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# Analogs of the additive divisor problem for Fourier coefficients of cusp forms

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The additive divisor problem is concerned with the asymptotic behaviour of the sum

$$D(x,m)=\sum_{n\le x} d(n)d(n+m), \tag{1}$$

where $m$ is a given positive integer. We consider its analogs for (holomorphic or non-holomorphic) cusp forms for the full modular group with a unified approach in mind.

First, if $F(z)$ is a holomorphic cusp form of weight $k$ with the Fourier series

$$F(z)=\sum_{n=1}^{\infty} a(n)e(nz),$$

then the corresponding sum is

$$A(x,m)=\sum_{n\le x} a(n)\overline{a(n+m)}. \tag{2}$$

Likewise, if $u(z)$ is a non-holomorphic cusp form which is an eigenfunction of the hyperbolic Laplacian with the eigenvalue $1/4+\kappa^2$, then its Fourier series

$$u(z)=y^{1/2}\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty} \rho(n)K_{i\kappa}(2\pi|n|y)e(nx)$$

gives rise to another analog for the additive divisor problem. Namely, supposing that $u(z)$ is an eigenfunction of all Hecke operators $T(n)$ with respective eigenvalues $t(n)$, and that it is an even or odd function of $x=$ Re $z$, then $\rho(n)=\rho(1)t(n)$ for $n\ge 1$, and the sum

$$T(x,m)=\sum_{n\le x} t(n)t(n+m) \tag{3}$$

correponds to (1) and (2).

The method of the generating Dirichlet series provides, at least in principle, an approach to all of the sums (1) – (3). In fact, such an argument has been developed by L. A. Tahtadjan and A. I. Vinogradov [**TV**] (see also [**J1**]) for the sum (1), and by A. Good [**G2**] for the sum (2). However, the method of Good was strictly specific to holomorphic cusp forms, the case of non-holomorphic forms being more problematic. In our recent paper [**J2**], we found an alternative and more general argument which applies even to non-holomorphic cusp forms, and yields the estimates

$$A(x,m)\ll x^{k-1/3+\varepsilon},$$
$$T(x,m)\ll x^{2/3+\varepsilon}$$

uniformly for $1 \leq m \ll x^{2/3}$. These should be compared with the estimate $\ll x^{2/3+\varepsilon}$ for the error term in the additive divisor problem; this was first obtained by J. - M. Deshouillers and H. Iwaniec [**DI**] for fixed $m$, and the range of uniformity has been recently extended by Y. Motohashi [**M**] even to $m \ll x^{20/27}$.

Turning to the generating Dirichlet series, we first recall the well-known spectral theoretic analysis of the function

$$\varphi_m(s) = \sum_{n=1}^{\infty} a(n)\overline{a(n+m)}(n+m)^{-s}$$

related to the sum (2). If $P_m(z,s)$ stands for the non-holomorphic Poincaré series, $\mathcal{F}$ is a fundamental domain for the modular group, $d\mu(z) = y^{-2}\,dx\,dy$ is the invariant hyperbolic measure, and

$$(f,g) = \int_{\mathcal{F}} f(z)\overline{g(z)}d\mu(z)$$

denotes the Petersson inner product, then it is easy to verify by the Rankin – Selberg unfolding method that

$$\varphi_m(s) = \frac{(4\pi)^s}{\Gamma(s)}(P_m(z,s+1-k), y^k|F(z)|^2) \quad \text{for } \sigma > k.$$

The next step is to apply Parseval's formula to the inner product. To formulate the result, we introduce some notation. Let $u_j(z)$ be the Maass wave form corresponding to the $j$'th eigenvalue $1/4+\kappa_j^2$ of the hyperbolic Laplacian, let $E(z,s)$ be the non-holomorphic Eisenstein series, and define

$$c_j = (u_j, y^k|F(z)|^2),$$
$$c(t) = (E(z,\tfrac{1}{2}+it), y^k|F(z)|^2).$$

Then, writing $z_j = 1/2 + i\kappa_j$ and $\sigma_\alpha(n) = \sum_{d|n} d^\alpha$, we have

$$\varphi_m(s) = \frac{(4\pi)^k m^{k-1/2-s}}{2\Gamma(s)\Gamma(s+1-k)}\sum_{j=1}^{\infty} c_j\overline{\rho_j(1)}t_j(m)\Gamma(s+1-k-z_j)\Gamma(s+1-k-\overline{z}_j)$$
$$+\frac{4^{k-1}\pi^{k-1/2}m^{k-1/2-s}}{\Gamma(s)\Gamma(s+1-k)}\int_{-\infty}^{\infty}\frac{c(u)\sigma_{2iu}(m)\Gamma(s+\frac{1}{2}-k+iu)\Gamma(s+\frac{1}{2}-k-iu)}{(\pi m)^{iu}\Gamma(\frac{1}{2}-iu)\zeta(1-2iu)}\,du.$$

This gives an analytic continuation of $\varphi_m(s)$ to a meromorphic function. For purposes of applications, we need a satisfactory mean value estimate for $c_j$ and $c(u)$. Such information is given the inequality

$$\text{(4)} \qquad \sum_{\kappa_j \leq K} |c_j|^2 \exp(\pi\kappa_j) + \int_{-K}^{K} |c(u)|^2 \exp(\pi|u|)\,du \ll K^{2k}.$$

due to A. Good [**G1**]. We gave another proof of this in [**J2**] (strictly speaking, in a slightly weaker form, with an extra $\varepsilon$ in the exponent). The point of our alternative proof lies in its generality: the same argument applies even to non-holomorphic cusp forms, as will be briefly outlined below.

