{-1}])^*) \\ &= \tau_W^*(v \otimes \mathbf{C}[t, t^{-1}]). \end{aligned} \tag{3.62}$$

In case $f(x) \in \mathbf{C}((x^{-1}))$, the x-expansion coefficients of $f(x)Y_W^*(v, x)$ span $\tau_W(v^* f(t^{-1})\mathbf{C}[t, t^{-1}]) = \tau_W^*(vf(t)\mathbf{C}[t, t^{-1}])$. (Cf. the comments after (3.9).)

Our action of the space $V \otimes \mathbf{C}[t, t^{-1}, (z+t)^{-1}]$ will be based on certain translation operations and on the $*$-operation. More precisely, it is the space $V \otimes \iota_+\mathbf{C}[t, t^{-1}, (z+t)^{-1}]$ whose action we shall define, where we use the notations

$$\begin{aligned} \iota_+ : \mathbf{C}(t) &\hookrightarrow \mathbf{C}((t)) \subset \mathbf{C}[[t, t^{-1}]] \\ \iota_- : \mathbf{C}(t) &\hookrightarrow \mathbf{C}((t^{-1})) \subset \mathbf{C}[[t, t^{-1}]] \end{aligned} \tag{3.63}$$

to denote the operations of expanding a rational function of the variable t in the indicated directions (as in Section 8.1 of [FLM2]). For $a \in \mathbf{C}$, we define the translation isomorphism

$$\begin{aligned} T_a : \mathbf{C}(t) &\xrightarrow{\sim} \mathbf{C}(t) \\ f(t) &\mapsto f(t+a) \end{aligned} \tag{3.64}$$

and we set

$$T_a^{\pm} = \iota_{\pm} \circ T_a \circ \iota_+^{-1} : \iota_+\mathbf{C}(t) \hookrightarrow \mathbf{C}((t^{\pm 1})). \tag{3.65}$$

(Note that the domains of these maps consist of certain series expansions of rational functions rather than of rational functions themselves.) We shall be interested in

$$T_{-z}^{\pm} : \iota_+\mathbf{C}[t, t^{-1}, (z+t)^{-1}] \hookrightarrow \mathbf{C}((t^{\pm 1})), \tag{3.66}$$

where z is an arbitrary nonzero complex number. The images of these two maps are $\iota_{\pm}\mathbf{C}[t, t^{-1}, (z-t)^{-1}]$.

Extend $T_{-z}^{\pm}$ to linear isomorphisms

$$T_{-z}^{\pm} : V \otimes \iota_+\mathbf{C}[t, t^{-1}, (z+t)^{-1}] \xrightarrow{\sim} V \otimes \iota_{\pm}\mathbf{C}[t, t^{-1}, (z-t)^{-1}] \tag{3.67}$$

given by $1 \otimes T_{-z}^{\pm}$ with $T_{-z}^{\pm}$ as defined above. Note that the domain of these two maps is described by (3.12)–(3.13), that the image of the map T_{-z}^{+} is described by (3.15)–(3.16) and that the image of the map T_{-z}^{-} is described by (3.18)–(3.19).

We have the two mutually inverse maps

$$\begin{aligned} V \otimes \iota_-\mathbf{C}[t, t^{-1}, (z-t)^{-1}] &\xrightarrow{*} V \otimes \iota_+\mathbf{C}[t, t^{-1}, (z^{-1}-t)^{-1}] \\ v \otimes f(t) &\mapsto v^* f(t^{-1}) \end{aligned} \tag{3.68}$$

and

$$\begin{aligned} V \otimes \iota_+\mathbf{C}[t, t^{-1}, (z^{-1}-t)^{-1}] &\xrightarrow{*} V \otimes \iota_-\mathbf{C}[t, t^{-1}, (z-t)^{-1}] \\ v \otimes f(t) &\mapsto v^* f(t^{-1}), \end{aligned} \tag{3.69}$$

which are both isomorphisms. We form the composition

$$T_{-z}^{*} = * \circ T_{-z}^{-} \tag{3.70}$$

to obtain another isomorphism

$$T^{*}_{-z} : V \otimes \iota_{+}\mathbf{C}[t, t^{-1}, (z+t)^{-1}] \xrightarrow{\sim} V \otimes \iota_{+}\mathbf{C}[t, t^{-1}, (z^{-1}-t)^{-1}].$$

The maps $T^{\pm}_{-z}$ and T^{*}_{-z} will be the main ingredients of our action. The following result asserts that T^{+}_{-z}, T^{-}_{-z} and T^{*}_{-z} transform the expression (3.12) into (3.15), (3.18) and the $*$-transform of (3.18), respectively:

Lemma 3.1 *We have*

$$T^{+}_{-z}\left(z^{-1}\delta\left(\frac{x_1-x_0}{z}\right)Y_t(v,x_0)\right) = x_0^{-1}\delta\left(\frac{z-x_1}{-x_0}\right)Y_t(v,x_1), \tag{3.71}$$

$$T^{-}_{-z}\left(z^{-1}\delta\left(\frac{x_1-x_0}{z}\right)Y_t(v,x_0)\right) = x_0^{-1}\delta\left(\frac{x_1-z}{x_0}\right)Y_t(v,x_1), \tag{3.72}$$

$$T^{*}_{-z}\left(z^{-1}\delta\left(\frac{x_1-x_0}{z}\right)Y_t(v,x_0)\right) = x_0^{-1}\delta\left(\frac{x_1-z}{x_0}\right)Y_t^{*}(v,x_1). \tag{3.73}$$

Proof We prove (3.71): From (3.12), the coefficient of $x_0^{-n-1}x_1^{-m-1}$ in the left-hand side of (3.71) is $T^{+}_{-z}(v \otimes (z+t)^m t^n)$. By the definitions,

$$T^{+}_{-z}(v \otimes (z+t)^m t^n) = v \otimes t^m(-(z-t))^n. \tag{3.74}$$

On the other hand, the right-hand side of (3.71) can be written as

$$v \otimes x_0^{-1}\delta\left(\frac{z-x_1}{-x_0}\right)x_1^{-1}\delta\left(\frac{t}{x_1}\right) = v \otimes x_0^{-1}\delta\left(\frac{z-t}{-x_0}\right)x_1^{-1}\delta\left(\frac{t}{x_1}\right) \tag{3.75}$$

where we have used (3.5) and the fundamental property (2.4) of the formal δ-function. The coefficient of $x_0^{-n-1}x_1^{-m-1}$ in the right-hand side of (3.75) is also $v \otimes t^m(-(z-t))^n$, proving (3.71). Formula (3.72) is proved similarly, and (3.73) is obtained from (3.72) by the application of the map $*$. □

4 The notions of $P(z)$- and $Q(z)$-tensor product of two modules

For any $\mathbf{C}$-graded vector space $W = \coprod W_{(n)}$ such that $\dim W_{(n)} < \infty$ for each $n \in \mathbf{C}$, we use the notation

$$\overline{W} = \prod_{n \in \mathbf{C}} W_{(n)} = W'^{*}, \tag{4.1}$$

where as usual $'$ denotes the graded dual space and * denotes the dual space of a vector space.

Let V be a vertex operator algebra and W a V-module. For any $v \in V$ and $n \in \mathbf{Z}$, v_n acts naturally on $\overline{W}$ because of (2.21) for modules (recall Remark 2.8) and v_n^* also acts natually on $\overline{W}$, in view of (2.21) and (3.20). Moreover, because of (2.21) and (2.44), for fixed $v \in V$, any infinite linear combination of the v_n of the form $\sum_{n<N} a_n v_n$ ($a_n \in \mathbf{C}$) acts on $\overline{W}$, and from (3.22), for example, we see that any infinite linear combination of the form $\sum_{n>N} a_n v_n^*$ also acts on $\overline{W}$.

Fix a nonzero number z and let (W_1, Y_1) and (W_2, Y_2) be two V-modules. In the present paper (Part I), we give the algebraic definitions and algebraic constructions of certain tensor products of (W_1, Y_1) and (W_2, Y_2), depending on z, but these have geometric meanings as well, which will be discussed and studied in other papers in this series. Let (W_3, Y_3) be another V-module. We call a *$P(z)$-intertwining map of type* $\binom{W_3}{W_1 W_2}$ (see Remark 4.3 below for the meaning of the symbol $P(z)$) a linear map $F : W_1 \otimes W_2 \to \overline{W}_3$ satisfying the condition

$$\begin{aligned} x_0^{-1}\delta\left(\frac{x_1 - z}{x_0}\right) & Y_3(v, x_1) F(w_{(1)} \otimes w_{(2)}) = \\ &= z^{-1}\delta\left(\frac{x_1 - x_0}{z}\right) F(Y_1(v, x_0) w_{(1)} \otimes w_{(2)}) \\ &\quad + x_0^{-1}\delta\left(\frac{z - x_1}{-x_0}\right) F(w_{(1)} \otimes Y_2(v, x_1) w_{(2)}) \end{aligned} \tag{4.2}$$

for $v \in V$, $w_{(1)} \in W_1$, $w_{(2)} \in W_2$ (cf. the identity (1.1) and the Jacobi identity (2.45) for intertwining operators). Note that the left-hand side of (4.2) is well defined in view of the comments in the preceding paragraph. A *$P(z)$-product of W_1 and W_2* is a V-module (W_3, Y_3) equipped with a $P(z)$-intertwining map F of type $\binom{W_3}{W_1 W_2}$. We denote it by $(W_3, Y_3; F)$ (or simply by (W_3, F)). Let $(W_4, Y_4; G)$ be another $P(z)$-product of W_1 and W_2. A *morphism* from $(W_3, Y_3; F)$ to $(W_4, Y_4; G)$ is a module map η from W_3 to W_4 such that

$$G = \overline{\eta} \circ F, \tag{4.3}$$

where $\overline{\eta}$ is the map from $\overline{W}_3$ to $\overline{W}_4$ uniquely extending η. We define the notion of $P(z)$-tensor product using a universal property as follows:

Definition 4.1 A *$P(z)$-tensor product of W_1 and W_2* is a $P(z)$-product $(W_1 \boxtimes_{P(z)} W_2, Y_{P(z)}; \boxtimes_{P(z)})$ such that for any $P(z)$-product $(W_3, Y_3; F)$, there is a unique morphism from $(W_1 \boxtimes_{P(z)} W_2, Y_{P(z)}; \boxtimes_{P(z)})$ to $(W_3, Y_3; F)$. The V-module $(W_1 \boxtimes_{P(z)} W_2, Y_{P(z)})$ is called a *$P(z)$-tensor product module* of W_1 and W_2.

Remark 4.2 As in the case of tensor products of modules for Lie algebras, it is clear from the definition that if a $P(z)$-tensor product of W_1 and W_2 exists, then it is unique up to unique isomorphism.

