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ON THE POSITIVITY OF THE SINGULAR INTEGRAL

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Introduction

In this note our main purpose is to present the positivity of the singular integral under a sufficient condition. The singular integral is the generalized Dirichlet integral which appears in the coefficient of asymptotic formula on the Waring problem and the Goldbach problem in algebraic number fields (see Y. Wang [4]).

First, we shall define a singular integral which can be apply to the Goldbach problem in algebraic number fields and show the positivity as Theorem 1. Here we notice that the positivity is not trivial if the algebraic number field K has the complex conjugates. Secondly, we shall explain an asymptotic formula as Thearem 2 following the generalized Vinogradov-Vaughan method introduced by Mitsui ([1], [2]). The asymptotic formula is a generalization of Sultanova [3] and Theorem 1 allow us to have a positive coefficient of the leading term on this problem.

1. Dirichlet integral, the rational field case

Let k, s be a positive rational integer with $s > 2k$. We define a singular integral in the case of rational field as follows:

$$F(\mu) = \int_D u_1^{\frac{1}{k}-1} u_2^{\frac{1}{k}-1} \cdots u_{s-1}^{\frac{1}{k}-1} (\mu - u_1 - \cdots - u_{s-1}) du_1 du_2 \cdots du_{s-1},$$

with

$$D = \left\{ (u_1, u_2, \cdots, u_{s-1}, \mu) \in \mathbb{R}^s \;\middle|\; \begin{array}{l} 0 \le u_j \le 1 \quad (j = 1, \cdots, s-1) \\ 0 \le \mu - u_1 - \cdots - u_{s-1} \le 1 \end{array} \right\}.$$

If μ is a real number with $0 < \mu \le 1$, $F(\mu)$ is called Dirichlet integral and we see

$$F(\mu) = \mu^{\frac{s}{k}-1} \frac{\Gamma(1/k)^s}{\Gamma(s/k)}.$$

In this case the positivity of the integral is trivial and we can see this integral in the famous asymptotic formula

$$R(N) = \mathfrak{S}(N)\frac{N^{s-1}}{(s-1)!} + O\left(\frac{N^{(s-1)}}{(\log N)^B}\right),$$

where $R(N) = \sum_{N=p_1+\cdots+p_s} \log N(p_1)\cdots\log N(p_s)$: the sum is taken over all the s-tuples $(p_1, p_2, \cdots, p_s)$ of positive prime numbers such that $N = p_1 + p_2 + \cdots + p_s$, and $\mathfrak{S}(N)$ is the singular series which is written as an infinite product:

$$\mathfrak{S}(N) = \prod_{p|N}\left(1 + \frac{(-1)^s}{(p-1)^{s-1}}\right)\prod_{p\nmid N}\left(1 + \frac{(-1)^{s+1}}{(p-1)^s}\right).$$

This is the asymptotic formula of the Goldbach type problem and here we put $\mu = 1$, $k = 1$. In the case of $k \geq 2$, $F(\mu)$ appears in the asymptotic formula of the Waring problem.

2. Statement of results

Let K be an algebraic number field of degree n. Let $K^{(q)}$ $(q = 1, 2, \ldots, r_1)$ be the real conjugates of K and $K^{(p)}, K^{(p+r_2)}$ $(p = r_1+1, \ldots, r_1+r_2)$ be the complex conjugates of K with $K^{(p+r_2)} = \bar{K}^{(p)}$. Let $\mathfrak{d}$ denote the different of K and $D = N(\mathfrak{d})$ (norm of $\mathfrak{d}$) the absolute value of the discriminant of K. Further, h denotes the ideal class number of K and R the regulator of K. Let γ be a number of K and put $\mathfrak{d}\gamma = \mathfrak{b}/\mathfrak{a}$ with integral ideals $\mathfrak{a}$ and $\mathfrak{b}$ such that $(\mathfrak{a}, \mathfrak{b}) = 1$. We write this relation by $\gamma \to \mathfrak{a}$.

