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Generalization of Anderson's t-motives and Tate conjecture

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§0. Introduction.

First we shall recall the Tate conjecture for abelian varieties.

Let B and B' be abelian varieties over a field k, and l a prime number $\neq \mathrm{char}(k)$. In [Tat], Tate asked:

If k is finitely generated over a prime field, is the natural homomorphism

$$\mathrm{Hom}_k(B', B) \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_l \to \mathrm{Hom}_{\mathbb{Z}_l[\mathrm{Gal}(k^{\mathrm{sep}}/k)]}(T_l(B'), T_l(B))$$

an isomorphism?

This is so-called *Tate conjecture.* It has been solved by Tate ([Tat]) for k finite, by Zarhin ([Z1], [Z2], [Z3]) and Mori ([M1], [M2]) in the case $\mathrm{char}(k) > 0$, and, finally, by Faltings ([F], [FW]) in the case $\mathrm{char}(k) = 0$.

We can formulate an analogue of the Tate conjecture for Drinfeld modules (=elliptic modules ([D1])) as follows. Let C be a proper smooth geometrically connected curve over the finite field $\mathbb{F}_q$, ∞ a closed point of C, and put $A = \Gamma(C - \{\infty\}, \mathcal{O}_C)$. Let k be an A-field, i.e. a field equipped with an A-algebra structure $\iota : A \to k$. Let G and G' be Drinfeld A-modules over k (with respect to ι), and v a maximal ideal $\neq \mathrm{Ker}(\iota)$. Then, if k is finitely generated over $\mathbb{F}_q$, is the natural homomorphism

$$\mathrm{Hom}_k(G', G) \underset{A}{\otimes} \widehat{A}_v \to \mathrm{Hom}_{\widehat{A}_v[\mathrm{Gal}(k^{\mathrm{sep}}/k)]}(T_v(G'), T_v(G))$$

an isomorphism? We can also formulate a similar conjecture for Anderson's abelian t-modules ([A1]) in the case $A = \mathbb{F}_q[t]$.

As for this analogue (which we will also refer as Tate conjecture), besides the result for Drinfeld modules over finite fields (implicitly [D2], [G], explicitly [Y]), we have Taguchi's important approach ([Tag1], [Tag2]). He studied the v-adic Galois representations attached to Drinfeld modules on the lines of Faltings, and succeeded in proving their semisimplicity. However, in proving the Tate conjecture for Drinfeld modules on these lines, we have to generalize his work for higher-dimensional t-modules (in order to apply Zarhin's arguments), which is difficult for several reasons. For example, no satisfactory theory of polarizations and moduli spaces of abelian t-modules has not been established.

Our aim is: (i) to introduce the notions of *A-premodules* and *A-premotives*, generalizing the notions of Anderson's (abelian) t-modules and t-motives ([A1]) respectively; and (ii) to investigate an analogue of the Tate conjecture for them.

As a special case, we obtain the affirmative answer to the conjecture above for Drinfeld modules and abelian t-modules. Our approach is purely algebraic, unlike Taguchi's (first) approach.

Remark. (i) The present article is a survey of [Tam].
(ii) Taguchi also has proved independently the Tate conjecture for Drinfeld modules and abelian t-modules, and his proof is close to ours ([Tag4]).

§1. Definition of A-premodules and A-premotives.

For simplicity, we assume that our finite base field is the prime field $\mathbb{F}_p$, although we can develop a similar theory over any finite field $\mathbb{F}_q$. Let A be a commutative $\mathbb{F}_p$-algebra, and k a field of characteristic p.

Definition. An *A-module scheme* over k is a pair (G, ϕ), where G is a commutative group scheme over k and $\phi : A \to \mathrm{End}_{(\text{group schemes}/k)}(G)$ is an $\mathbb{F}_p$-algebra homomorphism. (We often write G instead of (G, ϕ).)

Remark. This definition is temporary, and is different from the definition appearing in [Tag2], [Tag3].

The following proposition is fundamental.

Proposition. *The functor*

$$G \mapsto M(G) \stackrel{\text{def}}{=} \mathrm{Hom}_{(\text{group schemes}/k)}(G, \mathbb{G}_a)$$

gives an anti-equivalence between the category of affine A-module schemes over k whose Verschiebungs are zero and that of left $k\{\tau\} \underset{\mathbb{F}_p}{\otimes} A$-modules. Here $k\{\tau\}$ is the ring whose underlying abelian group is a k-vector space with basis $\{\tau^i\}_{i=0,1,\dots}$ and whose multiplication rule is:

$$\left(\sum_i a_i \tau^i\right)\left(\sum_j b_j \tau^j\right) = \sum_i \sum_j a_i b_j^{p^i} \tau^{i+j}.$$

(It is well-known that $k\{\tau\}$ can be identified with the endomorphism ring of the commutative group scheme $\mathbb{G}_a$ over k.) □