Remark 4.3 The symbol $P(z)$ in the definitions above in fact represents a geometric object. Geometrically, to define a tensor product of W_1 and W_2, we need to specify an element of the moduli space K of spheres with punctures and local coordinates vanishing at these punctures. (In this remark and in Remark 4.6 below, we invoke the detailed discussion of the moduli space K and its role in the geometric interpretation of the notion of vertex operator algebra given in [H1], [H2], [H3] or [H4]. The present remark and Remark 4.6 are not logically used in the algebraic treatment in Part I.) More precisely, we need to specify an element of the determinant line bundle over K raised to the power c, where c is the central charge of the vertex operator algebra. The definitions of intertwining map, product and tensor product above are those associated to the element $P(z)$ of K containing $\mathbb{C} \cup \{\infty\}$ with ordered punctures ∞, z, 0 and standard local coordinates $1/w$, $w-z$, w, vanishing at ∞, z, 0, respectively. Note that $P(z)$ is the geometric object corresponding to vertex operators or intertwining operators in the geometric interpretation of vertex operators and intertwining operators. The appropriate language describing tensor products defined using elements of K is that of operads, or more precisely, partial operads (see [M], [HL2], [HL3] and [H5]). These different tensor products will play important roles in the formulations and constructions of the associativity and commutativity isomorphisms.

Though it is natural to first consider $P(z)$-tensor products of two modules as defined above, in this paper (Part I) we shall instead construct another type of tensor product, the $Q(z)$-tensor product (see below), since the calculations involved in the direct construction of $Q(z)$-tensor products are simpler than those for $P(z)$-tensor products. Moreover, $P(z)$-tensor products can be obtained from $Q(z)$-tensor products by performing certain geometric transformations. We shall give the construction of a $P(z)$-tensor product in Part

III using the construction of the $Q(z)$-tensor product presented in Sections 5 and 6 below. The reader should observe that many of the considerations below concerning concepts based on $Q(z)$ carry over immediately to the analogous considerations related to $P(z)$; in Part I we focus only on $Q(z)$.

A *$Q(z)$-intertwining map of type* $\binom{W_3}{W_1 W_2}$ is a linear map $F : W_1 \otimes W_2 \to \overline{W}_3$ such that

$$\begin{aligned} z^{-1}\delta\left(\frac{x_1 - x_0}{z}\right) Y_3^*(v, x_0) F(w_{(1)} \otimes w_{(2)}) = \\ = x_0^{-1}\delta\left(\frac{x_1 - z}{x_0}\right) F(Y_1^*(v, x_1) w_{(1)} \otimes w_{(2)}) \\ - x_0^{-1}\delta\left(\frac{z - x_1}{-x_0}\right) F(w_{(1)} \otimes Y_2(v, x_1) w_{(2)}) \end{aligned} \tag{4.4}$$

for $v \in V$, $w_{(1)} \in W_1$, $w_{(2)} \in W_2$. As in the definition of $P(z)$-intertwining map, note that the left-hand side and both terms on the right-hand side of (4.4) are well defined. First replacing v by $(-x_0^2)^{L(0)} e^{-x_0 L(1)} v$ and then replacing x_0 by x_0^{-1} in (4.4), we see that (4.4) is equivalent to:

$$\begin{aligned} z^{-1}\delta\left(\frac{x_1 - x_0^{-1}}{z}\right) Y_3(v, x_0) F(w_{(1)} \otimes w_{(2)}) = \\ = x_0 \delta\left(\frac{x_1 - z}{x_0^{-1}}\right) F(Y_1(e^{x_1 L(1)} (x_1 x_0)^{-2L(0)} e^{-x_0^{-1} L(1)} v, x_1^{-1}) w_{(1)} \otimes w_{(2)}) \\ - x_0 \delta\left(\frac{z - x_1}{-x_0^{-1}}\right) F(w_{(1)} \otimes Y_2((-x_0^{-2})^{L(0)} e^{-x_0^{-1} L(1)} v, x_1) w_{(2)}). \end{aligned} \tag{4.5}$$

(The reverse procedure is given by first inverting x_0 and then replacing v by $e^{x_0 L(1)} (-x_0^{-2})^{L(0)} v$.)

We denote the vector space of $Q(z)$-intertwining maps of type $\binom{W_3}{W_1 W_2}$ by $\mathcal{M}[Q(z)]^{W_3}_{W_1 W_2}$ or simply by $\mathcal{M}^{W_3}_{W_1 W_2}$.

We define a *$Q(z)$-product of W_1 and W_2* to be a V-module (W_3, Y_3) together with a $Q(z)$-intertwining map F of type $\binom{W_3}{W_1 W_2}$ and we denote it by $(W_3, Y_3; F)$ (or (W_3, F_3)). Let $(W_3, Y_3; F)$ and $(W_4, Y_4; G)$ be two $Q(z)$-products of W_1 and W_2. A *morphism* from $(W_3, Y_3; F)$ to $(W_4, Y_4; G)$ is a module map η from W_3 to W_4 such that

$$G = \overline{\eta} \circ F \tag{4.6}$$

where, as in (4.3), $\overline{\eta}$ is the map from $\overline{W}_3$ to $\overline{W}_4$ uniquely extending η.

Definition 4.4 A *$Q(z)$-tensor product of W_1 and W_2* is a $Q(z)$-product $(W_1 \boxtimes_{Q(z)} W_2, Y_{Q(z)}; \boxtimes_{Q(z)})$ such that for any $Q(z)$-product $(W_3, Y_3; F)$, there is a unique morphism from $(W_1 \boxtimes_{Q(z)} W_2, Y_{Q(z)}; \boxtimes_{Q(z)})$ to $(W_3, Y_3; F)$. The V-module $(W_1 \boxtimes_{Q(z)} W_2, Y_{Q(z)})$ is called a *$Q(z)$-tensor product module* of W_1 and W_2.

Remark 4.5 As in the case of $P(z)$-tensor products, a $Q(z)$-tensor product is unique up to unique isomorphism if it exists.

Remark 4.6 In the definitions above, $Q(z)$ represents the element of K containing $\mathbb{C} \cup \{\infty\}$ with ordered punctures z, ∞, 0 and standard local coordinates vanishing at these punctures. (Recall Remark 4.3.) In fact, this is the same as the element of K containing $\mathbb{C} \cup \{\infty\}$ with ordered punctures ∞, $1/z$, 0 and local coordinates $z/(zw-1)$, $(zw-1)/z^2w$, $z^2w/(zw-1)$ vanishing at ∞, $1/z$, 0, respectively, and (4.5) corresponds to this canonical sphere with punctures and local coordinates.

The existence of a $Q(z)$-tensor product is not obvious. We shall prove the existence and give two constructions under certain assumptions on the vertex operator algebra in this and the next two sections. First we relate $Q(z)$-intertwining maps of type $\binom{W_3}{W_1 W_2}$ to intertwining operators of type $\binom{W_1'}{W_3' W_2}$.

Let $\mathcal{Y}$ be an intertwining operator of type $\binom{W_3}{W_1 W_2}$. For any complex number ζ and any $w_{(1)} \in W_1$, $\mathcal{Y}(w_{(1)}, x)\Big|_{x^n = e^{n\zeta},\ n \in \mathbb{C}}$ is a well-defined map from W_2 to $\overline{W}_3$, in view of formula (2.21) for intertwining operators. For brevity of notation, we shall write this map as $\mathcal{Y}(w_{(1)}, e^{\zeta})$, but note that $\mathcal{Y}(w_{(1)}, e^{\zeta})$ depends on ζ, not on just e^{ζ}, as the notation might suggest. In this paper we shall always choose $\log z$ so that

$$\log z = \log|z| + i \arg z \ \text{ with } \ 0 \leq \arg z < 2\pi. \tag{4.7}$$

Arbitrary values of the log function will be denoted

$$l_p(z) = \log z + 2p\pi i \tag{4.8}$$

for $p \in \mathbf{Z}$.

We now describe the close connection between intertwining operators of type $\binom{W_1'}{W_3' W_2}$ and $Q(z)$-intertwining maps of type $\binom{W_3}{W_1 W_2}$. Fix an integer p. Let $\mathcal{Y}$ be an intertwining operator of type $\binom{W_1'}{W_3' W_2}$. Then we have an element of $(W_1 \otimes W_3' \otimes W_2)^*$ whose value at $w_{(1)} \otimes w'_{(3)} \otimes w_{(2)}$ ($w_{(1)} \in W_1$, $w_{(2)} \in W_2$, $w'_{(3)} \in W_3'$) is

$$\langle w_{(1)}, \mathcal{Y}(w'_{(3)}, e^{l_p(z)})w_{(2)}\rangle_{W_1'}$$

where $\langle \cdot, \cdot \rangle_{W_1'}$ is the pairing between W_1 and $\overline{W_1'} = W_1^*$. Since any element of $(W_1 \otimes W_3' \otimes W_2)^*$ amounts exactly to a linear map from $W_1 \otimes W_2$ to $W_3'^* = \overline{W}_3$, our element of $(W_1 \otimes W_3' \otimes W_2)^*$ obtained from the intertwining operator $\mathcal{Y}$ gives us a linear map $F_{\mathcal{Y},p} : W_1 \otimes W_2 \to \overline{W}_3$ such that

$$\langle w'_{(3)}, F_{\mathcal{Y},p}(w_{(1)} \otimes w_{(2)})\rangle_{W_3} = \langle w_{(1)}, \mathcal{Y}(w'_{(3)}, e^{l_p(z)})w_{(2)}\rangle_{W_1'} \tag{4.9}$$

for all $w_{(1)} \in W_1$, $w_{(2)} \in W_2$, $w'_{(3)} \in W_3'$, where $\langle \cdot, \cdot \rangle_{W_3}$ is the pairing between W_3' and $\overline{W}_3$. (For any module W, we shall use the analogous notation $\langle \cdot, \cdot \rangle_W$ to denote the pairing between W' and $\overline{W}$.) The Jacobi identity for $\mathcal{Y}$ is equivalent to the identity

$$\begin{aligned}
x_2^{-1}\delta\left(\frac{x_1 - x_0}{x_2}\right) &\langle w_{(1)}, \mathcal{Y}(Y_3'(v, x_0)w'_{(3)}, x_2)w_{(2)}\rangle_{W_1'} \\
&= x_0^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right) \langle w_{(1)}, Y_1'(v, x_1)\mathcal{Y}(w'_{(3)}, x_2)w_{(2)}\rangle_{W_1'} \\
&\quad - x_0^{-1}\delta\left(\frac{x_2 - x_1}{-x_0}\right) \langle w_{(1)}, \mathcal{Y}(w'_{(3)}, x_2)Y_2(v, x_1)w_{(2)}\rangle_{W_1'}
\end{aligned} \tag{4.10}$$

for all $w_{(1)}$, $w_{(2)}$ and $w'_{(3)}$ (recall the notation (2.48)). Substituting $e^{nl_p(z)}$ for x_2^n, $n \in \mathbf{C}$, in (4.10), and noting that in case $n \in \mathbf{Z}$, we may simply write z^n for $e^{nl_p(z)}$, we obtain

$$\begin{aligned}
z^{-1}\delta\left(\frac{x_1 - x_0}{z}\right) &\langle w_{(1)}, \mathcal{Y}(Y_3'(v, x_0)w'_{(3)}, e^{l_p(z)})w_{(2)}\rangle_{W_1'} \\
&= x_0^{-1}\delta\left(\frac{x_1 - z}{x_0}\right) \langle w_{(1)}, Y_1'(v, x_1)\mathcal{Y}(w'_{(3)}, e^{l_p(z)})w_{(2)}\rangle_{W_1'} \\
&\quad - x_0^{-1}\delta\left(\frac{z - x_1}{-x_0}\right) \langle w_{(1)}, \mathcal{Y}(w'_{(3)}, e^{l_p(z)})Y_2(v, x_1)w_{(2)}\rangle_{W_1'}.
\end{aligned} \tag{4.11}$$

