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On the Gap Distribution of Prime Numbers.

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Abstract. A "theoretical" distribution of prime number gaps is proposed and compared with the actual distribution. Some probabilistic discussions are given.

1. Introduction

Let p_n be the n-th prime number and for $x > 0$, put $\pi(x) = \mathrm{Max}\{n | p_n \leq x\}$. The prime number theorem tells us that $\pi(x) \sim \frac{x}{\log x}$, or equivalently $p_n \sim n \log n$.

We call $d_n = p_{n+1} - p_n$ the n-th prime gap. On the order of the growth of d_n, we have two conjectures.

(1.1) $$\underline{\lim}\, d_n = 2$$

(1.2) $$\overline{\lim} \frac{d_n}{(\log p_n)^2} = 1,$$

or more weakly

$$\overline{\lim} \frac{d_n}{(\log p_n)^2} < \infty.$$

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The conjecture (1.1) is the famous twin prime conjecture, which has long been believed to be true, though not yet proved. Put $\pi_2(x) = \#\{n | p_n \le x \text{ and } d_n = 2\}$, then (1.1) is equivalent to $\lim_{x\to\infty} \pi_2(x) = \infty$. Much stronger Hardy-Littlewood conjecture[1] says that

$$\pi_2(x) \sim 2c\frac{x}{(\log x)^2} \tag{1.3}$$

with

$$c = \prod_{n=2}^{\infty}\left(1 - \frac{1}{(p_n-1)^2}\right) = 0.66016\cdots.$$

Some experiments on counting twin prime numbers by computers[2][3] seem to suggest that (1.3) is correct (at least up to $x = 10^{11}$)

Later we shall investigate (1.3) more closely.

Also the conjecture (1.2) has long been believed to be true[4][5][6], but the established results are much weaker: $d_n = O(p_n^\theta)$, $0 < {}^\exists\theta < 1$. The best record at present is $\theta = \frac{11}{20} - \frac{1}{384} \fallingdotseq 0.5473\cdots$[7]. Again by computers, (1.2) seems to be consistent with experiments up to $p_n \sim 10^{14}$.

Historically the studies on d_n have concentrated on the following two points: namely the frequency of twin primes and the occurrences of large gaps. In this paper, we shall discuss the distribution of d_n as a whole. There exists a belief (with no justification) that prime numbers distribute mutually independently except obvious inter-relations, such as $d_n (n \ge 2)$ must be even integers for instance. Under this "independence hypothesis", we can derive a "theoretical" distribution of d_n and compare it with the actual distribution obtained by counting them by computers. This is the purpose of the present paper. Especially, we show that the conjectures (1.2) and (1.3) are true with probability 1 under our "theoretical" distribution hypothesis.

2. Exponential distribution

Discussions in this section are not rigorous mathematically, but the authors' excuse is that the aim of this section is to find a simple and plausible "theoretical" distribution of d_n, not to prove something.

Consider the exponential distribution on $\mathbb{R}_+ = [0, \infty)$. It is the probability measure μ given by $\mu([a, \infty)) = e^{-\alpha a}$, or equivalently by

$$\mu(E) = \alpha \int_E e^{-\alpha t} dt \tag{2.1}$$

for a Borel set E of $[0, \infty)$. This is the distribution of the first occurrence time of the event which occurs with probability $\alpha \Delta t$ in an infinitesimal time interval Δt, independently of t.

Thanks to the "independence hypothesis", we shall assume that the exponential distribution can be applyed to the gaps of prime numbers. But the gaps are always even integers, so do not distribute continuously on $[0, \infty)$. Our excuse is that the smallest gap $d_n = 2$ may be regarded infinitesimal compared with the mean value $< d_n >\sim \log n$ after n primes. So, we shall apply the exponential distribution (of a continuous variable) to the gap distribution of prime numbers assumed to be sufficiently large.

