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WAVELET TRANSFORM ASSOCIATED TO THE EIGENVECTORS OF THE OPERATOR $Q-ikP^{-1}$

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I. Introduction

Recently, wavelet transform is often used for the 'time-frequency' analysis and for the 'multi-resolution' analysis in signal processing.[1-4] It is a kind of expansion of a square-integrable function into a superposition of 'wavelets' which have a constant shape but with various time-scales and with various time-shifts. In this field, several types of 'wavelets' are proposed for practical overcomplete systems whose (pseudo-)basis vectors are almost localized both in the time domain and in the frequency domain.

In terms of mathematical physics, this 'wavelet' is regarded as a generalized coherent state associated with the affine group[8] which is analoguous to the coherent state associated with the Weyl-Hisenberg group. The unitary representation of the affine group was investigated by Gelfand and Neumark[5] and Aslaksen and Klauder[6][7]. In terms of the Hermitean operators satisfying $[Q,P]=iI$, the affine transformation of the coordinate (or the eigenvalue of Q) $x \rightarrow rx-q$ is represented by the unitary operator $\exp(-iqP)\exp(isB)$ where $s=-\log r$ and $B=(PQ+QP)/2$ [7]. The operator B satisfies the commutation relation $[B,P]=iP$.

As is well known, the coherent state $|\alpha\rangle_a$ is the eigenvector of the annihilation operator $a=2^{-1/2}(Q+iP)$ [9] and satisfies the 'displacement' relation

$$\exp(ipQ-iqP)\,|\,\alpha\rangle_a = \exp[i\mathrm{Im}\{\gamma\,\alpha^*\}]\;|\,\alpha+\gamma\rangle_a$$
$$\text{where}\quad \gamma=2^{-1/2}(q+ip)\ . \qquad (1)$$

In this paper, for the affine group, we will find an analogous operator A whose eigenvector $|\,\alpha\rangle_A$ satisfies the relation

$$\exp(-iqP)\exp(isB)\,|\,\alpha\rangle_A = c(\alpha,s,q)\;|\,f(\alpha;s,q)\rangle_A \qquad (2)$$

where c and f are scalar functions of α,s and q. It is desirable that the system of eigenvectors is complete in the domain of definition of the operator A, and that the wavefunction of the eigenvector has useful property.

In Section III, we will discuss how to find such an operator, in respect of commutation relations. In the following sections we will discuss one of such operators

$$A_k \triangleq Q-ikP^{-1} \qquad (k\text{:positive integer})$$

which is defined in a subspace of $L^2_C(R)$ which is orthogonal to the eigenvector $|\,0\rangle_P$ of the operator P. For the operator A_k, relation (2) holds with $f(\alpha;s,q)=\exp(-s)(\alpha+q)$, for any complex eigenvalue α.

In Section IV, we will show that the wavefunction of the eigenvector of this operator (in 'position coordinate representation') is a very simple rational function which is square-integrable, and then we will investigate several properties of this function, such as the 'admissibility' and the degree of localization. It is also shown that the system of the eigenfunctions is over-complete in $L^2_C(R)$.

When k is large, this wavefunction has very similar shape to that of the squeezed state[10], or in particular cases to that of the coherent state (in terms of signal processing, to the Gabor's wavelet). In Section V, we will show this fact mathematically in terms of the norm $\| \quad \|_{L2}$.

II. WAVELETS

The wavelet transform was propoesd by Morlet et al.[1]. In this section we will mention the wavelet briefly. When a h(x) in $L^2(R)$ satisfies the 'admissibility condition'

$$C_h \triangleq \int_{-\infty}^{\infty} |y|^{-1} |H(y)|^2 \, dy < \infty \qquad (4)$$

where H(y) is the Fourier transform of h(x), then an arbitrary square-integrable function can be expressed as a linear superposition of the 'wavelets' with the 'similar shape to h(x)' in the following sense. Define

$$h^{(a,b)}(x) \triangleq |a|^{-1/2} h((x-b)/a) \quad (a,b\text{:real}) \quad . \qquad (5)$$