Let μ be a number of K; μ also denotes an n-dimensional complex vector $(\mu^{(1)}, \mu^{(2)}, \ldots, \mu^{(n)})$ with $\mu^{(i)} \in K^{(i)}$ $(i = 1, 2, \ldots, n)$. More generally we consider any n-dimensional complex vector $\xi = (\xi_1, \xi_2, \ldots, \xi_n)$ with real ξ_q $(q = 1, 2, \ldots, r_1)$ and complex $\xi_{p+r_2} = \bar{\xi}_p$ $(p = r_1+1, \ldots, r_1+r_2)$. We denote the set of ξ by E^{r_1,r_2}. For $\xi \in E^{r_1,r_2}$, we write

$$N(\xi) = \prod_{i=1}^n \xi_i, \qquad S(\xi) = \sum_{j=1}^n \xi_j \qquad \text{and} \qquad E(\xi) = e^{2\pi i S(\xi)}.$$

Let $x(\xi)$ denote the n-dimensional real vector $x(\xi) = (X_1(\xi), X_2(\xi), \ldots, X_n(\xi))$ with $X_q(\xi) = \xi_q$, $X_p(\xi) = (\xi_p + \bar{\xi}_p)/2$ and $X_{p+r_2}(\xi) = (\xi_p - \bar{\xi}_p)/2\sqrt{-1}$. We denote the map from E^{r_1,r_2} into $\mathbb{R}^n$ such that the image of ξ is $x(\xi)$ by ϕ.

Let $D(t)$ $(t > 0)$ be a set of $\xi \in E^{r_1,r_2}$ such that $0 < \xi_q \leq t$ $(q = 1, \cdots, r_1)$ and $|\xi_p| \leq t$ $(p = r_1+1, \cdots, r_1+r_2)$. Regarding $X_1(\xi), \cdots, X_n(\xi)$ as variables we define an integral

$$\Phi_k(z) = \frac{2^{r_2}}{\sqrt{D}}\int_{D(1)} E(z\xi^k)dx(\xi),$$

where k is a positive rational integer, $z\xi^k = (z_1\xi_1^k, \cdots, z_n\xi_n^k)$ with $z \in E^{r_1,r_2}$ and $dx(\xi) = dX_1(\xi)\cdots dX_n(\xi)$.

In the following we let μ be a totally positive integer and $a_k = ({a_k}^{(1)}, {a_k}^{(2)}, \cdots, {a_k}^{(n)})$ $(k = 1, 2, \cdots, s)$ be a point of E^{r_1,r_2} which satisfy the condition:

$${a_k}^{(i)} \in \mathbb{R}, \qquad {a_k}^{(i)} > 0 \qquad (i = 1, 2, \cdots, n; \quad k = 1, 2, \cdots, s),$$

$$0 < a_1{}^{(i)} \leq a_2{}^{(i)} \leq \cdots \leq a_s{}^{(i)} \leq 1 < 1 + c^{(i)} = a_1{}^{(i)} + a_2{}^{(i)} + \cdots + a_s{}^{(i)}$$

with a positive constant $c^{(i)}$. We define a singular integral as follows:

$$\Psi_1(\mu; \lambda_1, \lambda_2, \cdots, \lambda_s) = 2^{r_2}\sqrt{D} \int_{\mathbb{R}^n} \prod_{k=1}^{s} \Phi_1(\lambda_k z) E(-\mu z) dx(z)$$

with

$$\lambda_k = a_k \mu \qquad (k = 1, 2, \cdots, s).$$

Then we have the following theorem:

Theorem 1. *There is a positive constant c_1 which depends on $a_k{}^{(i)}$ $(i = 1, 2, \cdots, n; \quad k = 1, 2, \cdots, s)$ such that*

$$\Psi_1(\mu; \lambda_1 \lambda_2, \cdots, \lambda_s) \geq \frac{c_1}{N(\mu)}.$$

The case of $0 < \mu \leq 1$ with $a_k = 1$ $(k = 1, 2, \cdots, s)$ and K is a totally real number field, we can easily see the positivity of the integral $\Psi_1(\mu)$ using Dirichlet integral. The case of $0 < \mu < 1$ with $a_k = 1$ $(k = 1, 2, \cdots, s)$, we can find the positivity in the work of Mitsui [2]. Here we notice Theorem 1 is usefull for the case of $\mu = 1$.