However, "obvious inter-relations" should be taken into account. We observe that $d_n = 6$ is twice as frequent as $d_n = 2$ or $d_n = 4$. The reason is as follows: if $d_n = 2$, then we must have $3 \nmid p_n$ and $3 \nmid p_n + 2$, thus $p_n \equiv 2 \pmod 3$, while if $d_n = 6$, then $3 \nmid p_n$ assures automatically $3 \nmid p_n + 6$, whether $p_n \equiv 1$ or $\equiv 2$. Therefore, $d_n = 6$ is twice as probable as $d_n = 2$ or 4. Similar discussions can be applyed to $d_n = 2k$, and we see that $d_n = 2k$ is c_k times as probable as $d_n = 2$,where

$$c_k = \prod_{p|k} \frac{p-1}{p-2}, \tag{2.2}$$

the product being taken over all odd primes dividing k.

How should we include this effect in the exponential distribution? Suppose that we are challenging to some trial with success probability α. The probability that we succeed for the first time after n trials is $\alpha(1-\alpha)^n$. If a player is allowed to try twice after other person's n trials, the probability that he becomes the first success is $\alpha(1-\alpha)^n + \alpha(1-\alpha)^{n+1}$. This consideration suggests that in the case of the gap distribution of prime numbers, in order to include the above effect in the exponential distribution, it will suffice to take the time interval as $c_k \Delta t$ instead of Δt.

Thus, we obtain the following "theoretical" distribution of d_n.

(2.3)
$$\mathrm{Prob}(d_n = 2k) = \exp(-\alpha t_{k-1}) - \exp(-\alpha t_k)$$
$$\text{where } t_k = \sum_{j=1}^{k} c_j, \quad \alpha\text{:some constant} > 0.$$

Now, we must determine the value of α. The prime number theorem implies that the expectation value $< d_n >$ of d_n under our "theoretical" distribution should be of the order of $\log n$. From (2.3), we have

(2.4)
$$< d_n > = 2\alpha \int_0^\infty k(t) e^{-\alpha t} dt$$

where $k(t) = k$ for $t_{k-1} < t \leq t_k$.

We shall evaluate the order of $k(t)$, or equivalently the order of t_k. Again applying rough discussions, we shall suppose $t_k \sim ck$. Here c is the mean of c_j. For a given p, $p \nmid j$ is $(p-1)$-times as probable as $p \mid j$, so that

$$c = \prod_p \left(\frac{p-1}{p} + \frac{1}{p} \frac{p-1}{p-2} \right) = \prod_p \frac{(p-1)^2}{p(p-2)},$$

the product being taken over all odd primes.

(The discussions of this part can be made rigorous, namely we can prove that $\lim_{k \to \infty} \frac{t_k}{k} = c$.)

From $t_k \sim ck$, we have $k(t) \sim \frac{t}{c}$, so that

$$< d_n > \sim \frac{2\alpha}{c} \int_0^\infty t e^{-\alpha t} dt = \frac{2}{c\alpha}.$$

Combining this with $< d_n > \sim \log n$, we have $\alpha \sim \frac{2}{c \log n}$.

Note that $\frac{1}{c} = \prod_p \frac{p(p-2)}{(p-1)^2} = \prod_p \left(1 - \frac{1}{(p-1)^2}\right)$.

Thus we have determined our "theoretical" distribution as follows:

(2.5)
$$\mathrm{Prob}(d_n = 2k) = \exp(-\alpha_n t_{k-1}) - \exp(-\alpha_n t_k),$$

$$t_k = \sum_{j=1}^{k} c_j \ , \ c_j = \prod_{p|j} \frac{p-1}{p-2},$$

$$\alpha_n = \frac{2c}{\log n} \ , \ c = \prod_{p} \left(1 - \frac{1}{(p-1)^2}\right).$$

3. Conjectures (1.2) and (1.3)

Let $X_n (n = 1, 2, \dots)$ be mutually independent random variables whose distributions are given by (2.5), replacing d_n with X_n. In this situation, we shall prove that both conjectures (1.2) and (1.3) are true with probability 1.