An analog of $\varphi_m(s)$ for the non-holomorphic cusp form $u(z)$ is the function

$$\psi_m(s) = \frac{2(4\pi)^{s-1}s}{|\rho(1)|^2\Gamma(s)}(P_m(z,s), |u(z)|^2);$$

motivation for this definition can be found in [**J1**]. This is not exactly the generating Dirichlet series related to the sum (3), but plays anyway a similar role in practice. As analogs of $c_j$ and $c(t)$, we define

$$\tilde{c}_j = (u_j, |u(z)|^2),$$
$$\tilde{c}(t) = (E(z, \frac{1}{2} + it), |u(z)|^2).$$

Denote the Fourier coefficients of $u_j(z)$ by $\rho_j(n)$. Then, as above, Parseval's formula gives

$$\psi_m(s) = \frac{m^{1/2-s}s}{|\rho(1)|^2\Gamma^2(s)} \sum_{j=1}^{\infty} \tilde{c}_j\overline{\rho_j(1)}t_j(m)\Gamma(s - z_j)\Gamma(s - \overline{z}_j)$$
$$+ \frac{m^{1/2-s}s}{2\sqrt{\pi}|\rho(1)|^2\Gamma^2(s)} \int_{-\infty}^{\infty} \frac{\tilde{c}(u)\sigma_{2iu}(m)\Gamma(s - \frac{1}{2} + iu)\Gamma(s - \frac{1}{2} - iu)}{(\pi m)^{iu}\Gamma(\frac{1}{2} - iu)\zeta(1 - 2iu)}\, du$$

for $\sigma > 1/2$. Moreover, as an analog of (4), we have

$$\sum_{\kappa_j \le K} |\tilde{c}_j|^2 \exp(\pi\kappa_j) + \int_{-K}^{K} |\tilde{c}(u)|^2 \exp(\pi|u|)\, du \ll K^{\varepsilon}. \tag{5}$$

The actual novelty here is the inequality (5).

The basic idea in our unified approach to (4) and (5) is to use a method due to D. Zagier [**Z**]. Let us consider the estimation of $c_j$ and $\tilde{c}_j$; the continuous parts of (4) or (5) are less problematic. By definition, both $c_j$ and $\tilde{c}_j$ is of the form

$$\int_{\mathcal{F}} f(z)\, d\mu(z) \tag{6}$$

for a certain automorphic function $f$ of rapid decay as $y \to \infty$. We want to express this in terms of the integral of a suitable function over the strip $0 \le x \le 1$, $y > 0$.

A useful tool for this purpose is the function (in the notation of [**J2**])

$$R_0^*(f; s) = s(s-1) \int_{\mathcal{F}} f(z)E^*(z,s)\, d\mu(z) = s(s-1)\xi(2s) \int_0^{\infty} \int_0^1 f(z)y^{s-2}\, dx\, dy, \tag{7}$$

where $E^*(z,s) = \xi(2s)E(z,s)$ and $\xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$. Since $E^*(z,s)$ can be analytically continued to a meromorphic function of $s$ with simple poles at $s = 0$ and $s = 1$, the function $R_0^*(f;s)$ is entire. Moreover, the functional equation $E^*(z,s) = E^*(z, 1-s)$ implies the functional equation $R_0^*(f;s) = R_0^*(f;1-s)$.

We let now $s$ tend to 1 in (7) noting that $E(z,s)$ has a simple pole at $s = 1$ with residue $3/\pi$. Then the expression in the middle of (7) becomes one half of the integral (6), so

$$\int_{\mathcal{F}} f(z)\,d\mu(z) = 2R_0^*(f;1).$$

On the other hand, by Cauchy's integral formula, we have

$$R_0^*(f;1) = \frac{1}{2\pi i}\int_\gamma \frac{R_0^*(f;s)}{s-1}\,ds,$$

where $\gamma$ is any simple closed positively oriented contour encircling the point $s = 1$.

The last mentioned integral can be modified on inserting an extra factor $g(s)$ to the integrand, taken that $g(1) = 1$ and that $g(s)$ is holomorphic on $\gamma$ and its interior. This auxiliary function should decay rapidly in order to concentrate the relevant values of $s$ to a neighbourhood of the real axis.

Let us specify $\gamma$ to be a rectangle, the vertical sides of which lie on the lines $\sigma = a$ and $\sigma = 1 - a$ for $a > 1$, the horizontal sides lying on the lines $t = \pm T$ with $T \to \infty$. Since the function $\xi(s)$ decays exponentially as $|t|$ increases, and the function $g(s)$ accelerates the decay of the integrand, the contribution of the horizontal sides vanishes in the limit.

The contribution of the vertical sides can be combined in view of the functional equation for $R_0^*(f;s)$, and we end up with the formula

$$\int_{\mathcal{F}} f(z)\,d\mu(z) = \frac{1}{\pi i}\int_{(a)} \xi(2s)(sg(s) - (1-s)g(1-s))\left(\int_0^\infty \int_0^1 f(z)y^{s-2}\,dx\,dy\right) ds.$$

The inner products $c_j$ and $\tilde{c}_j$ are now rewritten by use of this formula, where the Fourier series of the cusp forms in question are substituted to the integrand. Since the range of the $xy$ – integration is a strip, the integrals can be easily evaluated. The resulting expressions involve double series, which can be truncated to finite sums. Finally, the remainig double sums are estimated non-trivially by appealing to a suitable version of the spectral large sieve inequality.

The mean value estimates (4) and (5) imply satisfactory estimates for the functions $\varphi_m(s)$ and $\psi_m(s)$, and by Perron's formula or similar arguments one may draw conclusions on the sum functions $A(x,m)$ and $T(x,m)$.

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