数理解析研究所講究録
第 887 巻 1994 年 216-225

Index for factors generated by direct sums of II_1 factors

Atsushi SAKURAMOTO

Department of Mathematics
Faculty of Science
Kyoto University

Introduction.

In 1983 Jones introduced in [**3**] the concept of an index for a pair of type II_1 factors, called Jones index nowadays, and he showed the importance such indices. With this as a momentum, the interests of research in the theory of operator algebras have been gradually extended from a single factor to a pair of factors. However it is not easy to calculate explicitly the index even for a pair of II_1 factors only from the definition. For this reason, useful index formulas are expected, but there are few except Wenzl's one [**4**]. Wenzl's formula is applicable only for pairs of approximately finite dimensional(=AFD) II_1 factors. Hence a formula for pairs of non-AFD factors has been expected. In this note we give a new index formula and its application, for a pair of non-AFD II_1 factors.

§1. Preliminaries.

1.1. Let $M = \bigoplus_{j=1}^{m} M_j$ be a finite direct sum of II_1 factors and q_j the minimal central projection corresponding to M_j. Since the normalized normal trace on II_1 factor is unique, a trace on M (denoted by tr) is specified by a numerical vector $(\mathrm{tr}(q_i))_{i=1,\cdots,m}$ called the trace vector.

Let $N = \bigoplus_{i=1}^{n} N_i \subset M$ be another finite direct sum of II_1 factors and p_i the corresponding minimal central projection. We assume that the trace on N is the restriction of the trace on M . The trace vector for M (resp. N) is denoted by $\vec{s}$ (resp. $\vec{t}$).

Now we define two matrices representing the inclusion relation $N \subset M$, one is the index matrix and another is the trace matrix. The index matrix $\Lambda_N^M = (\lambda_{ij})$ is given by

$$\lambda_{ij} = \begin{cases} [M_{p_iq_j} : N_{p_iq_j}]^{1/2} & \text{if } p_iq_j \neq 0, \\ 0 & \text{if } p_iq_j = 0, \end{cases}$$

and the trace matrix $T_N^M = (t_{ij})$ is given by $t_{ij} = \mathrm{tr}_{M_j}(p_iq_j)$, where tr_{M_j} is the unique normalized normal trace on M_j. The following properties (1.1)~(1.4) come from the very definitions.

(1.1) $\lambda_{ij} \in \{0\} \cup \{2\cos(\pi/n) \,; n \geq 3\} \cup [2, \infty]$

(1.2) Trace matrix T_N^M is column-stochastic, i.e., $t_{ij} \geq 0$ and $\sum_i t_{ij} = 1$ for all j.

(1.3) The equality $\vec{t} = T_N^M \vec{s}$ holds.

(1.4) If $N \subset M \subset L$ are finite direct sums of II_1 factors,then

$$T_N^L = T_N^M T_M^L.$$

1.2. We suppose that N is of finite index in M, i.e., there is a faithful representation π of M on a Hilbert space such that the commutant $\pi(N)'$ is finite. Then the algebra $\langle M, e_N \rangle$ obtained from the basic construction for $N \subset M$ is a finite direct sum of II_1 factors and the corresponding minimal central preojections are $Jq_1J, \cdots, Jq_mJ$, where J is the canonical conjugation on $L^2(M, \mathrm{tr})$.

As is shown in [**2**], the index matrix and the trace matrix for $M \subset \langle M, e_N\rangle$ have the following properties (1.5)~(1.7).

(1.5) $$\Lambda_M^{\langle M,e_N\rangle} = (\Lambda_N^M)^t$$

(1.6) $$T_M^{\langle M,e_N\rangle} = \tilde{T}_N^M F_N^M,$$

where $(\tilde{T}_N^M)_{ji} = \begin{cases} \dfrac{\lambda_{ij}^2}{t_{ij}} & p_iq_j \neq 0, \\ 0 & p_iq_j = 0, \end{cases}$ $F_N^M = \mathrm{diag}(\varphi_1, \cdots, \varphi_n)$,

$\varphi_i = (\sum_j (\tilde{T}_N^M)_{ji})^{-1}$.