In this note we call an integer ω of K a *prime number*, if the principal ideal (ω) is a prime ideal. Let $\Omega(\lambda_k)$ be a set of prime numbers ω_k of K such that

$$0 < \omega_k^{(q)} \leq \lambda_k^{(q)} \quad (q = 1, 2, \cdots, r_1), \qquad |\omega_k^{(p)}| \leq |\lambda_k^{(p)}| \quad (p = r_1 + 1, \cdots, r_1 + r_2).$$

We define a sum $R(\mu; \lambda_1, \lambda_2, \cdots, \lambda_s)$ as follows:

$$R(\mu; \lambda_1, \lambda_2, \cdots, \lambda_s) = \sum_{\mu = \omega_1 + \cdots + \omega_s,\ \omega_k \in \Omega(\lambda_k)} \log N(\omega_1) \cdots \log N(\omega_s),$$

where the sum is taken over all the s-tuples $(\omega_1, \omega_2, \cdots, \omega_s)$ of prime numbers such that

$$\mu = \omega_1 + \omega_2 + \cdots + \omega_s, \qquad \omega_k \in \Omega(\lambda_k) \qquad (k = 1, 2, \cdots, s).$$

Then we have

Theorem 2. *Let μ be a totally positive integer of K and s be a rational integer with $s \geq 3$. Then*

$$R(\mu; \lambda_1, \lambda_2, \cdots, \lambda_s) = \frac{\Psi_1(1; a_1, a_2, \cdots, a_s)}{W^s} \mathfrak{S}_G(\mu) \prod_{k=1}^{s} N(a_k) N(\mu)^{s-1} + O\left(\frac{N^{(s-1)n}}{(\log N)^{s+1}}\right),$$

where $N = \max\{\ |\lambda_s^{(i)}|\ \}_{(1 \leq i \leq n)}$, $W = 2^{r_1 + r_2} \pi^{r_2} hR / w\sqrt{D}$ with w the number of the roots of unity in K and $\mathfrak{S}_G(\mu)$ is the singular series which is written as an infinite product:

$$\mathfrak{S}_G(\mu) = \prod_{\mathfrak{p} | \mu} \left(1 + \frac{(-1)^s}{(N(\mathfrak{p}) - 1)^{s-1}}\right) \prod_{\mathfrak{p} \nmid \mu} \left(1 + \frac{(-1)^{s+1}}{(N(\mathfrak{p}) - 1)^s}\right).$$

3. Outline of proof

Applying the theory of Fourier integrals, $\Psi_1(\mu;\lambda_1,\lambda_2,\cdots,\lambda_s)$ is written as follows:

$$\Psi_1(\mu;\lambda_1,\lambda_2,\cdots,\lambda_s)=\frac{2^{r_2(s-1)}D^{\frac{1-s}{2}}}{N(\mu)\prod_{k=1}^{s}N(a_k)}\prod_{q=1}^{r_1}F_0^{(q)}(1)\prod_{p=r_1+1}^{r_1+r_2}G^{(p)}(1,0).$$

Here $F_0^{(q)}(1)$ and $G^{(q)}(1.0)$ denote the volumes of domains $B_0^{(q)}$ and $D_0^{(p)}$ in $(s-1)$ and $2(s-1)$-dimensional euclidian space, where $B_0^{(q)}$ and $D_0^{(p)}$ are given as follows:

$$B_0^{(q)}=\left\{(u_1,u_2,\cdots,u_{s-1})\in\mathbb{R}^{s-1}\;\middle|\;\begin{array}{l}0\le u_1\le a_1^{(q)},\cdots,0\le u_{s-1}\le a_{s-1}^{(q)}\\ 0\le 1-u_1-\cdots-u_{s-1}\le a_s^{(q)}\end{array}\right\},$$

$$D_0^{(p)}=\left\{\begin{array}{l}x_1,x_2,\cdots,x_{s-1}\\ y_1,y_2,\cdots,y_{s-1}\end{array}\;\middle|\;\begin{array}{l}x_1^2+y_1^2\le(a_1^{(p)})^2,\cdots,x_{s-1}^2+y_{s-1}^2\le(a_{s-1}^{(p)})^2\\ (1-x_1-\cdots-x_{s-1})^2+(y_1+\cdots+y_{s-1})^2\le(a_s^{(p)})^2\end{array}\right\}.$$

We shall give domains $B_1^{(q)}$ and $D_1^{(p)}$ such that $B_1^{(q)}\subset B_0^{(q)}$, $D_1^{(p)}\subset D_0^{(p)}$ and that the volumes of $B_1^{(q)}$ and $D_1^{(p)}$ are positive in each euclidian space. First, we consider two cases to define $B_1^{(q)}$.

