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Some asymptotic results on Hurwitz zeta-functions

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1 The discrete case

Let $s=\sigma+it$ be a complex variable and $\alpha>0$ be a parameter. Let $\zeta(s,\alpha)$ be the Hurwitz zeta-function defined by the analytic continuation of the Dirichlet series

$$\sum_{n=0}^{\infty}(n+\alpha)^{-s}.$$

Let q be a positive integer. The first object of this talk is the discrete mean square

$$J(s,q)=\sum_{a=1}^{q}\left|\zeta\left(s,\frac{a}{q}\right)\right|^{2}. \tag{1.1}$$

Let $\zeta(s)$, $\Gamma(s)$ be the Riemann zeta and the gamma-function respectively. Let $\psi(s)=(\Gamma'/\Gamma)(s)$ and let N be a positive integer. We define a contour $\mathcal{C}$ which starts from infinity, proceeds along the real axis to δ $(0<\delta<\pi)$, rounds the origin counter-clockwise, returns to infinity. Let $h^{(N)}(z)$ be the N-th derivative of the function

$$h(z)=\frac{e^{z}}{e^{z}-1}-\frac{1}{z},$$

and define

$$\begin{aligned} R_N(u,v;q) &= \frac{1}{\Gamma(u)\Gamma(v)(e^{2\pi iu}-1)(e^{2\pi iv}-1)} \\ &\quad\times\int_{\mathcal{C}}\frac{y^{v+N-1}}{e^{y}-1}\int_{\mathcal{C}}\int_{0}^{1}\frac{(1-\tau)^{N-1}}{(N-1)!}h^{(N)}(x+q^{-1}\tau y)x^{u-1}d\tau dx dy \end{aligned}$$

for $-N+1<\operatorname{Re}u<N+1$ and any $v\in\boldsymbol{C}$. For any integer n, let $(s)_n=\Gamma(s+n)/\Gamma(s)$ be a Pochhammer symbol. Then we can show

Theorem 1.1 ([9, Theorem 1]) *For any real t and any positive integers N and q, we have*

$$\begin{aligned} J(\tfrac{1}{2}+it,q) &= q\{\log(q/2\pi)+2\gamma+\operatorname{Re}\psi(\tfrac{1}{2}+it)\} \\ &\quad+2\sum_{n=0}^{N-1}\frac{(-1)^{n}q^{-n}}{n!}\operatorname{Re}\left\{q^{\frac{1}{2}+it}(\tfrac{1}{2}-it)_n\zeta(\tfrac{1}{2}+it-n)\zeta(\tfrac{1}{2}-it+n)\right\} \\ &\quad+2q^{-N}\operatorname{Re}\left\{q^{\frac{1}{2}+it}R_N(\tfrac{1}{2}+it,\tfrac{1}{2}-it;q)\right\}. \end{aligned}$$

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Since $\mathrm{Re}\,\psi(\frac{1}{2}+it)=\log t+O(t^{-2})$, the first term in the right-hand side can be written as $q\{\log(qt/2\pi)+2\gamma+O(t^{-2})\}$. Moreover, the remainder $R_N(u,v;q)$ has the following alternative infinte series expressions (see [6, Lemma 3] and [8, (1.13)]): Let $\sigma_a(n)$ denotes the a-th powers of positive divisors of n. Then for any complex variables u, v with $\mathrm{Re}\,u<N$, $\mathrm{Re}\,v>-N+1$ and $\mathrm{Re}(u+v)<2$, we have

$$\begin{aligned} R_N(u,v;q) &= (-1)^N(2\pi)^{u+v-1}\frac{\Gamma(N+1-u)}{\Gamma(v)}\int_0^1\frac{(1-\tau)^{N-1}}{(N-1)!}\sum_{l=1}^{\infty}\sigma_{u+v-1}(l) \\ &\quad\times\{e^{\frac{\pi i}{2}(u+v-1)}J_-(\tau,l;q)+e^{-\frac{\pi i}{2}(u+v-1)}J_+(\tau,l;q)\}d\tau, \end{aligned}$$

where

$$J_\pm(\tau,l;q)=\int_0^\infty y^{v+N-1}\Big(1+\frac{\tau y}{q}\Big)^{u-N-1}e^{\pm 2\pi i l y}dy.$$