Theorem 1

$\overline{\lim} \dfrac{X_n}{(\log n)^2} = 1$ *almost surely.*

Proof

Since $\text{Prob}(X_n > 2k) = \exp(-\alpha_n t_k)$, Borel-Cantelli's lemma implies that

$$\text{Prob}(\overline{\lim} \frac{X_n}{2k(n)} \geq 1) = 1 \text{ if } \sum_n \exp(-\alpha_n t_{k(n)}) = \infty$$

$$\text{Prob}(\overline{\lim} \frac{X_n}{2k(n)} \leq 1) = 1 \text{ if } \sum_n \exp(-\alpha_n t_{k(n)}) < \infty.$$

Put $k(n) = [\beta(\log n)^2]$, where $\beta > 0$ and $[\ \ ]$ is Gauss' symbol. Since $t_k \sim \frac{1}{c}k$, we have $\alpha_n t_{k(n)} \sim \frac{2c}{\log n}\frac{1}{c}\beta(\log n)^2 = 2\beta(\log n)$ so that $\exp(-\alpha_n t_{k(n)}) \sim n^{-2\beta}$. Thus if $\beta < \frac{1}{2}$, then $\overline{\lim} \dfrac{X_n}{(\log n)^2} \geq 2\beta$ almost surely, and if $\beta > \frac{1}{2}$, then $\overline{\lim} \dfrac{X_n}{(\log n)^2} \leq 2\beta$ almost surely. Combining these, we have $\overline{\lim} \dfrac{X_n}{(\log n)^2} = 1$ almost surely.

Theorem 2 (Probabilistic version of prime number theorem).

For $x > 0$, let $\pi(x)$ be a random variable defined by $\pi(x) = \text{Max}\{n | \sum_{k=1}^{n} X_k \leq x\}$. *Then* $\pi(x) \sim \frac{x}{\log x}$ *almost surely.*

(Remark: $\pi(x) \sim \frac{x}{\log x}$ *is equivalent to* $\sum_{k=1}^{n} X_k \sim n \log n$ *).*

Proof

Put $Y_n = \frac{X_n}{\log n}$, then $Y_n (n = 1, 2, \dots)$ are mutually independent random variables whose means and variances are bounded. So we can apply the strong law of large numbers. Since $< Y_n > \sim 1$, we have $\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} Y_k = 1$ almost surely. But $\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} Y_k = 1$ implies $\sum_{k=1}^{n} X_k \sim n \log n$ as proved below.

Since $Y_k = \frac{X_k}{\log k} \geq \frac{X_k}{\log n}$ for $k \leq n$, we have $\frac{1}{n} \sum_{k=1}^{n} Y_k \geq \frac{1}{n \log n} \sum_{k=1}^{n} X_k$, so that $\overline{\lim} \frac{1}{n \log n} \sum_{k=1}^{n} X_k \leq 1$. On the other hand, since $Y_k \leq \frac{X_k}{\log n}$ for $k \geq n$, we have

$$\sum_{k=[n^\alpha]}^{n} Y_k \leq \frac{1}{\log[n^\alpha]} \sum_{k=1}^{n} X_k \quad \text{for any } 0 < \alpha < 1.$$

The left hand side is equal to $\sum_{k=1}^{n} Y_k - \sum_{k=1}^{[n^\alpha]-1} Y_k$, so that $\lim_{n\to\infty} \frac{1}{n} \sum_{k=[n^\alpha]}^{n} Y_k = 1$, thus we have $\underline{\lim} \frac{1}{\alpha n \log n} \sum_{k=1}^{n} X_k \geq 1$. Letting $\alpha \to 1$, we obtain the desired result.