Then $\{ h^{(a,b)}(x) \ ; \ a,b\text{:real} \}$ is a complete system in $L^2(R)$. (See [2][3][4]) However, it is 'over-complete'. In terms of quantum mechanics, by denoting an arbitrary function in $L^2_C(R)$ by a vector $| \phi \rangle$ (not necessarily normalized) such that ${}_Q\langle x | \phi \rangle = f(x)$ (where $| x \rangle_Q$ denotes the eigenvector of the position coordinate operator Q associated with the eigenvalue x) and $h^{(a,b)}(x)$ by the vector $| a,b \rangle$ such that ${}_Q\langle x | a,b \rangle = h^{(a,b)}(x)$,

$$C_h^{-1} \int_{-\infty}^{\infty} |a|^{-2} \, da \int_{-\infty}^{\infty} db \ | a,b \rangle \langle a,b | \phi \rangle = | \phi \rangle \quad . \qquad (6)$$

It is analogous to the overcomplete system of the coherent states

[9][11] which satisfies

$$\pi^{-1}\int_{-\infty}^{\infty}d(\mathrm{Re}\,\alpha)\int_{-\infty}^{\infty}d(\mathrm{Im}\,\alpha)\quad|\,\alpha\,\rangle_{aa}\langle\,\alpha\,|\,\phi\,\rangle\;=\;|\,\phi\,\rangle\quad. \tag{7}$$

III. OPERATOR WHOSE EIGENVECTORS CONSTITUTE A WAVELET SYSTEM

It is well known that the relation (7) holds for more general cases and the system of coherent states is a special case of them. In a similar sense to this, the relation (6) holds whenever h(x) satisfies the admissibility condition (4). So a question arises: is there any simple operator such that the system of the eigenvectors would be a overcomplete system satisfing the relation (6) and for the eigenvectors a 'shift of the eigenvalue' would be performed by the unitary transformation represented by exp(-iqP)exp(isB) ? In this section we will discuss this question in terms of commutation relations.

In the case of the coherent state, the displacement of the eigenvalue in (1) can be derived from the relation

$$\exp(-ipQ+iqP)\ a\ \exp(ipQ-iqP)$$
$$=\exp(\gamma^{*}a-\gamma a^{\dagger})\ a\ \exp(\gamma a^{\dagger}-\gamma^{*}a)=a+\gamma I\quad, \tag{8}$$

which results from the commutation relation $[Q,P]=iI$ or $[a,a^{\dagger}]=I$. In a similar manner to this, if we can find a operator A such that

$$\exp(-isB)\exp(iqP)\ A\ \exp(-iqP)\exp(isB)\;=\;f(A;s,q) \tag{9}$$

(where B=(PQ+QP)/2 as defined above, and f(A;s,q) is the operator obtained by the substitution of the operator A into some function $f(\alpha;s,q)$) ,

then, from the relation with the eigenvector $|\,\alpha\,\rangle_A$ of of the operator A associated with the eigenvalue α

$$A\,[\,\exp(-iqP)\exp(isB)\,|\,\alpha\rangle_A\,] = \exp(-iqP)\exp(isB)f(A;s,q)\,|\,\alpha\rangle_A$$
$$= f(\alpha;s,q)\,[\,\exp(-iqP)\exp(isB)\,|\,\alpha\rangle_A\,]\,, \qquad (10)$$

the relation (2) is shown to hold. In comparison with the well-known relation for general operators S and T (See Chap.3 of [11])

$$\exp(zS)\,T\,\exp(-zS) = T + z[S,T] + (z^2/2!)[S,[S,T]]$$
$$+ (z^3/3!)[S,[S,[S,T]]] + \ldots \,, \qquad (11)$$

an operator satisfying the following commutation relations has the property (10);