Using (3.22) and (4.9), we see that (4.11) can be written as

$$z^{-1}\delta\left(\frac{x_1 - x_0}{z}\right) \langle w'_{(3)}, Y_3^*(v, x_0)F_{\mathcal{Y},p}(w_{(1)} \otimes w_{(2)})\rangle_{W_3}$$

$$= x_0^{-1}\delta\left(\frac{x_1 - z}{x_0}\right)\langle w'_{(3)}, F_{\mathcal{Y},p}(Y_1^*(v, x_1)w_{(1)} \otimes w_{(2)})\rangle_{W_3}$$
$$-x_0^{-1}\delta\left(\frac{z - x_1}{-x_0}\right)\langle w'_{(3)}, F_{\mathcal{Y},p}(w_{(1)} \otimes Y_2(v, x_1)w_{(2)}\rangle_{W_3}. \quad (4.12)$$

Thus $F_{\mathcal{Y},p}$ is a $Q(z)$-intertwining map of type $\binom{W_3}{W_1 W_2}$.

The only part of the definition of intertwining operator we have not yet used is the $L(-1)$-derivative property (2.46). (Recall that the lower truncation condition (2.44) has already been used in the formulation of the first term on the left-hand side of the Jacobi identity (2.45).) Since we have specialized x to z in $\mathcal{Y}(\cdot, x)$, there is no property of $F_{\mathcal{Y},p}$ corresponding the $L(-1)$-derivative property of $\mathcal{Y}$. Instead, the $L(-1)$-derivative property will enable us to recover $\mathcal{Y}(\cdot, x)$ from $F_{\mathcal{Y},p}$. Specifically, the $L(-1)$-derivative property enters into the proof of the formula

$$x^{L(0)}\mathcal{Y}(w'_{(3)}, x_0)x^{-L(0)} = \mathcal{Y}(x^{L(0)}w'_{(3)}, xx_0) \quad (4.13)$$

(recall [FHL], formula (5.4.22), and Lemma 5.2.3 and its proof), and this is equivalent to the formula

$$\langle x^{L(0)}w_{(1)}, \mathcal{Y}(x^{-L(0)}w'_{(3)}, x_0)x^{-L(0)}w_{(2)}\rangle_{W_1'} = \langle w_{(1)}, \mathcal{Y}(w'_{(3)}, xx_0)w_{(2)}\rangle_{W_1'} \quad (4.14)$$

for all $w_{(1)} \in W_1$. Substituting $e^{nl_p(z)}$ for x_0^n and $e^{-nl_p(z)}x^n$ for x^n, $n \in \mathbf{C}$, in (4.14), we obtain

$$\langle e^{-l_p(z)L(0)}x^{L(0)}w_{(1)}, \mathcal{Y}(e^{l_p(z)L(0)}x^{-L(0)}w'_{(3)}, e^{l_p(z)})e^{l_p(z)L(0)}x^{-L(0)}w_{(2)}\rangle_{W_1'}$$
$$= \langle w_{(1)}, \mathcal{Y}(w'_{(3)}, x)w_{(2)}\rangle_{W_1'}, \quad (4.15)$$

or equivalently, by (4.9),

$$\langle e^{l_p(z)L(0)}x^{-L(0)}w'_{(3)}, F_{\mathcal{Y},p}(e^{-l_p(z)L(0)}x^{L(0)}w_{(1)} \otimes e^{l_p(z)L(0)}x^{-L(0)}w_{(2)})\rangle_{W_3}$$
$$= \langle w_{(1)}, \mathcal{Y}(w'_{(3)}, x)w_{(2)}\rangle_{W_1'}. \quad (4.16)$$

Thus we have recovered $\mathcal{Y}$ from $F_{\mathcal{Y},p}$.

We shall also need the following alternative way of recovering $\mathcal{Y}$ from $F_{\mathcal{Y},p}$, using components. We write (4.9) as:

$$\langle w_{(1)}, \sum_{n\in\mathbf{C}}(w'_{(3)})_n w_{(2)}e^{(-n-1)l_p(z)}\rangle_{W_1'} = \langle w'_{(3)}, F_{\mathcal{Y},p}(w_{(1)} \otimes w_{(2)})\rangle_{W_3}. \quad (4.17)$$

This formula enables us to recover the components $(w'_{(3)})_n w_{(2)}$, $n \in \mathbf{C}$, of $\mathcal{Y}(w'_{(3)}, x)w_{(2)}$ from $F_{\mathcal{Y},p}$, assuming for convenience that $w_{(2)}$ and $w'_{(3)}$ are homogeneous vectors, in the following way: The map $F_{\mathcal{Y},p}$ gives an element of $(W_1 \otimes W'_3 \otimes W_2)^*$ whose value at $w_{(1)} \otimes w'_{(3)} \otimes w_{(2)}$ is equal to the right-hand side of (4.17). This element amounts to a map from $W'_3 \otimes W_2$ to W_1^*. By (4.17), the image of $w'_{(3)} \otimes w_{(2)}$ under this map is equal to $\sum_{n \in \mathbf{C}} (w'_{(3)})_n w_{(2)} e^{(-n-1)l_p(z)}$. Projecting this image to the homogeneous subspace of W'_1 of weight equal to

$$\mathrm{wt}\ w'_{(3)} - n - 1 + \mathrm{wt}\ w_{(2)},$$

we obtain $(w'_{(3)})_n w_{(2)} e^{(-n-1)l_p(z)}$. Mutiplying this by $e^{(n+1)l_p(z)}$, we recover the coefficient $(w'_{(3)})_n w_{(2)}$.

Motivated by this procedure, we would like to construct an intertwining operator of type $\binom{W'_1}{W'_3 W_2}$ from a $Q(z)$-intertwining map of type $\binom{W_3}{W_1 W_2}$. Let F be a $Q(z)$-intertwining map of type $\binom{W_3}{W_1 W_2}$. This linear map from $W_1 \otimes W_2$ to $\overline{W}_3$ gives us an element of $(W_1 \otimes W'_3 \otimes W_2)^*$ whose value at $w_{(1)} \otimes w'_{(3)} \otimes w_{(2)}$ is

$$\langle w'_{(3)}, F(w_{(1)} \otimes w_{(2)}) \rangle_{W_3}.$$

But since every element of $(W_1 \otimes W'_3 \otimes W_2)^*$ also amounts to a linear map from $W'_3 \otimes W_2$ to W_1^*, we have such a map as well. Let $w'_{(3)} \in W'_3$ and $w_{(2)} \in W_2$ be homogeneous elements. Since $W_1^* = \prod_{n \in \mathbf{C}} (W'_1)_{(n)}$, the image of $w'_{(3)} \otimes w_{(2)}$ under our map can be written as $\sum_{n \in \mathbf{C}} (w'_{(3)})_n w_{(2)} e^{(-n-1)l_p(z)}$ where for any $n \in \mathbf{C}$, $(w'_{(3)})_n w_{(2)} e^{(-n-1)l_p(z)}$ is the projection of the image to the homogeneous subspace of W'_1 of weight equal to

$$\mathrm{wt}\ w'_{(3)} - n - 1 + \mathrm{wt}\ w_{(2)}.$$

(Here we are defining elements denoted $(w'_{(3)})_n w_{(2)}$ of W'_1 for $n \in \mathbf{C}$.) We define

$$\mathcal{Y}_{F,p}(w'_{(3)}, x) w_{(2)} = \sum_{n \in \mathbf{C}} (w'_{(3)})_n w_{(2)} x^{-n-1} \in W'_1\{x\}$$

for all homogeneous elements $w'_{(3)} \in W'_3$ and $w_{(2)} \in W_2$. Using linearity, we extend $\mathcal{Y}_{F,p}$ to a linear map

$$\begin{array}{rcl} W'_3 \otimes W_2 & \to & W'_1\{x\} \\ w'_{(3)} \otimes w_{(2)} & \mapsto & \mathcal{Y}_{F,p}(w'_{(3)}, x) w_{(2)}. \end{array} \tag{4.18}$$

The correspondence $F \mapsto \mathcal{Y}_{F,p}$ is linear, and from the definitions and the discussion in the preceding paragraph, we have $\mathcal{Y}_{F_{\mathcal{Y},p},p} = \mathcal{Y}$ for an intertwining operator $\mathcal{Y}$ of type $\binom{W_1'}{W_3' W_2}$.