Theorem 3

For $x > 0$ *, let* $\pi_2(x)$ *be a random variable defined by* $\pi_2(x) = \#\{n | n \leq \pi(x), X_n = 2\}$. *Then* $\pi_2(x) \sim \frac{2cx}{(\log x)^2}$ *almost surely.*

Proof

Put

$$Y_n = \begin{cases} 0, & \text{if } X_n \neq 2; \\ [1 - \exp(-\alpha_n)]^{-1}, & \text{if } X_n = 2. \end{cases}$$

Then $Y_n (n = 1, 2, \dots)$ are mutually independent random variables with means 1. Since $< Y_n^2 > = [1 - \exp(-\alpha_n)]^{-1} \sim \frac{1}{\alpha_n} = \frac{\log n}{2c}$, we can apply the strong law of large numbers to obtain $\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} Y_k = 1$ almost surely. But $\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} Y_k = 1$ implies $\pi_2(x) \sim \frac{2cx}{(\log x)^2}$ as proved below.

Since $\lim_{x\to\infty} \pi(x) = \infty$ almost surely , we have

$$\lim_{x\to\infty} \frac{1}{\pi(x)} \sum_{\substack{k\le\pi(x)\\ X_k=2}} \left[1-\exp\left(-\frac{2c}{\log k}\right)\right]^{-1} = 1 \quad \text{almost surely.}$$

This can be rewritten as

$$\lim_{x\to\infty} \frac{1}{\pi(x)} \int_0^x \left[1-\exp\left(-\frac{2c}{\log \pi(t)}\right)\right]^{-1} d\pi_2(t) = 1$$

Since $\pi(x) \sim \frac{x}{\log x}$ almost surely , we have

$$\lim_{x\to\infty} \frac{\log x}{x} \int_0^x \frac{\log t}{2c} d\pi_2(t) = 1 \quad \text{almost surely.}$$

Replacing $\log t$ with $\log x$, we get

$$\underline{\lim} \frac{(\log x)^2}{2cx} \int_0^x d\pi_2(t) = \underline{\lim} \frac{(\log x)^2}{2cx} \pi_2(x) \ge 1.$$

Replacing $\log t$ with $\log x^\alpha$, we get

$$\int_{x^\alpha}^x \frac{\log t}{2c} d\pi_2(t) \ge \frac{\alpha \log x}{2c} \left(\pi_2(x) - \pi_2(x^\alpha)\right).$$

The left hand side is equal to

$$\int_0^x \frac{\log t}{2c} d\pi_2(t) - \int_0^{x^\alpha} \frac{\log t}{2c} d\pi_2(t),$$

so that $\sim \frac{x}{\log x}$, thus we get

$$\overline{\lim} \frac{\alpha(\log x)^2}{2cx} \left(\pi_2(x) - \pi_2(x^\alpha)\right) \le 1.$$

Since $\pi_2(x) \le \pi(x) \le \frac{x}{2}$, we have $\lim_{x\to\infty} \frac{(\log x)^2}{x} \pi_2(x^\alpha) = 0$ for $\alpha < 1$.
Therefore $\overline{\lim} \frac{\alpha(\log x)^2}{2cx} \pi_2(x) \le 1$. Letting $\alpha \to 1$, we obtain the desired result.

4. **Comparison with the actual distribution**

In the following Table 1, $\pi_{2k}(x) = \#\{n|\ p_{n+1} \le x\ ,\ d_n = 2k\}$ is given for $x = 10^3, 10^4, 10^5, 10^6, 10^7$ and 10^8. This is obtained by determining all prime numbers below x.

The corresponding expected value $\overline{\pi_{2k}(x)}$ under our "theoretical" distribution is given by

$$\overline{\pi_{2k}(x)} = \sum_{n=1}^{\pi(x)} \left[\exp\left(-\frac{2ct_{k-1}}{\log n}\right) - \exp\left(-\frac{2ct_k}{\log n}\right)\right]. \tag{4.1}$$

But since the derivation of the distribution (2.5) is not so rigorous, it does not seem necessary to carry out this complicated summation to check the validity of our "theoretical" distribution. Instead, we shall assume that all $d_n (1 \le n \le \pi(x))$ follow the same distribution as that of $n = \pi(x)/2$, thus we get

$$\overline{\pi_{2k}(x)} = \pi(x)\left[\exp\left(-\alpha t_{k-1}\right) - \exp\left(-\alpha t_k\right)\right], \tag{4.2}$$

$$\text{where } \alpha = \frac{2c}{\log(\pi(x)/2)}.$$

Hereafter $\overline{\pi_{2k}(x)}$ means the right hand side of (4.2). In the Table 1,we shall use the same notation $\overline{\pi_{2k}(x)}$ to denote its integral approximation, namely $\left[\overline{\pi_{2k}(x)} + 0.5\right]$.