$$[B,A] = g_1(A) \quad \text{and} \quad [P,A] = g_2(A) \qquad (12)$$

where g_1 and g_2 are functions of A. Since it is shown from the definition of B that $[B,P^n]=inP^n$ and $[B,Q^n]=-inQ^n$, we can find an example satisfying (12) easily. A trivial example is A=I. There are two other trivial cases, where $A=P^n$ or A=Q. A simple but nontrivial example is

$$A_k \triangleq Q - ikP^{-1} \quad (\text{ k:integer }) \qquad (13)$$

which is the main subject of this paper. For the well-definedness of the operator we restrict the space within a subspace where the the inner product

$$H(y) \triangleq {}_P\langle y\,|\,\phi\rangle$$

satisfies the condition

$$\int_{-\infty}^{\infty} |\,y\,|^{-2}\,|\,H(y)\,|^2\,dy < \infty \,.$$

(It is clear that this subspace is orthogonal to $|\,0\rangle_P$.)

The operator A_k has complex eigenvalues as shown in the next section, and it satisfies the following relations;

$$A_k^{\dagger} = A_{-k} \qquad (14)$$

$$[B,A_k] = -iA_k \tag{15}$$

$$[P,A_k] = -iI \tag{16}$$

$$[Q,A_k] = -kP^{-2} \tag{17}$$

$$[A_k,A_j] = (k-j)\,P^{-2} \tag{18}$$

From (11),(15) and (16),

$$\exp(-isB)\exp(iqP)\,A_k\,\exp(-iqP)\exp(isB) = \exp(-s)(A+qI), \tag{19}$$

so the wanted relation holds as

$$\exp(-iqP)\exp(isB)\,|\,\alpha\rangle_{A_k} = c(\alpha,s,q)\ \ |\exp(-s)(\alpha+q)\rangle_{A_k}\ . \tag{20}$$

Note that this relation holds even if k is not an integer. However, in the next section, it is shown that the eigenvector of A_k is square-integrable <u>when k is a positive integer and the eigenvalue is not real</u>.

IV. WAVEFUNCTIONS OF THE EIGENVECTORS OF $Q-ikP^{-1}$

First, we calculate the eigenfunctions of A_k in position-coordinate representation

$$h_k^{(\alpha)}(x) \triangleq {}_Q\langle x\,|\,\alpha\rangle_{A_k}\ . \tag{21}$$

Because of the restriction of the domain of definition of the operator which is mentioned in Section III, the relation ${}_P\langle 0\,|\,\alpha\rangle_{A_k} = 0$ must hold which means

$$\int_{-\infty}^{\infty} h_k^{(\alpha)}(x)\,dx = 0\ . \tag{22}$$

Since

$$(PQ-ikI)\,|\,\alpha\rangle_{A_k} = PA_k\,|\,\alpha\rangle_{A_k} = \alpha P\,|\,\alpha\rangle_{A_k}, \tag{23}$$

the wavefunction belongs to the class of the solutions of the differential equation

$$(x-\alpha)\frac{d}{dx}h_k^{(\alpha)}(x)+(k+1)\,h_k^{(\alpha)}(x)=0\ . \tag{24}$$

For positive k, the solution satisfying the condition (22) of this differential equation exists only when α is not real, as

$$h_k^{(\alpha)}(x)=C_k(x-\alpha)^{-(k+1)} \tag{25}$$

where C_k is a constant. We can easily verify this solution to be satisfying

$$A_k\,|\,\alpha\rangle_{A_k}=\alpha\,|\,\alpha\rangle_{A_k}.$$

Since a residue calculus results in

$$\int_{-\infty}^{\infty}(x-\mu^{*})^{-(m+1)}(x-\lambda)^{-(n+1)}dx=\begin{cases}0 & (\text{ if } (\mathrm{Im}\,\mu)(\mathrm{Im}\,\lambda)<0\ ) \\ 2\pi i(-1)^{n}\,{}_{m+n}C_{n}\ (\lambda-\mu^{*})^{-(m+n+1)} & (\text{ if } (\mathrm{Im}\,\mu)(\mathrm{Im}\,\lambda)>0\ )\ ,\end{cases} \tag{26}$$