While for $\mathrm{Re}\,u<N$, $\mathrm{Re}\,v>-N+1$ and $\mathrm{Re}(u+v)>0$, we have

$$\begin{aligned} R_N(u,v;q) &= (-1)^N\frac{\Gamma(v+N)}{\Gamma(v)}\int_0^1\frac{(1-\tau)^{N-1}}{(N-1)!}\sum_{l=1}^{\infty}\sigma_{1-u-v}(l) \\ &\quad\times\{\tilde{J}_-(\tau,l;q)+\tilde{J}_+(\tau,l;q)\}d\tau, \end{aligned}$$

where

$$\tilde{J}_\pm(\tau,l;q)=\int_0^1 y^{-u+N}\Big(1+\frac{\tau y}{q}\Big)^{-v-N}e^{\pm 2\pi i l y}dy.$$

Here the integrals $J_\pm(\tau,l;q)$ and $\tilde{J}_\pm(\tau,l;q)$ can be expressed in terms of a confluent hypergeometric function $\Psi(a,c;x)$ (cf. [4, p.256, (3)], and we can transfer from $J_\pm(\tau,l;q)$ to $\tilde{J}_\pm(\tau,l;q)$ by a transformation formula (cf. [4, p.257, (6)]) for $\Psi(a,c;x)$ (for details see [6, Sect.3]). An application of a saddle-point lemma of Atkinson [2, Lemma 1] to the integrals $J_\pm(\tau,l;q)$ and $\tilde{J}_\pm(\tau,l;q)$ yields the estimate

$$R_N(\sigma+it,\sigma-it;q)=O\{(|t|+1)^{2N+\frac{1}{2}-\sigma}\}$$

for $-N+1<\sigma<N$ and any real t, with the O-constant depending only on σ and N (see [7, Sect.3]). The above Theorem 1.1 therefore gives the asymptotic expansion of $J(\frac{1}{2}+it,q)$ with respect to q^{-1}.

In 1991, using the approximate functional equation of $\zeta(s,\alpha)$, Zhang [19] independently obtained

$$J(\tfrac{1}{2}+it,q)=q\{\log(qt/2\pi)+2\gamma\}+O(qt^{-1/12})+O\left((t^{5/6}+q^{1/2}t^{5/12})\log^3 t\right),$$

which should be compared with our Theorem 1.1.

Theorem 1.1 can be deduced as the limiting case $\sigma\to\frac{1}{2}$ in the following:

Theorem 1.2 ([9, Theorem 2]) *For any integers N, q and any real σ, t satisfying $-N+1<\sigma<N$, $\sigma+it\notin\boldsymbol{Z}$ and $2\sigma-1\notin\boldsymbol{Z}_{\le 1}$, we have*

$$\begin{aligned} J(\sigma+it,q) &= q^{2\sigma}\zeta(2\sigma)+2q\Gamma(2\sigma-1)\zeta(2\sigma-1)\,\mathrm{Re}\left\{\frac{\Gamma(1-\sigma+it)}{\Gamma(\sigma+it)}\right\} \\ &\quad+2\sum_{n=0}^{N-1}\frac{(-1)^n q^{-n}}{n!}\,\mathrm{Re}\left\{q^{\sigma+it}(\sigma-it)_n\zeta(\sigma+it-n)\zeta(\sigma-it+n)\right\} \\ &\quad+2q^{-N}\,\mathrm{Re}\left\{q^{\sigma+it}R_N(\sigma+it,\sigma-it;q)\right\}. \end{aligned}$$