Proposition 4.7 *For $p \in \mathbf{Z}$, the correspondence $\mathcal{Y} \mapsto F_{\mathcal{Y},p}$ is a linear isomorphism from the space $\mathcal{V}_{W_3' W_2}^{W_1'}$ of intertwining operators of type $\binom{W_1'}{W_3' W_2}$ to the space $\mathcal{M}_{W_1 W_2}^{W_3} = \mathcal{M}[Q(z)]_{W_1 W_2}^{W_3}$ of $Q(z)$-intertwining maps of type $\binom{W_3}{W_1 W_2}$. Its inverse is given by $F \mapsto \mathcal{Y}_{F,p}$.*

Proof We need only show that for any $Q(z)$-intertwining map F of type $\binom{W_3}{W_1 W_2}$, $\mathcal{Y}_{F,p}$ is an intertwining operator of type $\binom{W_1'}{W_3' W_2}$. From the discussion above and the definition of $\mathcal{Y}_{F,p}$, the lower truncation condition (2.44) holds for $\mathcal{Y}_{F,p}$ and we have the equality

$$\begin{aligned}
&\langle w_{(1)}, \mathcal{Y}_{F,p}(w'_{(3)}, x)w_{(2)}\rangle_{W_1'} = \\
&\quad = \langle e^{l_p(z)L(0)}x^{-L(0)}w'_{(3)}, F(e^{-l_p(z)L(0)}x^{L(0)}w_{(1)} \otimes \\
&\qquad \otimes e^{l_p(z)L(0)}x^{-L(0)}w_{(2)})\rangle_{W_3}
\end{aligned} \tag{4.19}$$

(cf. (4.16)). Now (4.4) gives

$$\begin{aligned}
&z^{-1}\delta\left(\frac{x_1 - x_0}{z}\right)\langle w'_{(3)}, Y_3^*(v, x_0)F(w_{(1)} \otimes w_{(2)})\rangle_{W_3} = \\
&\quad = x_0^{-1}\delta\left(\frac{x_1 - z}{x_0}\right)\langle w'_{(3)}, F(Y_1^*(v, x_1)w_{(1)} \otimes w_{(2)})\rangle_{W_3} \\
&\qquad - x_0^{-1}\delta\left(\frac{z - x_1}{-x_0}\right)\langle w'_{(3)}, F(w_{(1)} \otimes Y_2(v, x_1)w_{(2)})\rangle_{W_3}.
\end{aligned} \tag{4.20}$$

Changing the formal variables x_0 and x_1 in (4.20) to $zx_2^{-1}x_0$ and $zx_2^{-1}x_1$, respectively, and using (3.22), we obtain

$$\begin{aligned}
&x_2^{-1}\delta\left(\frac{x_1 - x_0}{x_2}\right)\langle Y_3'(v, zx_2^{-1}x_0)w'_{(3)}, F(w_{(1)} \otimes w_{(2)})\rangle_{W_3} = \\
&\quad = x_0^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right)\langle w'_{(3)}, F(Y_1^*(v, zx_2^{-1}x_1)w_{(1)} \otimes w_{(2)})\rangle_{W_3} \\
&\qquad - x_0^{-1}\delta\left(\frac{x_2 - x_1}{-x_0}\right)\langle w'_{(3)}, F(w_{(1)} \otimes Y_2(v, zx_2^{-1}x_1)w_{(2)})\rangle_{W_3}.
\end{aligned} \tag{4.21}$$

(Note that all powers of z occurring here are integral.) Using the formulas

$$\begin{aligned} Y_3'(v, zx_2^{-1}x_0) &= e^{l_p(z)L(0)}x_2^{-L(0)}Y_3'(e^{-l_p(z)L(0)}x_2^{L(0)}v, x_0)\cdot \\ &\quad \cdot e^{-l_p(z)L(0)}x_2^{L(0)}, \qquad (4.22) \\ Y_1^*(v, zx_2^{-1}x_1) &= e^{-l_p(z)L(0)}x_2^{L(0)}Y_1^*(e^{-l_p(z)L(0)}x_2^{L(0)}v, x_1)\cdot \\ &\quad \cdot e^{l_p(z)L(0)}x_2^{-L(0)}, \qquad (4.23) \\ Y_2(v, zx_2^{-1}x_1) &= e^{l_p(z)L(0)}x_2^{-L(0)}Y_2(e^{-l_p(z)L(0)}x_2^{L(0)}v, x_1)\cdot \\ &\quad \cdot e^{-l_p(z)L(0)}x_2^{L(0)}, \qquad (4.24) \end{aligned}$$

which follow from Lemma 5.2.3 together with formula (5.2.39) of [FHL] (note that the eigenvalues of $L(0)$ are not in general integral on the modules), we see that (4.21) becomes

$$\begin{aligned} &x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right)\langle e^{l_p(z)L(0)}x_2^{-L(0)}Y_3'(e^{-l_p(z)L(0)}x_2^{L(0)}v, x_0)\cdot \\ &\qquad \cdot e^{-l_p(z)L(0)}x_2^{L(0)}w'_{(3)}, F(w_{(1)}\otimes w_{(2)})\rangle_{W_3} \\ &= x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right)\langle w'_{(3)}, F(e^{-l_p(z)L(0)}x_2^{L(0)}\cdot \\ &\qquad \cdot Y_1^*(e^{-l_p(z)L(0)}x_2^{L(0)}v, x_1)e^{l_p(z)L(0)}x_2^{-L(0)}w_{(1)}\otimes w_{(2)})\rangle_{W_3} \\ &\quad -x_0^{-1}\delta\left(\frac{x_2-x_1}{-x_0}\right)\langle w'_{(3)}, F(w_{(1)}\otimes e^{l_p(z)L(0)}x_2^{-L(0)}\cdot \\ &\qquad \cdot Y_2(e^{-l_p(z)L(0)}x_2^{L(0)}v, x_1)e^{-l_p(z)L(0)}x_2^{L(0)}w_{(2)})\rangle_{W_3}. \qquad (4.25) \end{aligned}$$

Replacing v, $w_{(1)}$, $w_{(2)}$ and $w'_{(3)}$ in (4.25) by

$$e^{l_p(z)L(0)}x_2^{-L(0)}v,$$

$$e^{-l_p(z)L(0)}x_2^{L(0)}w_{(1)},$$

$$e^{l_p(z)L(0)}x_2^{-L(0)}w_{(2)}$$

and

$$e^{l_p(z)L(0)}x_2^{-L(0)}w'_{(3)},$$

respectively, we obtain

$$x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right)\langle e^{l_p(z)L(0)}x_2^{-L(0)}Y_3'(v, x_0)w'_{(3)},$$

$$\begin{aligned}
&F(e^{-l_p(z)L(0)}x_2^{L(0)}w_{(1)}\otimes e^{l_p(z)L(0)}x_2^{-L(0)}w_{(2)}))\rangle_{W_3}\\
&= x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right)\langle e^{l_p(z)L(0)}x_2^{-L(0)}w'_{(3)}, F(e^{-l_p(z)L(0)}x_2^{L(0)}\cdot\\
&\qquad\cdot Y_1^*(v,x_1)w_{(1)}\otimes e^{l_p(z)L(0)}x_2^{-L(0)}w_{(2)}))\rangle_{W_3}\\
&\quad -x_0^{-1}\delta\left(\frac{x_2-x_1}{-x_0}\right)\langle e^{l_p(z)L(0)}x_2^{-L(0)}w'_{(3)}, F(e^{-l_p(z)L(0)}\cdot\\
&\qquad\cdot x_2^{L(0)}w_{(1)}\otimes e^{l_p(z)L(0)}x_2^{-L(0)}Y_2(v,x_1)w_{(2)}))\rangle_{W_3}. \quad (4.26)
\end{aligned}$$

But using (4.19), we can write (4.26) as

$$\begin{aligned}
&x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right)\langle w_{(1)}, \mathcal{Y}_{F,p}(Y_3'(v,x_0)w'_{(3)},x_2)w_{(2)}\rangle_{W_1'}\\
&\quad = x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right)\langle Y_1^*(v,x_1)w_{(1)}, \mathcal{Y}_{F,p}(w'_{(3)},x_2)w_{(2)}\rangle_{W_1'}\\
&\qquad -x_0^{-1}\delta\left(\frac{x_2-x_1}{-x_0}\right)\langle w_{(1)}, \mathcal{Y}_{F,p}(w'_{(3)},x_2)Y_2(v,x_1)w_{(2)}\rangle_{W_1'}, \quad (4.27)
\end{aligned}$$

and (4.27) is equivalent to the Jacobi identity for $\mathcal{Y}_{F,p}$.

Finally, the Jacobi identity implies that

$$[L(0), \mathcal{Y}_{F,p}(w'_{(3)},x)] = \mathcal{Y}_{F,p}(L(0)w'_{(3)},x) + x\mathcal{Y}_{F,p}(L(-1)w'_{(3)},x),$$

and since by construction the weight of the operator $(w'_{(3)})_n$ ($n \in \mathbf{C}$) is $\mathrm{wt}\, w'_{(3)} - n - 1$ if $w'_{(3)}$ is homogeneous, the $L(-1)$-derivative property follows. □

The following immediate result relates module map from a tensor product module with intertwining maps and intertwining operators:

Proposition 4.8 *Suppose that $W_1 \boxtimes_{Q(z)} W_2$ exists. We have a natural isomorphism*

$$\begin{array}{rcl}
\mathrm{Hom}_V(W_1\boxtimes_{Q(z)}W_2, W_3) & \xrightarrow{\sim} & \mathcal{M}^{W_3}_{W_1W_2}\\
\eta & \mapsto & \overline{\eta}\circ\boxtimes_{Q(z)}
\end{array} \quad (4.28)$$

and for $p \in \mathbf{Z}$, a natural isomorphism

$$\begin{array}{rcl}
\mathrm{Hom}_V(W_1\boxtimes_{Q(z)}W_2, W_3) & \xrightarrow{\sim} & \mathcal{V}^{W_1'}_{W_3'W_2}\\
\eta & \mapsto & \mathcal{Y}_{\eta,p}
\end{array} \quad (4.29)$$

where $\mathcal{Y}_{\eta,p} = \mathcal{Y}_{F,p}$ with $F = \overline{\eta}\circ\boxtimes_{Q(z)}$. □

In Part II we shall prove the following:

Proposition 4.9 *For any integer r, there is a natural isomorphism*

$$B_r : \mathcal{V}^{W_3}_{W_1W_2} \to \mathcal{V}^{W'_1}_{W'_3W_2} \tag{4.30}$$

defined by the condition that for any intertwining operator $\mathcal{Y}$ in $\mathcal{V}^{W_3}_{W_1W_2}$ and $w_{(1)} \in W_1$, $w_{(2)} \in W_2$, $w'_{(3)} \in W'_3$,

$$\begin{aligned}\langle w_{(1)}, B_r(\mathcal{Y})(w'_{(3)}, x)w_{(2)}\rangle_{W'_1} &= \\ = \langle e^{-x^{-1}L(1)}w'_{(3)}, \mathcal{Y}(e^{xL(1)}w_{(1)}, x^{-1})&\cdot \\ \cdot e^{-xL(1)}e^{(2r+1)\pi i L(0)}x^{-2L(0)}w_{(2)}\rangle_{W_3}&.\end{aligned} \tag{4.31}$$

Combining the last two results, we obtain:

Corollary 4.10 *For any V-modules W_1, W_2, W_3 such that $W_1 \boxtimes_{Q(z)} W_2$ exists and any integers p and r, we have a natural isomorphism*

$$\begin{aligned}\mathrm{Hom}_V(W_1 \boxtimes_{Q(z)} W_2, W_3) &\xrightarrow{\sim} \mathcal{V}^{W_3}_{W_1W_2} \\ \eta &\mapsto B_r^{-1}(\mathcal{Y}_{\eta,p}). \qquad \square\end{aligned} \tag{4.32}$$

It is clear from Definition 4.4 that the tensor product operation distributes over direct sums in the following sense:

Proposition 4.11 *For V-modules $U_1, \dots, U_k$, $W_1, \dots, W_l$, suppose that each $U_i \boxtimes_{Q(z)} W_j$ exists. Then $(\coprod_i U_i) \boxtimes_{Q(z)} (\coprod_j W_j)$ exists and there is a natural isomorphism*

$$\left(\coprod_i U_i\right) \boxtimes_{Q(z)} \left(\coprod_j W_j\right) \xrightarrow{\sim} \coprod_{i,j} U_i \boxtimes_{Q(z)} W_j. \qquad \square \tag{4.33}$$

Now consider V-modules W_1, W_2 and W_3. The natural evaluation map

$$\begin{aligned}W_1 \otimes W_2 \otimes \mathcal{M}^{W_3}_{W_1W_2} &\to \overline{W}_3 \\ w_{(1)} \otimes w_{(2)} \otimes F &\mapsto F(w_{(1)} \otimes w_{(2)})\end{aligned} \tag{4.34}$$

gives a natural map

$$\mathcal{F}^{W_3}_{W_1W_2} : W_1 \otimes W_2 \to (\mathcal{M}^{W_3}_{W_1W_2})^* \otimes \overline{W}_3. \tag{4.35}$$

Suppose that dim $\mathcal{M}^{W_3}_{W_1W_2} < \infty$. Then $(\mathcal{M}^{W_3}_{W_1W_2})^* \otimes W_3$ is a V-module (with finite-dimensional weight spaces) in the obvious way, and the map $\mathcal{F}^{W_3}_{W_1W_2}$ is clearly a $Q(z)$-intertwining map, where we make the identification

$$(\mathcal{M}^{W_3}_{W_1W_2})^* \otimes \overline{W}_3 = \overline{(\mathcal{M}^{W_3}_{W_1W_2})^* \otimes W_3}. \tag{4.36}$$

This gives us a natural $Q(z)$-product. Moreover, we have a natural linear injection

$$\begin{array}{rcl} i : \mathcal{M}^{W_3}_{W_1W_2} & \to & \mathrm{Hom}_V((\mathcal{M}^{W_3}_{W_1W_2})^* \otimes W_3, W_3) \\ F & \mapsto & (f \otimes w_{(3)} \mapsto f(F)w_3) \end{array} \tag{4.37}$$

which is an isomorphism if W_3 is irreducible, since in this case,

$$\mathrm{Hom}_V(W_3, W_3) \simeq \mathbf{C}$$

(see [FHL], Remark 4.7.1). On the other hand, the natural map

$$\begin{array}{rcl} h : \mathrm{Hom}_V((\mathcal{M}^{W_3}_{W_1W_2})^* \otimes W_3, W_3) & \to & \mathcal{M}^{W_3}_{W_1W_2} \\ \eta & \mapsto & \overline{\eta} \circ \mathcal{F}^{W_3}_{W_1W_2} \end{array} \tag{4.38}$$

given by composition clearly satisfies the condition that

$$h(i(F)) = F, \tag{4.39}$$

so that if W_3 is irreducible, the maps h and i are mutually inverse isomorphisms and we have the universal property that for any $F \in \mathcal{M}^{W_3}_{W_1W_2}$, there exists a unique η such that

$$F = \overline{\eta} \circ \mathcal{F}^{W_3}_{W_1W_2} \tag{4.40}$$

(cf. Definition 4.4).

Now we consider a special but important class of vertex operator algebras satisfying certain finiteness and semisimplicity conditions.

Definition 4.12 A vertex operator algebra V is *rational* if it satisfies the following conditions:

1. There are only finitely many irreducible V-modules (up to equivalence).

2. Every V-module is completely reducible (and is in particular a *finite* direct sum of irreducible modules).

3. All the fusion rules for V are finite (for triples of irreducible modules and hence arbitrary modules).

The next result shows that tensor products exist for the category of modules for a rational vertex operator algebra. Note that there is no need to assume that W_1 and W_2 are irreducible in the formulation or proof, but by Proposition 4.11, the case in which W_1 and W_2 are irreducible is in fact sufficient, and the tensor product operation is canonically described using only the spaces of intertwining maps among triples of *irreducible* modules.

Proposition 4.13 *Let V be rational and let W_1, W_2 be V-modules. Then $(W_1 \boxtimes_{Q(z)} W_2, Y_{Q(z)}; \boxtimes_{Q(z)})$ exists, and in fact*

$$W_1 \boxtimes_{Q(z)} W_2 = \coprod_{i=1}^{k} (\mathcal{M}_{W_1 W_2}^{M_i})^* \otimes M_i, \tag{4.41}$$

where $\{M_1, \ldots, M_k\}$ is a set of representatives of the equivalence classes of irreducible V-modules, and the right-hand side of (4.41) is equipped with the V-module and $Q(z)$-product structure indicated above. That is,

$$\boxtimes_{Q(z)} = \sum_{i=1}^{k} \mathcal{F}_{W_1 W_2}^{M_i}. \tag{4.42}$$

Proof From the comments above and the definitions, it is clear that we have a $Q(z)$-product. Let $(W_3, Y_3; F)$ be any $Q(z)$-product. Then $W_3 = \coprod_j U_j$ where j ranges through a finite set and each U_j is irreducible. Let $\pi_j : W_3 \to U_j$ denote the j-th projection. A module map $\eta : \coprod_{i=1}^{k} (\mathcal{M}_{W_1 W_2}^{M_i})^* \otimes M_i \to W_3$ amounts to module maps

$$\eta_{ij} : (\mathcal{M}_{W_1 W_2}^{M_i})^* \otimes M_i \to U_j$$

for each i and j such that $U_j \simeq M_i$, and $F = \overline{\eta} \circ \boxtimes_{Q(z)}$ if and only if

$$\overline{\pi}_j \circ F = \overline{\eta}_{ij} \circ \mathcal{F}_{W_1 W_2}^{M_i}$$

for each i and j, the bars having the obvious meaning. But $\overline{\pi}_j \circ F$ is a $Q(z)$-intertwining map of type $\binom{U_j}{W_1 W_2}$, and so $\overline{\iota} \circ \overline{\pi}_j \circ F \in \mathcal{M}_{W_1 W_2}^{M_i}$, where

$\iota : U_j \xrightarrow{\sim} M_i$ is a fixed isomorphism. Denote this map by τ. Thus what we finally want is a unique module map

$$\theta : (\mathcal{M}^{M_i}_{W_1 W_2})^* \otimes M_i \to M_i$$

such that

$$\tau = \overline{\theta} \circ \mathcal{F}^{M_i}_{W_1 W_2}.$$

But we in fact have such a unique θ, by (4.39)–(4.40). □

Remark 4.14 *By combining Proposition 4.13 with Proposition 4.7 or Proposition 4.9, we can express $W_1 \boxtimes_{Q(z)} W_2$ in terms of $\mathcal{V}^{W_1'}_{M_i' W_2}$ or $\mathcal{V}^{M_i}_{W_1 W_2}$ in place of $\mathcal{M}^{M_i}_{W_1 W_2}$.*

The construction in Proposition 4.13 is tautological, and we view the argument as essentially an existence proof. In the next two sections, we present "first and second constructions" of a $Q(z)$-tensor product.

5 First construction of $Q(z)$-tensor product

Here and in the next section, we give two constructions of a $Q(z)$-tensor product of two modules for a vertex operator algebra V, in the presence of a certain hypothesis which holds in case V is rational. In this section, we first define an action of $V \otimes \iota_+ \mathbb{C}[t, t^{-1}, (z+t)^{-1}]$ on $(W_1 \otimes W_2)^*$ motivated by the definition (4.4) of $Q(z)$-intertwining map. We establish some basic properties of this action, deferring the proof of a commutator formula (Proposition 5.2) to Part II. Then we take the sum of all "compatible modules" in $(W_1 \otimes W_2)^*$. Under the assumption that this sum is again a module, we construct the $Q(z)$-tensor product as its contragredient module equipped with the restriction to $W_1 \otimes W_2$ of the adjoint of the embedding map of this sum in $(W_1 \otimes W_2)^*$. In the next section we observe that every element in the sum of compatible modules in $(W_1 \otimes W_2)^*$ satisfies a certain set of conditions, and we show that, modulo two important results stated there but whose proofs are deferred to Part II, the subspace of $(W_1 \otimes W_2)^*$ consisting of all the elements satisfying these conditions is equal to this sum of compatible modules. In this way we obtain another construction of the $Q(z)$-tensor product.

Fix a nonzero complex number z and V-modules (W_1, Y_1) and (W_2, Y_2) as before. We first want to define an action of $V \otimes \iota_+\mathbb{C}[t, t^{-1}, (z+t)^{-1}]$ on $(W_1 \otimes W_2)^*$, that is, a linear map

$$\tau_{Q(z)} : V \otimes \iota_+\mathbb{C}[t, t^{-1}, (z+t)^{-1}] \to \mathrm{End}\,(W_1 \otimes W_2)^*.$$

Recall the maps

$$\tau_{W_i} : V \otimes \mathbb{C}((t)) \to \mathrm{End}\, W_i, \quad i = 1, 2,$$

from (3.2). We define $\tau_{Q(z)}$ by

$$(\tau_{Q(z)}(\xi)\lambda)(w_{(1)} \otimes w_{(2)}) = \lambda(\tau_{W_1}(T^*_{-z}\xi)w_{(1)} \otimes w_{(2)}) - \lambda(w_{(1)} \otimes \tau_{W_2}(T^+_{-z}\xi)w_{(2)}) \tag{5.1}$$

for $\xi \in V \otimes \iota_+\mathbb{C}[t, t^{-1}, (z+t)^{-1}]$, $\lambda \in (W_1 \otimes W_2)^*$, $w_{(1)} \in W_1$, $w_{(2)} \in W_2$. Using (3.12)–(3.13), (3.60) and Lemma 3.1, we see that the definition (5.1) can be written using generating functions as:

$$\begin{aligned}\left(\tau_{Q(z)}\left(z^{-1}\delta\left(\frac{x_1 - x_0}{z}\right)Y_t(v, x_0)\right)\lambda\right)&(w_{(1)} \otimes w_{(2)})\\ = x_0^{-1}\delta\left(\frac{x_1 - z}{x_0}\right)&\lambda(Y_1^*(v, x_1)w_{(1)} \otimes w_{(2)})\\ -x_0^{-1}\delta\left(\frac{z - x_1}{-x_0}\right)&\lambda(w_{(1)} \otimes Y_2(v, x_1)w_{(2)}).\end{aligned} \tag{5.2}$$

Write

$$Y'_{Q(z)}(v, x) = \tau_{Q(z)}(Y_t(v, x)). \tag{5.3}$$

Using (2.6) and the fundamental property of the formal δ-function, we have

$$\begin{aligned}(Y'_{Q(z)}(v, x_0)\lambda)(w_{(1)} \otimes w_{(2)}) &=\\ = \ &\mathrm{Res}_{x_1} x_0^{-1}\delta\left(\frac{x_1 - z}{x_0}\right)\lambda(Y_1^*(v, x_1)w_{(1)} \otimes w_{(2)})\\ &- \mathrm{Res}_{x_1} x_0^{-1}\delta\left(\frac{z - x_1}{-x_0}\right)\lambda(w_{(1)} \otimes Y_2(v, x_1)w_{(2)})\\ = \ &\lambda(Y_1^*(v, x_0 + z)w_{(1)} \otimes w_{(2)})\\ &- \mathrm{Res}_{x_1} x_0^{-1}\delta\left(\frac{z - x_1}{-x_0}\right)\lambda(w_{(1)} \otimes Y_2(v, x_1)w_{(2)}),\end{aligned} \tag{5.4}$$

where we have used the notation Res_{x_1}, which means taking the coefficient of x_1 in a formal series. We have the following results for $Y'_{Q(z)}$:

Proposition 5.1 *The action $Y'_{Q(z)}$ has the property*

$$Y'_{Q(z)}(\mathbf{1},x)=1, \tag{5.5}$$

where 1 *on the right-hand side is the identity map of* $(W_1\otimes W_2)^*$, *and the* $L(-1)$*-derivative property*

$$\frac{d}{dx}Y'_{Q(z)}(v,x)=Y'_{Q(z)}(L(-1)v,x) \tag{5.6}$$

for $v\in V$.