So, when α is not real, we redefine the eigenfunctions as

$$h_k^{(\alpha)}(x)=i^{k+1}\,2^{k}\,[\ \pi\ {}_{2k}C_{k}\ |\mathrm{Im}\,\alpha|^{-(2k+1)}\ ]^{-1/2}\ (x-\alpha)^{-(k+1)} \tag{27}$$

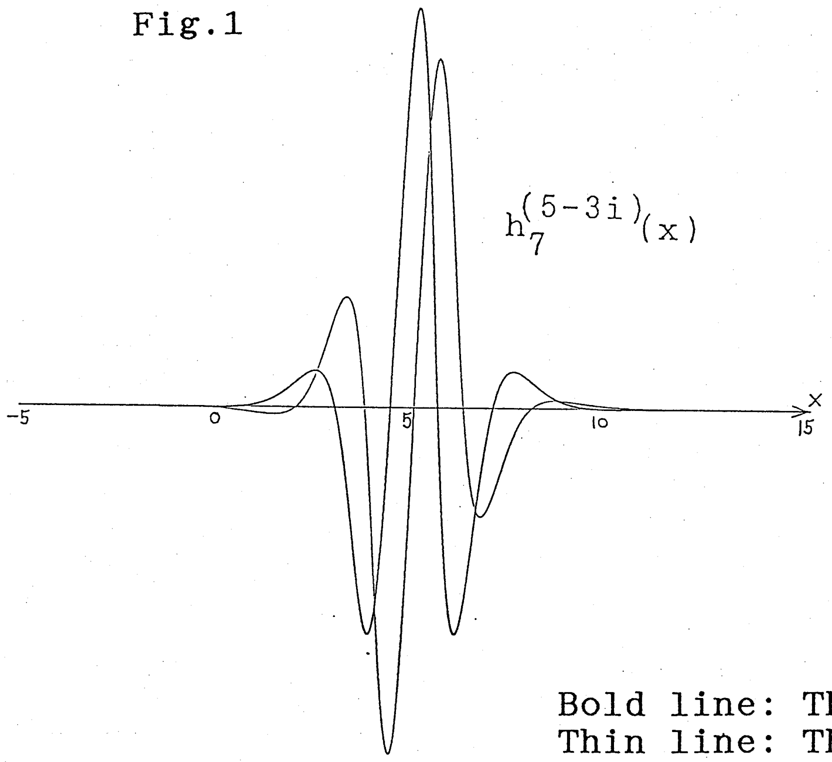
so that the normalization can be made as

$$\int_{-\infty}^{\infty}|h_k^{(\alpha)}(x)|^{2}\,dx=1\ . \tag{28}$$

Thus the eigenfunctions associated with non-real eigenvalues are square integrable for k=1,2,3,... .