Dividing both sides of the above formula by q and letting $q \to +\infty$, we have when $\sigma < \frac{1}{2}$,

$$\begin{aligned}\int_0^1 |\zeta(\sigma+it,\alpha)|^2 d\alpha &= 2\Gamma(2\sigma-1)\zeta(2\sigma-1)\,\mathrm{Re}\,\frac{\Gamma(1-\sigma+it)}{\Gamma(\sigma+it)} \\ &= 2(2\pi)^{2\sigma-2}\zeta(2\sigma-2)|\Gamma(1-\sigma-it)|^2\cosh(\pi t),\end{aligned}$$

which was originally proved by Mikolás [15] in 1956 by using Parseval's identity, where the last equality follows from the functional equation of the Riemann zeta-function.

2 The continuous case

Let $\zeta(s,\alpha)$ be the Hurwitz zeta-function as in the previous section, and define

$$\zeta_1(s,\alpha) = \zeta(s,\alpha) - \alpha^{-s} = \zeta(s,\alpha+1).$$

The second object of this talk is the continuous mean square

$$I(s) = \int_0^1 |\zeta_1(s,\alpha)|^2 d\alpha. \tag{2.1}$$

The asymptotic behaviour of $I(\frac{1}{2}+it)$ was first studied by Koksma-Lekkerkerker [13] in 1952, who proved

$$I(\tfrac{1}{2}+it) = O(\log t)$$

for any $t \geq 2$. After their pioneering work, the following several inprovements have been obtained:

$$\begin{aligned}I(\tfrac{1}{2}+it) &= \log t + O(\log\log t), \\ &\qquad \text{by Balasubramanian [3] in 1979,} \\ &= \log t + O(1), \\ &\qquad \text{by Rane [17] in 1983,} \\ &= \log(t/2\pi) + \gamma + O(t^{-3/16}(\log t)^{3/8}), \\ &\qquad \text{by Sitaramachandrarao [18] in 1987,} \\ &= \log(t/2\pi) + \gamma + O(t^{-7/36}(\log t)^{25/18}), \\ &\qquad \text{by Zhang [21] in 1991,}\end{aligned}$$

where γ is Euler's constant. Zhang further conjectured in [21] that

$$I(\tfrac{1}{2}+it) = \log(t/2\pi) + \gamma + O(t^{-1/4}).$$

All these results and improvements were achieved by applying the approximate functional equation of $\zeta(s,\alpha)$, and the exponent $-1/4$ in the error term of Zhang's conjecture is a certain infimum when one applies this method for $I(\frac{1}{2}+it)$. But this tool is insufficient for the problem of evaluating (2.1). Indeed, by an ingenious simple argument based on the functional equation of $\zeta(s,\alpha)$, Zhang [22] proved a remarkable formula

$$I(\tfrac{1}{2}+it)=\log(t/2\pi)+\gamma-2\operatorname{Re}\frac{\zeta(\frac{1}{2}+it)}{\frac{1}{2}+it}+O(t^{-1}). \tag{2.2}$$

This was independently obtained by Andersson [1] in 1992, who applied Mikolás' idea of using Parseval's identity.

By a different method from both of Zhang's and Andersson's, we have obtained the following two kinds of refinements for (2.2).

Theorem 2.1 ([10, Theorem 1] and [12, Theorem 1]) *For any integer $K\geq 0$ and any real $t\geq 1$, we have the asymptotic expansion*

$$\begin{aligned}I(\tfrac{1}{2}+it) &= \gamma-\log 2\pi+\operatorname{Re}\psi(\tfrac{1}{2}+it)-2\operatorname{Re}\frac{\zeta(\frac{1}{2}+it)-1}{\frac{1}{2}+it}\\ &\quad -2\operatorname{Re}\sum_{k=1}^{K}\frac{(-1)^{k-1}(k-1)!}{(\frac{3}{2}-k+it)(\frac{5}{2}-k+it)\cdots(\frac{1}{2}+it)}\sum_{l=1}^{\infty}l^{-k}(l+1)^{-\frac{3}{2}+k-it}\\ &\quad +O(t^{-K-1}),\end{aligned}$$

where the O-constant depends only on K, and the empty sum is to be considered as 0.