Definition. (i) An *A-premotive* over k is a left $k\{\tau\} \underset{\mathbb{F}_p}{\otimes} A$-module M satisfying the following two conditions:

(a) M is a finitely presented $k \underset{\mathbb{F}_p}{\otimes} A$-module;

(b) $\tau_{\text{linear}} : M^{(p)} \to M$ is injective, where

$$M^{(p)} = k \underset{\sigma, k}{\otimes} M$$

($\sigma : k \to k$ is the p-th power map), and τ_{linear} is the linearization of τ:

$$\tau_{\text{linear}}(a \otimes x) = a\tau x.$$

(ii) An *A-premodule* over k is an affine A-module scheme over k whose Verschiebung is zero and such that $M(G)$ is an A-premotive.

Remark. (i) By the proposition above, the category of A-premodules over k is anti-equivalent to that of A-premotives over k.
(ii) Drinfeld modules and Anderson's abelian t-modules are examples of A-premodules (for suitable A).
(iii) Unlike the cases of Drinfeld modules and Anderson's abelian t-modules, we do not assume that k is an A-algebra. In particular, we do not (or cannot!) assume any compatibility of A-actions on $\mathrm{Lie}(G)$.
(iv) We do not assume that G is algebraic over k, or, equivalently, that $M(G)$ is finitely generated as a $k\{\tau\}$-module.

§2. Tate conjecture.

From now on, we assume that A is a finitely generated $\mathbb{F}_p$-algebra. (In particular, A is noetherian.)

Definition. Let $G = (G, \phi)$ be an A-premotive over k, and put $M = M(G)$, the A-premotive associated with G.
(i) Let J be an ideal of A. We define an A-module scheme $G[J]$ by:

$$G[J] = \bigcap_{f \in J} \mathrm{Ker}(\phi_f : G \to G),$$

which is regarded also as an A/J-module scheme. Note that

$$M(G[J]) = M/JM.$$

(ii) Let I be an ideal of A. We define an ind-A-module scheme $\mathcal{T}_I(G)$ by:

$$\mathcal{T}_I(G) = \bigcup_{n=0}^{\infty} G[I^n],$$

and call it *the I-divisible Tate module scheme of G.*

Let v be a maximal ideal of A, and G and G' be A-premodules over k étale along v, that is to say, $G[v^n]$ and $G'[v^n]$ are étale over k for all $n \geq 0$ (or, equivalently, for some $n > 0$). Then the Tate conjecture in our situation is as follows:

If k is finitely generated over $\mathbb{F}_p$, is the natural homomorphism

$$(\sharp) \qquad \mathrm{Hom}_k(G', G) \otimes_A A_v \to \mathrm{Hom}_{A_v[\mathrm{Gal}(k^{\mathrm{sep}}/k)]}(\mathcal{T}_v(G')(k^{\mathrm{sep}}), \mathcal{T}_v(G)(k^{\mathrm{sep}}))$$

an isomorphism?

Remark. This is a v-divisible formulation of the Tate conjecture. We also have a v-adic formulation of the right-hand side under a certain condition.

From now on, we always assume that k is finitely generated over $\mathbb{F}_p$. When k is finite, the Tate conjecture is easy to prove:

Theorem. *If k is finite, then ($\sharp$) is an isomorphism.* $\square$

Unfortunately, ($\sharp$) is not an isomorphism in general.

Example. Let $A = \mathbb{F}_p[t]$, and v a maximal ideal of A. For $\theta \in k$, the Carlitz module with respect to θ is an A-premodule $G = (G, \phi)$, where

$$G = \mathbb{G}_a$$

and

$$\phi : A \to \mathrm{End}_{(\text{group schemes}/k)}(\mathbb{G}_a) = k\{\tau\},\ t \mapsto \theta + \tau.$$

Let G' be the Carlitz module with respect to θ^p.

Assume that θ is transcendental over $\mathbb{F}_p$. (Note that G is then étale along v.) Then we can prove

$$\mathrm{Hom}_k(G', G) = 0,$$

and

$$\mathrm{Hom}_k(\mathcal{T}_v(G'), \mathcal{T}_v(G)) \simeq \widehat{A}_v.$$

The example above shows that our definition of 'pre-'modules and 'pre-'motives is not good enough for the Tate conjecture. So, we introduce the concept of admissibility for the pair (G, G'), so that the Tate conjecture is valid. For simplicity, we assume, from now on, that the ring A is regular at v.