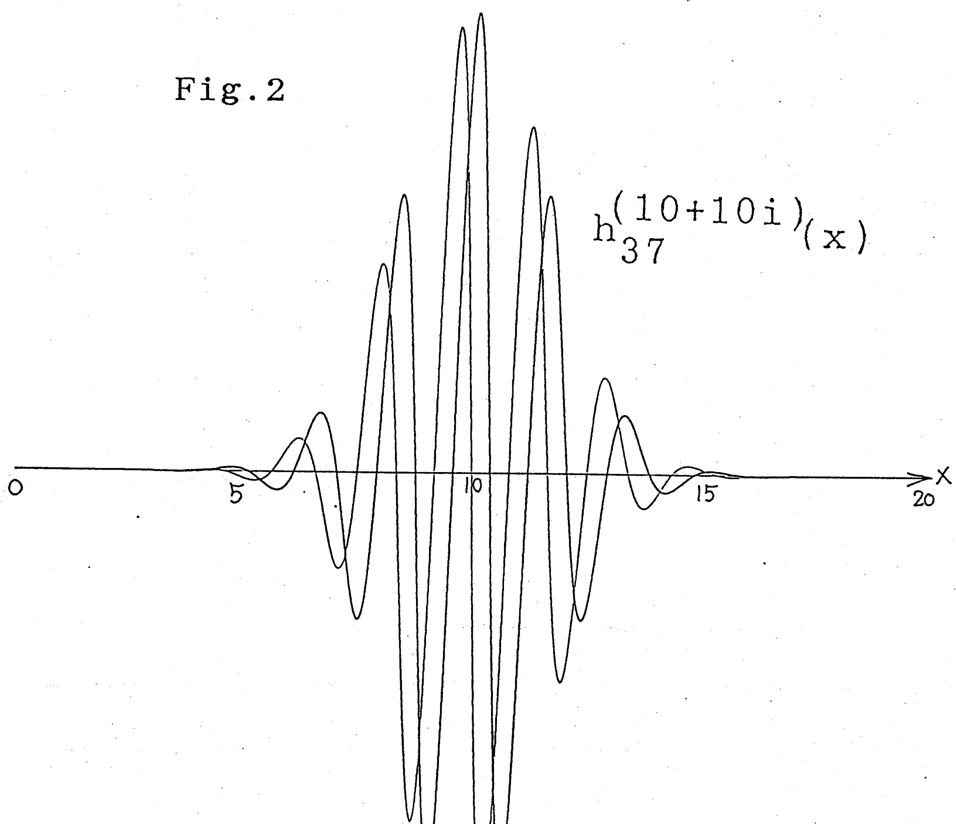
The wavefunctions are almost localized and their shape are 'wavelet-like'. Some examples are given in Figs.1 and 2 (where the maximum values are normalized). This wave-like behavior results from the fact that arg$(x-\alpha)$ is almost linear around $x=\mathrm{Re}\,\alpha$. When k is large, the 'phase' of the eigenfunction increses at an alomst constant rate in a wide range of x near the

Fig.1



Bold line: The real part
Thin line: The imaginary part

Fig.2



Bold line: The real part
Thin line: The imaginary part

center of the wavelet, and then the 'envelope' of the wavelet is similar to the Gaussian shape. So the shape seems to be very close to the wavefunctions of coherent states or squeezed states. This resemblance is mathematically verified, in the next section. The center of the wavelet is located at $x=\mathrm{Re}\,\alpha$ and the wavelet is more wide-spread as $|\mathrm{Im}\,\alpha|$ is larger. The number of 'large peaks' is nearly aslarge as $k^{1/2}$, which can be shown by an easy calculation. As is expected from the relation (20), for the affine transformation of the coordinate $x \to rx-q$ the eigenfunction satisfies the relation

$$h_k^{(\alpha)}(rx-q) = r^{1/2}\, h_k^{((\alpha+q)/r)}(x) \quad (r>0). \tag{29}$$

As a very useful property, the real part of the eigenfunction is an even function of $(x-\mathrm{Re}\,\alpha)$, its imaginary part is an odd function of $(x-\mathrm{Re}\,\alpha)$.

Next, we investigate the Fourier transform of $h_k^{(\alpha)}(x)$ or the wavefunction in the momentum representation

$$H_k^{(\alpha)}(y) \triangleq {}_P\langle y \mid \alpha \rangle_{A_k} = (2\pi)^{-1/2} \int_{-\infty}^{\infty} h_k^{(\alpha)}(x)\, \exp(-iyx)\, dx \,. \tag{30}$$

By a residue calculus,

$$H_k^{(\alpha)}(y) = \begin{cases} 0 & (\text{ If } y(\mathrm{Im}\,\alpha) \geqq 0 \text{ }) \\ 2^{k+1/2}\, [(2k)!]^{-1/2}\, |\mathrm{Im}\,\alpha|^{k+1/2}\, y^k \, \exp(-i\alpha y) & (\text{ If } y(\mathrm{Im}\,\alpha)<0 \text{ }). \end{cases} \tag{31}$$

When k is large, the eigenfunction in the momentum representation is almost localized around $y=-k/\mathrm{Im}\,\alpha$. So, in terms of signal processing, the wavelets associated to these eigenvectors are almost localized both in the time domain and in the frequency domain.