Since $\operatorname{Re}\psi(\frac{1}{2}+it)=\log t+O(t^{-2})$, Theorem 2.1 implies Andersson-Zhang's formula (2.2).

Theorem 2.2 ([10, Corollary 2] and [12, Theorem 2]) *For any real t we have*

$$I(\tfrac{1}{2}+it)=\gamma-\log 2\pi+\operatorname{Re}\psi(\tfrac{1}{2}+it)-2\operatorname{Re}\sum_{n=0}^{\infty}\frac{\zeta(\frac{1}{2}+n+it)-1}{\frac{1}{2}+n+it}.$$

We note that Theorem 2.2 has been proved in Andersson [1] by using a different method, and the special case $t=0$ in Theorem 2.2 is also given in Zhang [21].

Both of the above two theorems give a kind of refinements of Andersson-Zhang's formula. It is interesting to point out that we can unify these two directions of refinements. In fact, Theorems 2.1 and 2.2 can be deduced from the following more general formula:

Theorem 2.3 ([10, Theorem 3] and [12, Theorm 3]) *Let u, v be complex variables, and let E be the set of (u,v) such that $u+v\in \boldsymbol{Z}_{\leq 1}$ or $u\in\boldsymbol{Z}$ or $v\in\boldsymbol{Z}$. Let $N\geq 1$ be an integer, $-N+1<\operatorname{Re}u<N+1$, $-N+1<\operatorname{Re}v<N+1$, and $(u,v)\notin E$. Then it holds*

$$\begin{aligned}&\int_0^1\zeta_1(u,\alpha)\zeta_1(v,\alpha)d\alpha\\ &= \frac{1}{u+v-1}+\Gamma(u+v-1)\zeta(u+v-1)\left\{\frac{\Gamma(1-v)}{\Gamma(u)}+\frac{\Gamma(1-u)}{\Gamma(v)}\right\}\\ &\quad -S_N(u,v)-S_N(v,u)-T_N(u,v)-T_N(v,u),\end{aligned}$$

where

$$S_N(u,v)=\sum_{n=0}^{N-1}\frac{(u)_n}{(1-v)_{n+1}}\{\zeta(u+n)-1\},$$

$$T_N(u,v)=\frac{(u)_N}{(1-v)_N}\sum_{l=1}^{\infty}l^{1-u-v}\int_l^{\infty}\beta^{u+v-2}(1+\beta)^{-u-N}d\beta.$$

Moreover, $T_N(u,v)$ has the expression

$$\begin{aligned}
&T_N(u,v)\\
&=\sum_{k=1}^{K}(-1)^{k-1}\frac{(2-u-v)_{k-1}(u)_{N-k}}{(1-v)_N}\sum_{l=1}^{\infty}l^{-k}(l+1)^{-u-N+k}\\
&\quad+(-1)^K\frac{(2-u-v)_K(u)_{N-K}}{(1-v)_N}\sum_{l=1}^{\infty}l^{1-u-v}\int_l^{\infty}\beta^{u+v-K-2}(1+\beta)^{-u-N+K}d\beta
\end{aligned}$$

for any integer $K\geq 0$.