Case 1. $a_1^{(q)}+a_2^{(q)}+\cdots+a_{s-1}^{(q)}<1$. Let us define

$$B_1^{(q)}=\left\{u_1,u_2,\cdots,u_{s-1}\;\middle|\;a_i^{(q)}-\delta^{(q)}\le u_i\le a_i^{(q)}\quad(i=1,2,\cdots,s-1)\right\},$$

where

$$\delta^{(q)}=\min\left(a_1^{(q)},\ c^{(q)}/(s-1)\right).$$

Case 2. $1\le a_1^{(q)}+a_2^{(q)}+\cdots+a_{s-1}^{(q)}$. Let us define

$$B_1^{(q)}=\left\{u_1,u_2,\cdots,u_{s-1}\;\middle|\;a_i^{(q)}-h_i^{(q)}\le u_i\le a_i^{(q)}-h_i^{(q)}+\delta_i^{(q)}\quad(i=1,2,\cdots,s-1)\right\}$$

with

$$h_i^{(q)}=s_i^{(q)}c^{(q)},$$
$$\delta_i^{(q)}=h_i^{(q)}+s_i^{(q)}(1-a_1^{(q)}-\cdots-a_{s-1}^{(q)}),$$

where we take positive constants $s_i^{(q)}$ $(i=1,2,\cdots,s-1)$ which satisfy following conditions:

$$s_1^{(q)}+s_2^{(q)}+\cdots+s_{s-1}^{(q)}=1,$$
$$s_i^{(q)}c^{(q)}\le a_i^{(q)}.$$

Secondly we consider two cases to define $D_1^{(p)}$.

Case 1. Suppose $2a_s^{(q)} \geq c^{(q)}$. Let us define

$$D_1^{(p)} = \left\{ \begin{matrix} x_1, x_2, \cdots, x_{s-1} \\ y_1, y_2, \cdots, y_{s-1} \end{matrix} \middle| \begin{matrix} a_j^{(p)} - s\delta^{(p)}/(s-1) \leq x_j \leq a_j^{(p)} - \delta^{(p)} \\ 0 \leq y_j \leq \delta^{(p)}/(s-1) \quad (j = 1, 2, \cdots, s-1) \end{matrix} \right\},$$

where

$$\delta^{(p)} = \min\left(a_1^{(p)}/2,\ c^{(p)}/s\right).$$

Case 2. Suppose $c^{(p)} > 2a_s^{(p)}$. Let $t_j^{(p)}$ be positive constants which satisfy the following conditions:

$$\begin{aligned} &t_1^{(p)} + t_2^{(p)} + \cdots + t_{s-1}^{(p)} = 1, \qquad 0 < t_1^{(p)} < t_2^{(p)} < \cdots < t_{s-1}^{(p)} < 1, \\ &t_j^{(p)} \leq s a_j^{(p)}/(s+1) \qquad\qquad (j = 1, 2, \cdots, s-1). \end{aligned}$$

We define

$$D_1^{(p)} = \left\{ \begin{matrix} x_1, x_2, \cdots, x_{s-1} \\ y_1, y_2, \cdots, y_{s-1} \end{matrix} \middle| \begin{matrix} t_j^{(p)} - \delta^{(p)} \leq x_j \leq t_j^{(p)} \\ 0 \leq y_j \leq \delta^{(p)} \quad (j = 1, 2, \cdots, s-1) \end{matrix} \right\}$$

with

$$\delta^{(p)} = \min\left(a_1^{(p)}/(s+1),\ t_1^{(p)}\right).$$

Now we consider the constant c_1 defined by

$$c_1 = \frac{2^{r_2(s-1)} D^{\frac{1-s}{2}}}{\prod_{k=1}^{s} N(a_k)} \prod_{q=1}^{r_1} \left\{ \prod_{i=1}^{s-1} \delta_i^{(q)} \right\} \prod_{p=r_1+1}^{r_1+r_2} \left\{ \delta^{(p)}/(s-1) \right\}^{2(s-1)},$$

which allow us to establish Theorem 1.

On the proof of Theorem 2, we follow the Vinogradov-Vaughan method generalized to the case of algebraic number fields. We can find the most simple way to derive the asymptotic formula on this problem in Mitsui [2].

References

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