Proof From (5.4), (3.20) and (2.7),

$$\begin{aligned}(Y(\mathbf{1},x)\lambda)(w_{(1)}\otimes w_{(2)}) &= \\ &= \mathrm{Res}_{x_1}x^{-1}\delta\left(\frac{x_1-z}{x}\right)\lambda(w_{(1)}\otimes w_{(2)}) \\ &\quad -\mathrm{Res}_{x_1}x^{-1}\delta\left(\frac{z-x_1}{-x}\right)\lambda(w_{(1)}\otimes w_{(2)}) \\ &= \mathrm{Res}_{x_1}x_1^{-1}\delta\left(\frac{z+x}{x_1}\right)\lambda(w_{(1)}\otimes w_{(2)}) \\ &= \lambda(w_{(1)}\otimes w_{(2)}),\end{aligned} \tag{5.7}$$

proving (5.5). We now prove the $L(-1)$-derivative property. From (5.4),

$$\begin{aligned}&\left(\left(\frac{d}{dx}Y'_{Q(z)}(v,x)\right)\lambda\right)(w_{(1)}\otimes w_{(2)}) = \\ &\quad = \frac{d}{dx}\lambda(Y_1^*(v,x+z)w_{(1)}\otimes w_{(2)}) \\ &\quad\quad -\mathrm{Res}_{x_1}\left(\frac{d}{dx}z^{-1}\delta\left(\frac{-x+x_1}{z}\right)\right)\lambda(w_{(1)}\otimes Y_2(v,x_1)w_{(2)}).\end{aligned} \tag{5.8}$$

Note that for any formal Laurent series $f(x)$, we have

$$\frac{d}{dx}f\left(\frac{-x+x_1}{z}\right)=-\frac{d}{dx_1}f\left(\frac{-x+x_1}{z}\right) \tag{5.9}$$

and if $f(x)$ involves only finitely many negative powers of x,

$$\mathrm{Res}_{x_1}\left(\frac{d}{dx_1}z^{-1}\delta\left(\frac{-x+x_1}{z}\right)\right)f(x_1)=-\,\mathrm{Res}_{x_1}z^{-1}\delta\left(\frac{-x+x_1}{z}\right)\frac{d}{dx_1}f(x_1) \tag{5.10}$$

(since the residue of a derivative is 0). From (3.22) and the $L(-1)$-derivative property for the contragredient module W_1', we have

$$\frac{d}{dx}Y_1^*(v,x) = Y_1^*(L(-1)v,x).$$

Thus the right-hand side of (5.8) is equal to

$$\begin{aligned}
&\lambda(Y_1^*(L(-1)v,x+z)w_{(1)}\otimes w_{(2)})\\
&\quad - \mathrm{Res}_{x_1} z^{-1}\delta\left(\frac{-x+x_1}{z}\right)\frac{d}{dx_1}\lambda(w_{(1)}\otimes Y_2(v,x_1)w_{(2)})\\
&= \lambda(Y_1^*(L(-1)v,x+z)w_{(1)}\otimes w_{(2)})\\
&\quad - \mathrm{Res}_{x_1} z^{-1}\delta\left(\frac{-x+x_1}{z}\right)\lambda(w_{(1)}\otimes Y_2(L(-1)v,x_1)w_{(2)})\\
&= (Y'_{Q(z)}(L(-1)v,x)\lambda)(w_{(1)}\otimes w_{(2)}),
\end{aligned} \tag{5.11}$$

completing the proof. $\square$

Proposition 5.2 *The action $Y'_{Q(z)}$ satisfies the commutator formula for vertex operators, that is, on $(W_1\otimes W_2)^*$,*

$$\begin{aligned}
&[Y'_{Q(z)}(v_1,x_1), Y'_{Q(z)}(v_2,x_2)] =\\
&= \mathrm{Res}_{x_0} x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right)Y'_{Q(z)}(Y(v_1,x_0)v_2,x_2)
\end{aligned} \tag{5.12}$$

for $v_1, v_2 \in V$.

The proof of this proposition will be given in Part II.

From these results and the relation (2.22), we see that the coefficient operators of $Y'_{Q(z)}(\omega,x)$ satisfy the Virasoro algebra commutator relations, that is, writing

$$Y'_{Q(z)}(\omega,x) = \sum_{n\in\mathbf{Z}} L'_{Q(z)}(n)x^{-n-2}, \tag{5.13}$$

we have

$$[L'_{Q(z)}(m), L'_{Q(z)}(n)] = (m-n)L'_{Q(z)}(m+n) + \frac{1}{12}(m^3-m)\delta_{m+n,0}c. \tag{5.14}$$

We call the eigenspaces of the operator $L'_{Q(z)}(0)$ the *weight subspaces* or *homogeneous subspaces* of $(W_1\otimes W_2)^*$, and we have the corresponding notions of *weight vector* (or *homogeneous vector*) and *weight*. When there is no confusion, we shall simply write $L'_{Q(z)}(n)$ as $L(n)$.

Let W_3 be another V-module. Note that $V \otimes \iota_+ \mathbb{C}[t, t^{-1}, (z+t)^{-1}]$ acts on W'_3 in the obvious way. The following result, which follows immediately from the definitions (4.4) and (5.2), provides further motivation for the definition of our action on $(W_1 \otimes W_2)^*$:

Proposition 5.3 *Under the natural isomorphism*

$$\mathrm{Hom}(W'_3, (W_1 \otimes W_2)^*) \xrightarrow{\sim} \mathrm{Hom}(W_1 \otimes W_2, \overline{W}_3), \tag{5.15}$$

the maps in $\mathrm{Hom}(W'_3, (W_1 \otimes W_2)^*)$ *intertwining the two actions of* $V \otimes \iota_+ \mathbb{C}[t, t^{-1}, (z+t)^{-1}]$ *on* W'_3 *and* $(W_1 \otimes W_2)^*$ *correspond exactly to the* $Q(z)$*-intertwining maps of type* $\binom{W_3}{W_1 W_2}$. □

Remark 5.4 Combining the last result with Proposition 4.7, we see that the maps in $\mathrm{Hom}(W'_3, (W_1 \otimes W_2)^*)$ intertwining the two actions on W'_3 and $(W_1 \otimes W_2)^*$ also correspond exactly to the intertwining operators of type $\binom{W'_1}{W'_3\, W_2}$. In particular, given any integer p, the map $F'_{\mathcal{Y},p} : W'_3 \to (W_1 \otimes W_2)^*$ defined by

$$F'_{\mathcal{Y},p}(w'_{(3)})(w_{(1)} \otimes w_{(2)}) = \langle w_{(1)}, \mathcal{Y}(w'_{(3)}, e^{l_p(z)}) w_{(2)} \rangle_{W'_1} \tag{5.16}$$

(recall (4.9)) intertwines the actions of $V \otimes \iota_+ \mathbb{C}[t, t^{-1}, (z+t)^{-1}]$ on W'_3 and $(W_1 \otimes W_2)^*$.

Suppose that $G \in \mathrm{Hom}(W'_3, (W_1 \otimes W_2)^*)$ intertwines the two actions as in Proposition 5.3. Then for $w'_{(3)} \in W'_3$,

$$\begin{aligned}
&\tau_{Q(z)}\left(z^{-1}\delta\left(\frac{x_1 - x_0}{z}\right) Y_t(v, x_0)\right) G(w'_{(3)}) = \\
&= G\left(\tau_{W'_3}\left(z^{-1}\delta\left(\frac{x_1 - x_0}{z}\right) Y_t(v, x_0)\right) w'_{(3)}\right) \\
&= G\left(z^{-1}\delta\left(\frac{x_1 - x_0}{z}\right) Y'_3(v, x_0) w'_{(3)}\right) \\
&= z^{-1}\delta\left(\frac{x_1 - x_0}{z}\right) G(Y'_3(v, x_0) w'_{(3)}) \\
&= z^{-1}\delta\left(\frac{x_1 - x_0}{z}\right) Y'_{Q(z)}(v, x_0) G(w'_{(3)}).
\end{aligned} \tag{5.17}$$

Thus $G(w'_{(3)})$ satisfies the following nontrivial and subtle condition on $\lambda \in (W_1 \otimes W_2)^*$: The formal Laurent series $Y'_{Q(z)}(v, x_0)\lambda$ involves only finitely many negative powers of x_0 and

$$\tau_{Q(z)}\left(z^{-1}\delta\left(\frac{x_1 - x_0}{z}\right) Y_t(v, x_0)\right)\lambda =$$
$$= z^{-1}\delta\left(\frac{x_1 - x_0}{z}\right) Y'_{Q(z)}(v, x_0)\lambda \quad \text{for all } v \in V. \tag{5.18}$$

(Note that the two sides are not *a priori* equal for general $\lambda \in (W_1 \otimes W_2)^*$.) We call this the *compatibility condition* on $\lambda \in (W_1 \otimes W_2)^*$, for the action $\tau_{Q(z)}$.

Let W be a subspace of $(W_1 \otimes W_2)^*$. We say that W is *compatible for* $\tau_{Q(z)}$ if every element of W satisfies the compatibility condition. Also, we say that W is ($\mathbb{C}$-)*graded* if it is $\mathbb{C}$-graded by its weight subspaces, and that W is a V-*module* (respectively, *generalized module*) if W is graded and is a module (respectively, generalized module) when equipped with this grading and with the action of $Y'_{Q(z)}(\cdot, x)$ (recall Definition 2.11). A sum of compatible modules or generalized modules is clearly a generalized module. The weight subspace of a subspace W with weight $n \in \mathbb{C}$ will be denoted $W_{(n)}$.