By specializing $u=\sigma+it$ and $v=\sigma-it$ in Theorem 2.3, we have

Corollary 2.1 ([12, Corollary 1]) *Let N, K be integers with $N\geq 1$ and $K\geq 0$. Then, for any σ satisfying $-N+1<\sigma<N+1$, $2\sigma-1\notin\{1,0,-1,-2,\cdots\}$ and any $t\geq 1$, we have*

$$\begin{aligned}
I(\sigma+it) &= \frac{1}{2\sigma-1}+2\Gamma(2\sigma-1)\zeta(2\sigma-1)\operatorname{Re}\frac{\Gamma(1-\sigma+it)}{\Gamma(\sigma+it)}\\
&\quad-2\operatorname{Re}\sum_{n=0}^{N-1}\frac{(\sigma+it)_n}{(1-\sigma+it)_{n+1}}\{\zeta(\sigma+it+n)-1\}\\
&\quad-2\operatorname{Re}\sum_{k=1}^{K}(-1)^{k-1}\frac{(2-2\sigma)_{k-1}(\sigma+it)_{N-k}}{(1-\sigma+it)_N}\sum_{l=1}^{\infty}l^{-k}(l+1)^{-\sigma-N+k-it}\\
&\quad+O(t^{-K-1}),
\end{aligned}$$

where the O-constant depends only on N, K and σ.

The error estimate $O(t^{-K-1})$ in the above lemma follows from the facts that

$$\frac{(2-2\sigma)_K(\sigma+it)_{N-K}}{(1-\sigma+it)_N}=O(t^{-K})$$

and

$$\sum_{l=1}^{\infty}l^{1-2\sigma}\int_l^{\infty}\beta^{2\sigma-K-2}(1+\beta)^{-\sigma-it-N+K}d\beta=O(t^{-1}),$$

where the last estimate can be proved by integration by parts.

Taking $N=1$ and letting $u\to\frac{1}{2}+it$, $v\to\frac{1}{2}+it$ in Theorem 2.3, we obtain Theorem 2.1.

Other exceptional cases can also be treated as the limiting cases. For example, since $(2-u-v)_{k-1}=0$ for $k\geq 2$ if $u+v=2$, taking the limit $u\to 1+it$ and $v\to 1-it$ in Theorem 2.3, we have

Corollary 2.2 ([12, Corollary 2]) *For any $N\geq 1$, and any real $t\geq 1$, we have*

$$\begin{aligned}
I(1+it) &= 1-t^{-2}-2\operatorname{Re}\frac{\psi(1+it)}{it}\\
&\quad-2\operatorname{Re}\frac{1}{it}\sum_{n=0}^{N-1}\{\zeta(1+n+it)-1\}-2\operatorname{Re}\frac{1}{it}\sum_{l=1}^{\infty}\frac{1}{l(l+1)^{N+it}}.
\end{aligned}$$

Taking the limit $N \to \infty$ in Theorem 2.3, we obtain the following explicit formula, since $T_N(u,v) \to 0$ as $N \to \infty$.

Corollary 2.3 ([10, Corollary 1] and [12, Corollary 3]) *For any complex u, v with $(u,v) \notin E$, we have*

$$\begin{aligned}
&\int_0^1 \zeta_1(u,\alpha)\zeta_1(v,\alpha)d\alpha \\
&= \frac{1}{u+v-1} + \Gamma(u+v-1)\zeta(u+v-1)\left\{\frac{\Gamma(1-v)}{\Gamma(u)} + \frac{\Gamma(1-u)}{\Gamma(v)}\right\} \\
&\quad - \sum_{n=0}^{\infty} \frac{(u)_n}{(1-v)_{n+1}}\{\zeta(u+n)-1\} - \sum_{n=0}^{\infty} \frac{(v)_n}{(1-u)_{n+1}}\{\zeta(v+n)-1\}.
\end{aligned}$$

We next mention the results when u and v are integers.