Definition. Put $M = M(G)$ and $M' = M(G')$. We say that the (ordered) pair (G, G') is *admissible at* v, if the following conditions are satisfied:

(a) $M \underset{A}{\otimes} A_v$ and $M' \underset{A}{\otimes} A_v$ are projective as $k \underset{\mathbb{F}_p}{\otimes} A_v$-modules, where

$$A_v = (A - v)^{-1}A.$$

(b)

$$\mathrm{Hom}_{k \underset{\mathbb{F}_p}{\otimes} A_v}(\overline{M}^{(p^i)} \underset{A}{\otimes} A_v, \overline{M}' \underset{A}{\otimes} A_v) = 0$$

for each $i > 0$, where $\overline{N}$ means $N/k\{\tau\}\tau N$.

The following is our main theorem:

Theorem. *If the pair (G, G') is admissible at v, then ($\sharp$) is an isomorphism.* $\square$

Corollary. *The Tate conjecture for Drinfeld A-modules and abelian t-modules (see §0) is true.* $\square$

§3. Outline of proof.

For simplicity, we assume that (A is regular at v and) A/v, which is a finite extension of $\mathbb{F}_p$, coincides with $\mathbb{F}_p$. Accordingly, we have

$$\widehat{A}_v \simeq \mathbb{F}_p[[t_1, \ldots, t_n]],$$

where $n = \dim(\widehat{A}_v)$. Now, in terms of premotives, ($\sharp$) corresponds to

$$\mathrm{Hom}_{(k \underset{\mathbb{F}_p}{\otimes} \widehat{A}_v)\{\tau\}}(M \underset{A}{\otimes} \widehat{A}_v, M' \underset{A}{\otimes} \widehat{A}_v) \to \mathrm{Hom}_{(k \underset{\mathbb{F}_p}{\otimes} A)^{\wedge}_v\{\tau\}}(\widehat{M}_v, \widehat{M'}_v).$$

The problem is the difference of two rings

$$k \underset{\mathbb{F}_p}{\otimes} \widehat{A}_v \simeq k \underset{\mathbb{F}_p}{\otimes} (\mathbb{F}_p[[t_1, \ldots, t_n]])$$

and

$$(k \underset{\mathbb{F}_p}{\otimes} A)_v^{\widehat{}} \simeq k[[t_1, \ldots, t_n]].$$

(Thereby the case where k is finite is clear.)

More concretely, assume that M and M' are free $k \underset{\mathbb{F}_p}{\otimes} A$-module of rank r and r', respectively. (We can reduce the general case to this case.) Let $C \in M_r(k \underset{\mathbb{F}_p}{\otimes} A)$ and $C' \in M_{r'}(k \underset{\mathbb{F}_p}{\otimes} A)$ be the matrices representing the τ-actions on M and M', respectively. Then the problem is: If $X \in M_{r',r}(k[[t_1, \ldots, t_n]])$ satisfies the equation

$$XC = C'X^{(p)},$$

does X belong to $M_{r',r}(k \underset{\mathbb{F}_p}{\otimes} (\mathbb{F}_p[[t_1, \ldots, t_n]]))$ automatically? Here,

$$f^{(p)} \stackrel{\text{def}}{=} \sum a^p_{i_1 \cdots_n} t_1^{i_1} \ldots t_n^{i_n}$$

for $f = \sum a_{i_1 \cdots_n} t_1^{i_1} \ldots t_n^{i_n} \in k[[t_1, \ldots, t_n]]$, and

$$X^{(p)} \stackrel{\text{def}}{=} (f^{(p)}_{i',i})$$

for $X = (f_{i',i}) \in M_{r',r}(k[[t_1, \ldots, t_n]])$.

The proof goes as follows. First, we develop a local theory of premotives, which can be regarded as a generalization of Anderson's recent work [A2]. The conclusion is that, for each discrete valuation ring R whose field of fractions is k, the solution X belongs to $M_{r',r}(k \underset{R}{\otimes} (R[[t_1, \ldots, t_n]]))$ and it belongs to $M_{r',r}(R[[t_1, \ldots, t_n]])$ for 'most' R. Then, the following is a key lemma to collect the local results into the global result (which follows from finiteness of the global sections of the coherent $\mathcal{O}_{\mathfrak{X}}$-module $\mathcal{O}_{\mathfrak{X}}(D)$ for a suitable (Weil) divisor D whose support is in Σ_0.)

Lemma. *Let $\mathfrak{X}$ be a proper normal model of k over $\mathbb{F}_p$, and Σ the set of points of codimension 1 in $\mathfrak{X}$. Let Σ_0 be a finite subset of Σ. Then we have*

$$\bigcap_{x \in \Sigma - \Sigma_0} \mathcal{O}_{\mathfrak{X},x}[[t_1, \ldots, t_n]] \bigcap \bigcap_{x \in \Sigma_0} k \underset{\mathcal{O}_{\mathfrak{X},x}}{\otimes} (\mathcal{O}_{\mathfrak{X},x}[[t_1, \ldots, t_n]]) \subset k \underset{\mathbb{F}_p}{\otimes} (\mathbb{F}_p[[t_1, \ldots, t_n]]). \quad \square$$

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