We can show that the eigenfunctions satisfy the admissibility condition (4) as

$$C_{h_k}^{(\alpha)} \triangleq \int_{-\infty}^{\infty} |y|^{-1} \; |H_k^{(\alpha)}(y)| \quad dx$$

$$= 2^{2k+1}[(2k)!]^{-1} \,|\mathrm{Im}\,\alpha\,|^{2k+1} \int_0^{\infty} y^{2k-1} \exp(-2y\,|\mathrm{Im}\,\alpha\,|)\, dy$$

$$= 4k \;|\mathrm{Im}\,\alpha\,| \quad < \infty \;. \tag{32}$$

According to the argument of Section II, the system $\{ h_k^{(\alpha)}(x) \; ; \; \alpha:\text{complex and } \mathrm{Im}\,\alpha \neq 0 \}$ constitutes an over-complete system in $L_C^2(R)$ because the relation (29) and the relation

$$h_k^{(-\alpha)}(x) = (-1)^{k+1} h_k^{(\alpha)}(-x) \tag{33}$$

hold. If we wish to apply these eigenfunction for the wavelet transform of real- valued signals, we can use the real part or the imaginary part of them. It is easily verified that the admissibility condition is satisfied for each part. Because of the 'parity' of each part, the systems

$$\{ \mathrm{Re}\; h_k^{(\alpha)}(x); \; \alpha:\text{complex and } \mathrm{Im}\,\alpha > 0 \}$$

$$\{ \mathrm{Im}\; h_k^{(\alpha)}(x); \; \alpha:\text{complex and } \mathrm{Im}\,\alpha > 0 \}$$

are over-complete in $L^2(R)$.

V. RESEMBLANCE TO THE WAVEFUNCTIONS OF SQUEEZED STATES

As has been pointed out in the above section, the eigenfunctions has a resemblance to the wavefunctions of some kinds of squeezed states when k is large. In a rough arrroximation, this resemblance can be expected also from the following facts;

1. $| h_k^{(\pm ik^{1/2})}(x) |$ is proportional to the probability density function of Student's t-distribution with the degree of freedom k which converges to the standard Gaussian distribution as $k \to \infty$.

2. $\arg h_k^{(\alpha)}(\mathrm{Re}\,\alpha + u) = -(k+1)u + O(u^3)$.

In this section, we will prove this property more exactly. For r:real($r \neq 0$) , define

$$M(r) \triangleq (r+r^{-1})/2 \tag{34}$$

$$N(r) \triangleq (r-r^{-1})/2 \tag{35}$$

$$s(x;\alpha,r) \triangleq {}_Q\langle x \mid \alpha; M(r), N(r)\rangle_b \tag{36}$$

and

$$S(y;\alpha,r) \triangleq \int_{-\infty}^{\infty} s(x;\alpha,r)\exp(-iyx)\,dx = {}_P\langle y \mid \alpha; M(r), N(r)\rangle_b \ , \tag{37}$$

where $| \alpha ; \mu , \nu \rangle_b$ denotes the vector of the squeezed state[10] (normalized). Then,

$$s(x;\alpha,r) = (2/\pi)^{1/4} r^{1/2} \exp[\,-r^2(x-\mathrm{Re}\,\alpha)^2 + 2i(\mathrm{Im}\,\alpha)x - i(\mathrm{Re}\,\alpha)(\mathrm{Im}\,\alpha)\,] \tag{38}$$

$$S(y;\alpha,r) = (2\pi)^{-1/4} r^{-1/2} \exp[-(2r)^{-2}(y-2\mathrm{Im}\,\alpha)^2 - i(\mathrm{Re}\,\alpha)y + i(\mathrm{Re}\,\alpha)(\mathrm{Im}\,\alpha)] \ . \tag{39}$$