Corollary 2.4 ([10, Corollary 4]) *For any integer $m \geq 0$, we have*

$$\begin{aligned}
&\int_0^1 \zeta_1(-m,\alpha)^2 d\alpha \\
&= -\frac{1}{2m+1} + (-1)^{m+1}\frac{(m!)^2}{(2m+1)!}\zeta(-2m-1) \\
&\quad - 2(-1)^m \frac{(m!)^2}{(2m+2)!} - 2\sum_{n=0}^{m} \frac{(-1)^n (m!)^2}{(m+n+1)!(m-n)!}\{\zeta(n-m)-1\}.
\end{aligned}$$

The closed form of Corollary 2.4 can also be proved from the well-known formula

$$\zeta(-m,\alpha) = -B_{m+1}(\alpha)/(m+1),$$

where $B_{m+1}(\alpha)$ is the $(m+1)$-th Bernoulli polynomial. This is achieved by the same method as in [11, Sect.4] of using generating functions for Bernoulli polynomials.

It is also possible to deduce the asymptotic formula for positive integers:

Corollary 2.5 ([14, p.13]) *For any integer $m \geq 2$, we have*

$$\begin{aligned}
&\int_0^1 \zeta_1(m,\alpha)^2 d\alpha \\
&= \frac{1}{2m-1} + \frac{2(-1)^m}{\{(m-1)!\}^2}\{\Gamma'(2m-1)\zeta(2m-1) + \Gamma(2m-1)\zeta'(2m-1) \\
&\quad - \Gamma(2m-1)\zeta(2m-1)\psi(m)\} - 2\sum_{n=0}^{m-2} \frac{(m)_n}{(1-m)_{n+1}}\{\zeta(m+n)-1\} \\
&\quad + 2\sum_{n=m-1}^{\infty} \frac{(-1)^{m-1}(m)_n}{(m-1)!(n-m+1)!}\{\psi(n+2-m)-\psi(m)\}\{\zeta(m+n)-1\}.
\end{aligned}$$

3 The derivative case

We finally consider the mean square of the derivative $\zeta_1'(s,\alpha) = (\partial/\partial s)\zeta_1(s,\alpha)$. Zhang [20] proved in 1990 that there exist constants A and B, for which

$$\int_0^1 |\zeta_1'(\tfrac{1}{2}+it,\alpha)|^2 d\alpha = \frac{1}{3}\log^3(t/2\pi) + \gamma\log^2(t/2\pi) - 2B\log(t/2\pi) + A + \rho(t)$$

holds, where $\rho(t) = O(t^{-1/6}(\log t)^{10/3})$. Zhang defined A and B as certain integrals, but we found that the actual values of A and B are $2\gamma_2$ and $-\gamma_1$ respectively, where γ_1 and γ_2 are generalized Euler's constants defined by the Laurent series

$$\zeta(1+s) = s^{-1} + \gamma + \gamma_1 s + \gamma_2 s^2 + \cdots.$$

Our method gives more precise information on $\rho(t)$. Namely we have

Theorem 3.1 ([10, Theorem 4])

$$\begin{aligned}\rho(t) &= -2\,\mathrm{Re}\,\frac{\zeta'(\frac{1}{2}+it)}{(\frac{1}{2}+it)^2} - 2\,\mathrm{Re}\,\frac{\partial^2}{\partial u\partial v}T_1(u,v)\bigg|_{(u,v)=(\frac{1}{2}+it,\frac{1}{2}-it)} \\ &\quad + O(t^{-2}(\log 2t)^2),\end{aligned}$$

where $T_1(u,v)$ is defined in Theorem 2.3.

This in particular implies $\rho(t) = O(t^{-1})$, which improves Zhang's estimate for $\rho(t)$ mentioned above.

4 The method of proof

We can prove all of our results in this article by using Atkinson's devise, which was first applied by himself to treat the product $\zeta(u)\zeta(v)$ as a function of two complex variables u and v. We denote by $L(s,\chi)$ the Dirichlet L-function attached to a Dirichlet character χ mod q. Atkinson's devise was enhanced from a complex function theoretical viewpoint by Motohashi [16], who applied this method for the functon $\sum_{\chi(\mathrm{mod}\, q)} L(u,\chi)L(v,\bar{\chi})$, where the summation is taken over all the characters mod q.