Table 1, Actual and Expected Number of Gaps.

$x = 10^3$

$2k$	$\pi_{2k}(x)$	$\overline{\pi_{2k}(x)}$
2	35	43
4	40	32
6	44	42
8	15	13
10	16	12
12	7	11
14	7	4
16	0	3
18	1	3
20	1	1
22		1
24		1

$x = 10^4$

$2k$	$\pi_{2k}(x)$	$\overline{\pi_{2k}(x)}$
2	205	228
4	202	186
6	299	275
8	101	100
10	119	105
12	105	113
14	54	48
16	33	32
18	40	48
20	15	22
22	16	15
24	15	19
26	3	8
28	5	7
30	11	10
32	1	3
34	2	2
36	1	3
38		1
40		1
42		1

$x = 10^5$

$2k$	$\pi_{2k}(x)$	$\overline{\pi_{2k}(x)}$	$2k$	$\pi_{2k}(x)$	$\overline{\pi_{2k}(x)}$
2	1224	1384	42	19	36
4	1215	1184	44	5	13
6	1940	1880	46	4	10
8	773	742	48	3	15
10	916	826	50	5	8
12	964	957	52	7	5
14	484	447	54	4	8
16	339	313	56	1	4
18	514	498	58	4	3
20	238	255	60	1	5
22	223	176	62	1	1
24	206	249	64	1	1
26	88	106	66	0	2
28	98	98	68	0	1
30	146	162	70	0	1
32	32	45	72	1	1
34	33	41	74		0
36	54	61	76		0
38	19	25	78		1
40	28	27			

$x = 10^6$

$2k$	$\pi_{2k}(x)$	$\overline{\pi_{2k}(x)}$	$2k$	$\pi_{2k}(x)$	$\overline{\pi_{2k}(x)}$	$2k$	$\pi_{2k}(x)$	$\overline{\pi_{2k}(x)}$
2	8169	9211	52	77	108	102	0	2
4	8143	8130	54	140	163	104	0	1
6	13549	13511	56	53	80	106	0	1
8	5569	5591	58	54	60	108	0	1
10	7079	6448	60	96	123	110	0	1
12	8005	7866	62	16	38	112	1	0
14	4233	3859	64	24	32	114	1	1
16	2881	2802	66	48	59			
18	4909	4657	68	13	23			
20	2401	2518	70	22	29			
22	2172	1801	72	13	29			
24	2682	2674	74	12	12			
26	1175	1200	76	6	11			
28	1234	1145	78	13	19			
30	1914	2006	80	3	9			
32	550	596	82	5	6			
34	557	559	84	6	12			
36	767	867	86	4	4			
38	330	378	88	1	4			
40	424	411	90	4	7			
42	476	587	92	1	2			
44	202	218	94	0	2			
46	155	179	96	2	3			
48	196	284	98	1	2			
50	106	153	100	2	1			