Given G as above, it is clear that $G(W'_3)$ is a V-module since G intertwines the two actions of $V \otimes \mathbb{C}[t, t^{-1}]$. We have in fact established that $G(W'_3)$ is in addition a compatible V-module since G intertwines the full actions. Moreover, if $H \in \mathrm{Hom}(W'_3, (W_1 \otimes W_2)^*)$ intertwines the two actions of $V \otimes \mathbb{C}[t, t^{-1}]$, then H intertwines the two actions of $V \otimes \iota_+\mathbb{C}[t, t^{-1}, (z + t)^{-1}]$ if and only if the V-module $H(W'_3)$ is compatible.

Define

$$W_1 \boxdot_{Q(z)} W_2 = \sum_{W \in \mathcal{W}_{Q(z)}} W = \bigcup_{W \in \mathcal{W}_{Q(z)}} W \subset (W_1 \otimes W_2)^*, \tag{5.19}$$

where $\mathcal{W}_{Q(z)}$ is the set all compatible modules for $\tau_{Q(z)}$ in $(W_1 \otimes W_2)^*$. Then $W_1 \boxdot_{Q(z)} W_2$ is a compatible generalized module and coincides with the sum (or union) of the images $G(W'_3)$ of modules W'_3 under the maps G as above. Moreover, for any V-module W_3 and any map $H : W'_3 \to W_1 \boxdot_{Q(z)} W_2$ of generalized modules, $H(W'_3)$ is compatible and hence H intertwines the two actions of $V \otimes \iota_+\mathbb{C}[t, t^{-1}, (z + t)^{-1}]$. Thus we have:

Proposition 5.5 *The subspace $W_1 \boxdot_{Q(z)} W_2$ of $(W_1 \otimes W_2)^*$ is a generalized module with the following property: Given any V-module W_3, there is a nat-*

ural linear isomorphism determined by (5.15) between the space of all $Q(z)$-intertwining maps of type $\binom{W_3}{W_1\,W_2}$ and the space of all maps of generalized modules from W_3' to $W_1 \boxdot_{Q(z)} W_2$. $\square$

Proposition 5.6 *Let V be a rational vertex operator algebra and W_1, W_2 two V-modules. Then $W_1 \boxdot_{Q(z)} W_2$ is a module.*

Proof Since $W_1 \boxdot_{Q(z)} W_2$ is the sum of all compatible modules for $\tau_{Q(z)}$ in $(W_1 \otimes W_2)^*$ and since by assumption every module is completely reducible, the generalized V-module $W_1 \boxdot_{Q(z)} W_2$ is a direct sum of irreducible modules. If it is an infinite direct sum, it must include infinitely many copies of at least one irreducible V-module, say, W_3, since a rational vertex operator algebra has only finitely many irreducible modules. From Proposition 5.5, the space of $Q(z)$-intertwining maps of type $\binom{W_3'}{W_1\,W_2}$ must be infinite-dimensional, and by Proposition 4.7, this contradicts the assumed finiteness of the fusion rules. Thus $W_1 \boxdot_{Q(z)} W_2$ is a finite direct sum of irreducible modules and hence is a module. $\square$

Now we assume that $W_1 \boxdot_{Q(z)} W_2$ is a module (which occurs if V is rational, by the last proposition). In this case, we define a V-module $W_1 \boxtimes_{Q(z)} W_2$ by

$$W_1 \boxtimes_{Q(z)} W_2 = (W_1 \boxdot_{Q(z)} W_2)' \tag{5.20}$$

($\boxdot' = \boxtimes$!) and we write the corresponding action as $Y_{Q(z)}$. Applying Proposition 5.5 to the special module $W_3 = W_1 \boxtimes_{Q(z)} W_2$ and the identity map $W_3' \to W_1 \boxdot_{Q(z)} W_2$ (recall Theorem 2.10), we obtain using (5.15) a canonical $Q(z)$-intertwining map of type $\binom{W_1 \boxtimes_{Q(z)} W_2}{W_1 W_2}$, which we denote

$$\begin{aligned} \boxtimes_{Q(z)} : W_1 \otimes W_2 &\to \overline{W_1 \boxtimes_{Q(z)} W_2} \\ w_{(1)} \otimes w_{(2)} &\mapsto w_{(1)} \boxtimes_{Q(z)} w_{(2)}. \end{aligned} \tag{5.21}$$

This is the unique linear map such that

$$\langle \lambda, w_{(1)} \boxtimes_{Q(z)} w_{(2)} \rangle_{W_1 \boxtimes_{Q(z)} W_2} = \lambda(w_{(1)} \otimes w_{(2)}) \tag{5.22}$$

for all $w_{(1)} \in W_1$, $w_{(2)} \in W_2$ and $\lambda \in W_1 \boxdot_{Q(z)} W_2$. Moreover, we have:

Proposition 5.7 *The $Q(z)$-product $(W_1 \boxtimes_{Q(z)} W_2, Y_{Q(z)}; \boxtimes_{Q(z)})$ is a $Q(z)$-tensor product of W_1 and W_2.*

Proof Let $(W_3, Y_3; F)$ be a $Q(z)$-product of W_1 and W_2. By Proposition 5.5, there is a unique V-module map

$$\eta' : W_3' \to W_1 \boxbslash_{Q(z)} W_2$$

such that

$$\langle w'_{(3)}, F(w_{(1)} \otimes w_{(2)})\rangle_{W_3} = \eta'(w'_{(3)})(w_{(1)} \otimes w_{(2)})$$

for any $w_{(1)} \in W_1$, $w_{(2)} \in W_2$ and $w'_{(3)} \in W_3'$. But by (5.22), this equals

$$\langle \eta'(w'_{(3)}), w_{(1)} \boxtimes_{Q(z)} w_{(2)} \rangle_{W_1 \boxtimes_{Q(z)} W_2} = \langle w'_{(3)}, \overline{\eta}(w_{(1)} \boxtimes_{Q(z)} w_{(2)} \rangle_{W_3},$$

where $\eta \in \mathrm{Hom}_V(W_1 \boxtimes_{Q(z)} W_2, W_3)$ and η' are mutually adjoint maps. In particular, there is a unique η such that

$$\langle w'_{(3)}, F(w_{(1)} \otimes w_{(2)})\rangle_{W_3} = \langle w'_{(3)}, \overline{\eta}(w_{(1)} \boxtimes_{Q(z)} w_{(2)} \rangle_{W_3},$$

i.e., such that

$$F = \overline{\eta} \circ \boxtimes_{Q(z)} : W_1 \otimes W_2 \to \overline{W}_3,$$

and this establishes the desired universal property. □

More generally, dropping the assumption that $W_1 \boxbslash_{Q(z)} W_2$ is a module, we have:

Proposition 5.8 *The $Q(z)$-tensor product of W_1 and W_2 exists (and is given by (5.20)) if and only if $W_1 \boxbslash_{Q(z)} W_2$ is a module.*

Proof It is sufficient to show that if the $Q(z)$-tensor product exists, then $W_1 \boxbslash_{Q(z)} W_2$ is a module. Consider the module

$$W_0 = (W_1 \boxtimes_{Q(z)} W_2)'.$$

Applying Proposition 5.5 to the $Q(z)$-product $W_1 \boxtimes_{Q(z)} W_2$, we have a unique map

$$i : W_0 \to W_1 \boxbslash_{Q(z)} W_2$$

of generalized modules such that

$$i(w_{(0)})(w_{(1)} \otimes w_{(2)}) = \langle w_{(0)}, w_{(1)} \boxtimes_{Q(z)} w_{(2)} \rangle_{W_1 \boxtimes_{Q(z)} W_2}$$

for $w_{(0)} \in W_0$, $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$. It suffices to show that i is a surjection.

Let $W \in \mathcal{W}_{Q(z)}$ (recall (5.19)) and set $W_3 = W'$. By Proposition 5.5, the injection $W_3' \hookrightarrow W_1 \boxbackslash_{Q(z)} W_2$ induces a unique $Q(z)$-intertwining map F of type $\binom{W'}{W_1 W_2}$ such that

$$w(w_{(1)} \otimes w_{(2)}) = \langle w, F(w_{(1)} \otimes w_{(2)})\rangle_{W'}$$

for $w \in W$, $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$. But by the universal property of $W_1 \boxtimes_{Q(z)} W_2$, there is a unique module map $\eta' : W_1 \boxtimes_{Q(z)} W_2 \to W'$ such that $F = \overline{\eta'} \circ \boxtimes_{Q(z)}$, and hence a unique module map $\eta : W \to W_0$ such that

$$\langle \eta(w), w_{(1)} \boxtimes_{Q(z)} w_{(2)}\rangle_{W_1 \boxtimes_{Q(z)} W_2} = \langle w, F(w_{(1)} \otimes w_{(2)})\rangle_{W'}.$$

Thus

$$\begin{aligned} w(w_{(1)} \otimes w_{(2)}) &= \langle \eta(w), w_{(1)} \boxtimes_{Q(z)} w_{(2)}\rangle_{W_1 \boxtimes_{Q(z)} W_2} \\ &= i(\eta(w))(w_{(1)} \otimes w_{(2)}) \end{aligned}$$

and so $w = i(\eta(w))$ for all $w \in W$, showing that W lies in the image of the map i and hence that i is surjective. $\square$

6 Second construction of $Q(z)$-tensor product

Let V be a vertex operator algebra and W_1, W_2 two V-modules. From the definition (5.19) of $W_1 \boxbackslash_{Q(z)} W_2$, we see that any element of $W_1 \boxbackslash_{Q(z)} W_2$ is an element λ of $(W_1 \otimes W_2)^*$ satisfying:

The compatibility condition (recall (5.18)): **(a)** The *lower truncation condition*: For all $v \in V$, the formal Laurent series $Y'_{Q(z)}(v, x)\lambda$ involves only finitely many negative powers of x.

(b) The following formula holds:

$$\begin{aligned} &\tau_{Q(z)}\left(z^{-1}\delta\left(\frac{x_1 - x_0}{z}\right) Y_t(v, x_0)\right)\lambda = \\ &\quad = z^{-1}\delta\left(\frac{x_1 - x_0}{z}\right) Y'_{Q(z)}(v, x_0)\lambda \quad \text{for all } v \in V. \qquad (6.1) \end{aligned}$$

The local grading-restriction condition: **(a)** The *grading condition*: λ is a (finite) sum of weight vectors of $(W_1 \otimes W_2)^*$.