Let

$$h(z;\alpha) = \frac{e^{(1-\alpha)z}}{e^z - 1} - \frac{1}{z},$$

and define

$$g(u,v;\alpha) = \frac{1}{\Gamma(u)\Gamma(v)(e^{2\pi iu}-1)(e^{2\pi iv}-1)}\int_C \frac{y^{v-1}}{e^y - 1}\int_C h(x+y;\alpha)x^{u-1}dx\,dy, \tag{4.1}$$

where the integral is absolutely convergent for $\mathrm{Re}\,u < 1$ and any complex v. By applying Atkinson-Motohashi's method for $\zeta(u,\alpha)\zeta(v,\alpha)$, we can show the formula

$$\begin{aligned}(4.2)\qquad & \zeta(u,\alpha)\zeta(v,\alpha) \\ &= \zeta(u+v,\alpha) + \Gamma(u+v-1)\zeta(u+v-1)\left\{\frac{\Gamma(1-u)}{\Gamma(v)} + \frac{\Gamma(1-v)}{\Gamma(u)}\right\} \\ &\quad + g(u,v,\alpha) + g(v,u,\alpha)\end{aligned}$$

for $\mathrm{Re}\, u < 1$, $\mathrm{Re}\, v < 1$ and $\alpha > 0$, which corresponds to [2, (3.3)] and [16, Lemma 1], and plays the fundamental role in the proofs of Theorems 1.2 and 2.3. We further define

$$\begin{aligned} r_N(u,v;\alpha) &= \frac{1}{\Gamma(u)\Gamma(v)(e^{2\pi i u}-1)(e^{2\pi i v}-1)} \\ &\quad \times \int_0^1 \frac{(1-\tau)^{N-1}}{(N-1)!} \int_C \frac{y^{v+N-1}}{e^y-1} \int_C h^{(N)}(x+\tau y;\alpha) x^{u-1} dx dx d\tau, \end{aligned}$$

where $h^{(N)}(z;\alpha) = (\partial/\partial z)^N h(z;\alpha)$, and the integral is absolutely convergent for $\mathrm{Re}\, u < N+1$ and any complex v. Substituting

$$h(x+y;\alpha) = \sum_{n=0}^{N-1} \frac{h^{(n)}(x;\alpha)}{n!} y^n + y^N \int_0^1 \frac{(1-\tau)^{N-1}}{(N-1)!} h^{(N)}(x+\tau y;\alpha) d\tau,$$

which follows from Taylor's formula, into (4.1), we obtain the expression

$$g(u,v;\alpha) = \sum_{n=0}^{N-1} \frac{(-1)^n (v)_n}{n!} \zeta(u-n,\alpha)\zeta(v+n) + r_N(u,v;\alpha), \tag{4.3}$$

and this gives the meromorphic continuation of $g(u,v;\alpha)$ into the region $\{(u,v);\ \mathrm{Re}\, u < N+1,\ v \in \boldsymbol{C}\}$ (see [12, (4.3)]). From the formulas (4.2) and (4.3), together with the relations

$$\sum_{a=1}^{q} \zeta\Big(s, \frac{a}{q}\Big) = q^s \zeta(s)$$

and

$$\sum_{a=1}^{q} r_N\Big(u, v; \frac{a}{q}\Big) = q^{u-N} R_N(u,v;q),$$

we get the assertion of Theorem 1.2. For the deduction of Theorem 2.3, we make use of the formulas (4.2) and (4.3), and it is essential to notice that the vanishing properties

$$\int_0^1 \zeta(s,\alpha) d\alpha = 0 \quad \text{for} \quad \mathrm{Re}\, s < 1,$$

and

$$\int_0^1 r_N(u,v;\alpha) d\alpha = 0 \quad \text{for} \quad \mathrm{Re}\, u < N+1,\ \forall v \in \boldsymbol{C}$$

hold.

The detailed proofs for our results in Sections 1 and 2 are given in [9] and [12] respectively.

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