$x = 10^7$

$2k$	$\pi_{2k}(x)$	$\overline{\pi_{2k}(x)}$	$2k$	$\pi_{2k}(x)$	$\overline{\pi_{2k}(x)}$	$2k$	$\pi_{2k}(x)$	$\overline{\pi_{2k}(x)}$
2	58980	65554	56	1072	1304	110	11	24
4	58621	59087	58	1052	1003	112	11	17
6	99987	101265	60	1834	2135	114	11	25
8	42352	43270	62	543	681	116	7	10
10	54431	51129	64	559	593	118	4	9
12	65513	64568	66	973	1116	120	10	20
14	35394	32772	68	358	451	122	3	6
16	25099	24357	70	524	589	124	4	6
18	43851	41744	72	468	611	126	8	11
20	22084	23383	74	218	268	128	2	4
22	19451	17159	76	194	248	130	1	5
24	27170	26311	78	362	432	132	5	6
26	12249	12208	80	165	220	134	1	2
28	13255	11924	82	100	150	136	2	2
30	21741	21733	84	247	294	138	2	4
32	6364	6719	86	66	105	140	2	2
34	6721	6438	88	71	102	142	0	1
36	10194	10307	90	141	201	144	0	2
38	4498	4649	92	37	65	146	1	1
40	5318	5172	94	39	57	148	2	1
42	7180	7683	96	65	95	150	0	2
44	2779	2958	98	29	48	152	1	1
46	2326	2493	100	36	47	154	1	1
48	3784	4069	102	34	63	156		1
50	2048	2279	104	21	27	158		0
52	1449	1644	106	12	23	160		0
54	2403	2570	108	26	38	162		1

$x = 10^8$

$2k$	$\pi_{2k}(x)$	$\overline{\pi_{2k}(x)}$	$2k$	$\pi_{2k}(x)$	$\overline{\pi_{2k}(x)}$
2	440312	489399	56	16595	17659
4	440257	447828	58	14611	13817
6	768752	784766	60	28439	30204
8	334180	343127	62	8496	9922
10	430016	412597	64	8823	8763
12	538382	534172	66	15579	16900
14	293201	277833	68	6200	7002
16	215804	209958	70	8813	9334
18	384738	367927	72	8453	9950
20	202922	211398	74	4316	4469
22	175945	158024	76	3580	4194
24	257548	247981	78	6790	7492
26	119465	117814	80	3281	3913
28	129567	117075	82	2362	2710
30	222847	219546	84	4668	5456
32	68291	69824	86	1597	1997
34	71248	67955	88	1637	1971
36	114028	111305	90	3337	4007
38	51756	51397	92	1083	1332
40	60761	58215	94	971	1186
42	86637	88902	96	1641	2031
44	34881	35167	98	851	1056
46	29327	30126	100	878	1049
48	49824	50285	102	1059	1440
50	27522	28892	104	494	638
52	20595	21223	106	404	543
54	33593	33953	108	711	932

$2k$	$\pi_{2k}(x)$	$\overline{\pi_{2k}(x)}$	$2k$	$\pi_{2k}(x)$	$\overline{\pi_{2k}(x)}$
110	454	591	166	1	10
112	330	425	168	8	20
114	487	648	170	6	10
116	191	276	172	1	6
118	181	247	174	3	11
120	433	550	176	5	5
122	131	178	178	4	4
124	145	165	180	4	10
126	204	329	182	1	4
128	76	118	184	1	3
130	78	154	186	0	5
132	132	200	188	0	2
134	50	79	190	0	3
136	40	76	192	0	3
138	93	129	194	0	1
140	57	84	196	1	2
142	30	47	198	1	2
144	51	82	200	0	1
146	22	36	202	0	1
148	34	33	204	0	2
150	37	74	206	0	1
152	20	25	208	0	1
154	13	28	210	2	2
156	23	39	212	0	0
158	10	16	214	0	0
160	11	19	216	0	1
162	8	24	218	0	
164	5	11	220	1	

From the Table 1, we observe that the expected number $\overline{\pi_{2k}(x)}$ is in good accordance with the actual one $\pi_{2k}(x)$, at least qualitatively. Though the accordance is not good numerically for some k, the tendencies of both distributions coincide, thus we ascertain the exponential feature of the actual distribution of d_n. The number of twin primes $\pi_2(x)$ is about 10% smaller than $\overline{\pi_2(x)}$. The maximum gap $\max_{n \leq \pi(x)} d_n$, which is given in the Table 2, shows remarkably good accordance, intensifying the belief that the conjecture (1.2) should be true.

Table 2, Maximum gap below x.