(b) Let W_λ be the smallest subspace of $(W_1 \otimes W_2)^*$ containing λ and stable under the component operators $\tau_{Q(z)}(v \otimes t^n)$ of the operators $Y'_{Q(z)}(v, x)$ for $v \in V$, $n \in \mathbf{Z}$. Then the weight spaces $(W_\lambda)_{(n)}$, $n \in \mathbf{C}$, of the (graded) space W_λ have the properties

$$\dim\,(W_\lambda)_{(n)} < \infty \quad \text{for } n \in \mathbf{C}, \tag{6.2}$$

$$(W_\lambda)_{(n)} = 0 \quad \text{for } n \text{ whose real part is sufficiently small.} \tag{6.3}$$

Note that the set of the elements of $(W_1 \otimes W_2)^*$ satisfying any one of the lower truncation condition, the compatibility condition, the grading condition or the local grading-restriction condition forms a subspace.

In Part II, we shall prove the following two basic results:

Theorem 6.1 *Let λ be an element of $(W_1 \otimes W_2)^*$ satisfying the compatibility condition. Then when acting on λ, the Jacobi identity for $Y'_{Q(z)}$ holds, that is,*

$$\begin{aligned} &x_0^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right) Y'_{Q(z)}(u, x_1) Y'_{Q(z)}(v, x_2)\lambda \\ &\quad - x_0^{-1}\delta\left(\frac{x_2 - x_1}{-x_0}\right) Y'_{Q(z)}(v, x_2) Y'_{Q(z)}(u, x_1)\lambda \\ &= x_2^{-1}\delta\left(\frac{x_1 - x_0}{x_2}\right) Y'_{Q(z)}(Y(u, x_0)v, x_2)\lambda \end{aligned} \tag{6.4}$$

for $u, v \in V$.

Proposition 6.2 *The subspace consisting of the elements of $(W_1 \otimes W_2)^*$ satisfying the compatibility condition is stable under the operators $\tau_{Q(z)}(v \otimes t^n)$ for $v \in V$ and $n \in \mathbf{Z}$, and similarly for the subspace consisting of the elements satisfying the local grading-restriction condition.*

These results give us another construction of $W_1 \boxtimes_{Q(z)} W_2$:

Theorem 6.3 *The subspace of $(W_1 \otimes W_2)^*$ consisting of the elements satisfying the compatibility condition and the local grading-restriction condition, equipped with $Y'_{Q(z)}$, is a generalized module and is equal to $W_1 \boxtimes_{Q(z)} W_2$.*

Proof Let W_0 be the space of vectors satisfying the two conditions. We have already observed that $W_1 \boxtimes_{Q(z)} W_2 \subset W_0$, and it suffices to show that W_0 is a generalized module which is a union of compatible modules. But W_0 is a compatible generalized module by Theorem 6.1 and Proposition 6.2, together with Proposition 5.1 and formula (5.14), and every element of W_0 generates a compatible module contained in W_0, by the local grading-restriction condition. □

The following result follows immediately from Proposition 5.8, the theorem above and the definition of $W_1 \boxtimes_{Q(z)} W_2$:

Corollary 6.4 *The $Q(z)$-tensor product of W_1 and W_2 exists if and only if the subspace of $(W_1 \otimes W_2)^*$ consisting of the elements satisfying the compatibility condition and the local grading-restriction condition, equipped with $Y'_{Q(z)}$, is a module. In this case, this module coincides with the module $W_1 \boxtimes_{Q(z)} W_2$, and the contragredient module of this module, equipped with the $Q(z)$-intertwining map $\boxtimes_{Q(z)}$, is a $Q(z)$-tensor product of W_1 and W_2, equal to the structure $(W_1 \boxtimes_{Q(z)} W_2, Y_{Q(z)}; \boxtimes_{Q(z)})$ constructed in Section 5.*

From this result and Propositions 5.6 and 5.7, we have:

Corollary 6.5 *Let V be a rational vertex operator algebra and W_1, W_2 two V-modules. Then the $Q(z)$-tensor product $(W_1 \boxtimes_{Q(z)} W_2, Y_{Q(z)}; \boxtimes_{Q(z)})$ may be constructed as described in Corollary 6.4.*

References

[BPZ] A. A. Belavin, A. M. Polyakov and A. B. Zamolodchikov, Infinite conformal symmetries in two-dimensional quantum field theory, *Nucl. Phys.* **B241** (1984), 333–380.

[B1] R. E. Borcherds, Vertex algebras, Kac-Moody algebras, and the Monster, *Proc. Natl. Acad. Sci. USA* **83** (1986), 3068–3071.

[B2] R. E. Borcherds, Monstrous moonshine and monstrous Lie superalgebras, *Invent. Math.* **109** (1992), 405–444.

[CN] J. H. Conway and S. P. Norton, Monstrous moonshine, *Bull. London Math. Soc.* **11** (1979), 308–339.

[Do] C. Dong, Representations of the moonshine module vertex operator algebra, to appear.

[DL] C. Dong and J. Lepowsky, *Generalized Vertex Algebras and Relative Vertex Operators*, Progress in Math., Vol. 112, Birkhäuser, Boston, 1993.

[Dr1] V. Drinfeld, On quasi-cocommutative Hopf algebras, *Algebra and Analysis* **1** (1989), 30–46.

[Dr2] V. Drinfeld, On quasitriangular quasi-Hopf algebras and a certain group closely related to Gal($\bar{\mathbf{Q}}/\mathbf{Q}$), *Algebra and Analysis* **2** (1990), 149–181.

[F] M. Finkelberg, Fusion categories, Ph.D. thesis, Harvard University, 1993.

[FHL] I. B. Frenkel, Y.-Z. Huang and J. Lepowsky, On axiomatic approaches to vertex operator algebras and modules, preprint, 1989; *Memoirs Amer. Math. Soc.* **104**, 1993.

[FLM1] I. B. Frenkel, J. Lepowsky and A. Meurman, A natural representation of the Fischer-Griess Monster with the modular function J as character, *Proc. Natl. Acad. Sci. USA* **81** (1984), 3256–3260.

[FLM2] I. B. Frenkel, J. Lepowsky and A. Meurman, *Vertex Operator Algebras and the Monster*, Pure and Appl. Math., Vol. 134, Academic Press, Boston, 1988.

[FZ] I. B. Frenkel and Y. Zhu, Vertex operator algebras associated to representations of affine and Virasoro algebras, *Duke Math. J.* **66** (1992), 123–168.

[FS] D. Friedan and S. Shenker, The analytic geometry of two-dimensional conformal field theory, *Nucl. Phys.* **B281** (1987), 509–545.

[H1] Y.-Z. Huang, *On the geometric interpretation of vertex operator algebras*, Ph.D. thesis, Rutgers University, 1990; Operads and the geometric interpretation of vertex operator algebras, I, to appear.

[H2] Y.-Z. Huang, Geometric interpretation of vertex operator algebras, *Proc. Natl. Acad. Sci. USA* **88** (1991), 9964–9968.

[H3] Y.-Z. Huang, Applications of the geometric interpretation of vertex operator algebras, in: *Proc. 20th International Conference on Differential Geometric Methods in Theoretical Physics, New York, 1991*, ed. S. Catto and A. Rocha, World Scientific, Singapore, 1992, Vol. 1, 333–343.

[H4] Y.-Z. Huang, Vertex operator algebras and conformal field theory, *Intl. J. Mod. Phys.* **A7** (1992), 2109–2151.

[H5] Y.-Z. Huang, Operads and the geometric interpretation of vertex operator algebras, II, in preparation.

[HL1] Y.-Z. Huang and J. Lepowsky, Toward a theory of tensor products for representations of a vertex operator algebra, in: *Proc. 20th International Conference on Differential Geometric Methods in Theoretical Physics, New York, 1991*, ed. S. Catto and A. Rocha, World Scientific, Singapore, 1992, Vol. 1, 344–354.

[HL2] Y.-Z. Huang and J. Lepowsky, Vertex operator algebras and operads, in: *The Gelfand Mathematical Seminars, 1990–1992*, ed. L. Corwin, I. Gelfand and J. Lepowsky, Birkhäuser, Boston, 1993, 145–161.

[HL3] Y.-Z. Huang and J. Lepowsky, Operadic formulation of the notion of vertex operator algebra, to appear.

[J] V. F. R. Jones, Hecke algebra representations of braid groups and link polynomials, *Ann. Math.* **126** (1987), 335–388.

[KL1] D. Kazhdan and G. Lusztig, Affine Lie algebras and quantum groups, *International Mathematics Research Notices* (in *Duke Math. J.*) **2** (1991), 21–29.

[KL2] D. Kazhdan and G. Lusztig, Tensor structures arising from affine Lie algebras, I, *J. Amer. Math. Soc.*, to appear.

[KL3] D. Kazhdan and G. Lusztig, Tensor structures arising from affine Lie algebras, II, *J. Amer. Math. Soc.*, to appear.

[KL4] D. Kazhdan and G. Lusztig, Tensor structures arising from affine Lie algebras, III, to appear.

[KL5] D. Kazhdan and G. Lusztig, Tensor structures arising from affine Lie algebras, IV, to appear.

[KZ] V. G. Knizhnik and A. B. Zamolodchikov, Current algebra and Wess-Zumino models in two dimensions, *Nucl. Phys.* **B247** (1984), 83–103.

[K1] T. Kohno, Linear representations of braid groups and classical Yang-Baxter equations, in: *Braids, Santa Cruz, 1986*, *Contemporary Math.* **78** (1988), 339–363.

[K2] T. Kohno, Monodromy representations of braid groups and Yang-Baxter equations, *Ann. Inst. Fourier* **37** (1987), 139–160.

[L] J. Lepowsky, Remarks on vertex operator algebras and moonshine, in: *Proc. 20th International Conference on Differential Geometric Methods in Theoretical Physics, New York, 1991*, ed. S. Catto and A. Rocha, World Scientific, Singapore, 1992, Vol. 1, 362–370.

[M] J. P. May, *The geometry of iterated loop spaces*, Lecture Notes in Mathematics, No. 271, Springer-Verlag, 1972.

[MS] G. Moore and N. Seiberg, Classical and quantum conformal field theory, *Comm. Math. Phys.* **123** (1989), 177–254.

[RT] N. Reshetikhin and V. G. Turaev, Invariants of 3-manifolds via link polynomials and quantum groups, *Invent. Math.* **103** (1991), 547–597.

[TK] A. Tsuchiya and Y. Kanie, Vertex operators in conformal field theory on $\mathbf{P}^1$ and monodromy representations of braid group, in: *Conformal*

Field Theory and Solvable Lattice Models, *Advanced Studies in Pure Math.*, Vol. 16, Kinokuniya Company Ltd., Tokyo, 1988, 297–372.

[V] E. Verlinde, Fusion rules and modular transformations in 2D conformal field theory, *Nucl. Phys.* **B300** (1988), 360–376.

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