For simplicity, define

$$K(k) \triangleq [(k+1)/2]^{-1/2} \ . \tag{40}$$

Then, using the formula

$$\int_0^{\infty} t^{n-1}\exp(-ct^2)\,dt = 2^{-1}c^{-n/2}\Gamma(n/2) \qquad (\Gamma\text{:Gamma func.}) \ , \tag{41}$$

we can show from (31) and (39) that the following relation holds for any real numbers r and v;

$$\exp[-iK(k)rv]\ {}_{A_k}\langle v-iK(k)/r \mid v+iK(k)r;M(r),N(r)\rangle_b$$

$$= \exp[-iK(k)rv]\ \int_{-\infty}^{\infty} H_k^{(v-iK(k)/r)^*}(y)\ S(y;v+iK(k)r,r)\ dy$$

$$= 2^{k/2}\ \pi^{-1/4}\, r^{-k-1}\,(k+1)^{k/2+1/4} \exp[-(k+1)/2]\ [\Gamma(2k+1)]^{-1/2}$$

$$\times \int_0^{\infty}\ y^k \exp[\ -(2r)^2\ y^2\ ]\ dy$$

$$= 2^{3k/2}\ \pi^{-1/4}(k+1)^{k/2+1/4}\ \ \Gamma((k+1)/2)\ [\Gamma(2k+1)]^{-1/2}\ . \qquad (42)$$

From the relation known as Stirling's formula

$$\lim_{t\to\infty}\ (2\pi)^{1/2}\exp(-t)\ t^{t-1/2}\ [\Gamma(t)]^{-1} = 1 \qquad (43)$$

and the relation

$$\lim_{k\to\infty}\ (dk+1)^{k+c}[\ (dk)^{k+c} e^{1/d}\ ]^{-1} = 1$$

$$(\text{ c: a real const. ; d=1,2 }) , \qquad (44)$$

we have

$$\lim_{k\to\infty}\ \exp[-iK(k)rv]\ {}_{A_k}\langle v-iK(k)/r \mid v+iK(k)r;M(r),N(r)\rangle_b = 1\ . \qquad (45)$$

Since the eigenvectors $|\,\alpha\,\rangle_{A_k}$ and $|\,\alpha\,;\,\mu\,,\,\nu\,\rangle_b$ are normalized, the relation (45) implies that

$$\lim_{k\to\infty}\ \|\ \ |\,v-iK(k)/r\rangle_{A_k} - \exp[-iK(k)rv]\ |\,v+iK(k)r;M(r),N(r)\rangle_b\ \ \|_{L2}$$

$$= 0 \qquad (46)$$

or

$$\lim_{k\to\infty} \int_{-\infty}^{\infty} \left|\ h_k^{(v-iK(k)/r)}(x)\ -\ \exp[-iK(k)rv]\ s(x;v+iK(k)r,r)\ \right|^2 dx$$

$$= 0 \qquad (47)$$

Thus we can verify that the eigenvector of A_k gradually approaches a sequence of squeezed-state vectors with respect to the norm $\| \; \|_{L2}$. As a special case where r=1, this approaches destinated to a sequence of coherent-state vectors because $|\alpha;1,0\rangle_b = |\alpha\rangle_a$.

VII. CONCLUSIONS

We have discussed the system of the eigenvectors of the operator $Q-ikP^{-1}$. The system constitutes an over-complete wavelet system, where the eigenvector associated with a non-real eigenvalue is transformed to the eigenvalue associated with another non-real eigenvalue by the affine transformation of the coordinate. We have shown this fact in terms of the operator algebra. The eigenfunctions in position coordinate representation are simple rational functions and have a localized 'wavelet-like' shape. They satisfy the 'admissibility condition'. It has been proved that in a limit the eigenvector gradually approaches a sequence of sqeezed-state vectors with respect to $\| \; \|_{L2}$ as $k \to \infty$.

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