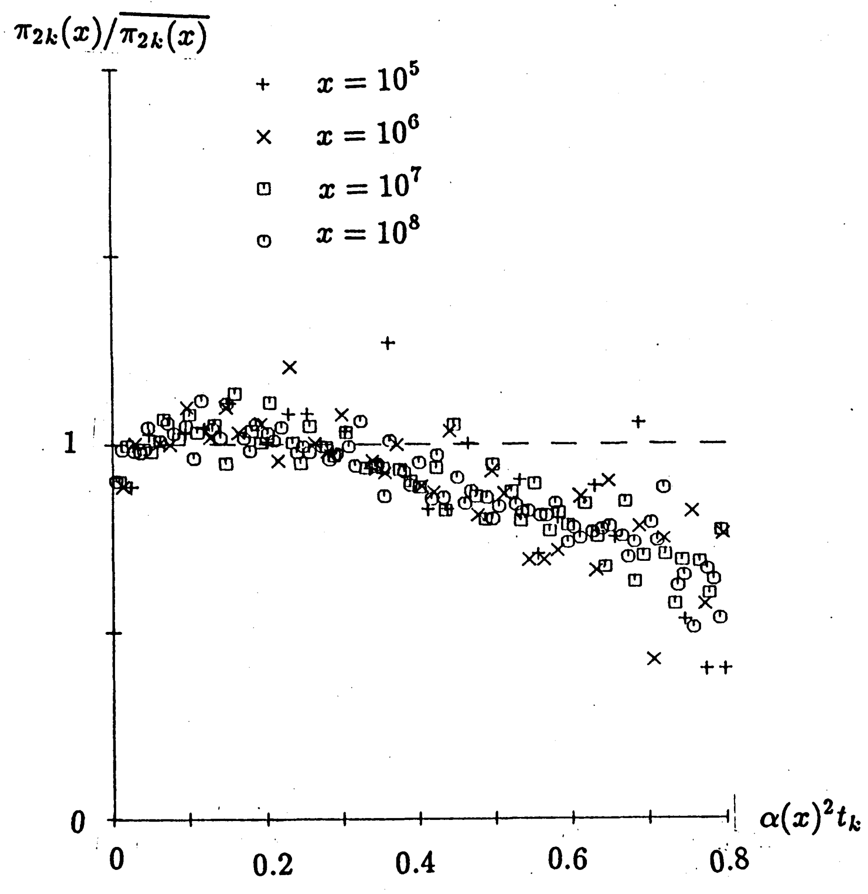
x	actual	expected
10^3	20	24
10^4	36	42
10^5	72	78
10^6	114	114
10^7	154	162
10^8	220	216
10^9	282	282
10^{10}	354	360
10^{11}	464	450
10^{12}	540	546
10^{13}	674	660
10^{14}	804	762

(Actual value for $x \geq 10^9$ is cited from [8]. Expected value is computed by (4.2).)

Our "theoretical" distribution (2.5) is rather simple. More elaborate and complicated "theoretical" distribution may be useful to obtain better accordance. From the Table 1, we observe that for small k and large k, we have $\overline{\pi_{2k}(x)} \geq \pi_{2k}(x)$, while for the medium value of k, we have $\pi_{2k}(x) \geq \overline{\pi_{2k}(x)}$. This does not seem to be accidental, and suggests that we should replace the exponential distribution with some other one to get better accordance.

The following Figure 1 is the graphs of $\pi_{2k}(x)/\overline{\pi_{2k}(x)}$ as the functions of $\alpha(x)^2 t_k$ for $x = 10^5, 10^6, 10^7$ and 10^8, where $\alpha(x) = 2c/\log(\pi(x)/2)$

Figure 1, $\pi_{2k}(x)/\overline{\pi_{2k}(x)}$ as functions of $\alpha^2 t_k$.



We observe that these graphs seem to converge to some curve as $x \to \infty$, but the limit is apparently not the constant 1. Note also that the absissa is $\alpha^2 t_k$, not αt_k. The reason for this deviation is not known, but some effect depending on $(\log \pi(x))^2$ seem to exist.

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