(1.7) For any trace Tr on $\langle M, e_N\rangle$, $\mathrm{Tr}(e_Np_i) = \varphi_i \mathrm{Tr}(Jp_iJ)$.

The index $[M : N]$ is defined as follows,

(1.8) $[M : N] = r(\tilde{T}_N^M T_N^M)$, where $r(T)$ is the spectral radius of T.

1.3. We conclude this section by recalling the trace on the relative commutant.

Let $M_0 \subset M_1$ be an irreducible pair, that is $M_0' \cap M_1 = \mathbb{C}$, of II_1 factors with finite index. By the basic construction, we obtain a tower of II_1 factors $M_0 \subset M_1 \subset M_2 \subset \cdots \subset M_n \subset \cdots$ and denote by E_n the trace preserving conditional expectation of M_n onto M_{n-1}. (For the definition of a conditional expectation, see e.g. [**2**] §2.1.) Since the relative commutant $M_{n-1}' \cap M_n$ is trivial, the conditional expectation E_n is minimal and $E_{M_0}^{M_n} = E_nE_{n-1}\cdots E_1$ is also minimal. Therefore we obtain that

$$\mathrm{tr}_{M_n}(x) = \mathrm{tr}_{M_0'}(x) \quad \text{for } x \in M_0' \cap M_n. \tag{1.9}$$

§2. Factors generated by direct sums of II_1 factors.

In this section, we construct a pair of factors from two increasing sequences of finite direct sums of II_1 factors and caluculate the index for the pair.

LEMMA 2.1. *Let $N \subset M$ be a pair of II_1 von Neumann algebras with finite dimensional centers acting on a Hilbert space H. Let* tr *be a faithful finite trace on M and E_N be the trace preserving conditional expectation of M onto N. Suppose a projection $e \in B(H)$ satisfies the following conditions:*

$exe = E_N(x)e$ for all $x \in M$ and the map $N \ni x \longmapsto xe \in Ne$ is $*$-isomorhic.

Then,

(1) $\langle M, e\rangle = A \oplus B$, with two von Neumann algebras $A \cong \langle M, e_N\rangle$, and B isomorphic to an ultraweakly closed subalgebra of M.

(2) Let $z \in \langle M, e\rangle$ be the central projection onto A. Then z is equal to the central support of e.

(3) Let Tr *be a trace on $\langle M, e\rangle$ such that $\mathrm{Tr}|_M = \mathrm{tr}$, then*

$$\mathrm{Tr}(e) \geq d \cdot \mathrm{Tr}(z), \quad \textit{where } \ d = \min\{\varphi_i = (F_N^M)_{ii}\ ;\ i = 1, \cdots, n\}.$$

Let $\{M_n\}_{n\in\mathrm{N}}$ and $\{N_n\}_{n\in\mathrm{N}}$ be two increasing sequences of direct sums of II_1 factors such that, for each $n \in \mathbb{N}$, the following diagram

$$
(2.1) \qquad \begin{array}{ccc} M_n & \subset & M_{n+1} \\ \cup & & \cup \\ N_n & \subset & N_{n+1} \end{array}
$$

is a commuting square.

DEFINITION 2.2: Let A, B, C and D are finite von Neumann algebras such that $A \supset B \supset D$ and $A \supset C \supset D$, and tr be a finite trace on A. Denote by E_N^M the trace preserving conditional expectation of M onto N. The diagram

$$
\begin{array}{ccc} A & \supset & B \\ \cup & & \cup \\ C & \supset & D \end{array}
$$

is called a *commuting square* if the diagram with the mappings

$$
\begin{array}{ccc} A & \xrightarrow{E_B^A} & B \\ {\scriptstyle E_C^A}\downarrow & & \downarrow{\scriptstyle E_D^B} \\ C & \xrightarrow[E_D^C]{} & D \end{array}
$$

commutes.

Moreover we deal with the following two conditions.

CONDITION I (Periodicity): There exist $n_0 \geq 1$ and $p \geq 1$ such that for any $n \geq n_0$,

(1) $T_{N_n}^{N_{n+1}}$, $T_{M_n}^{M_{n+1}}$ and $F_{N_n}^{M_n}$ are periodic modulo p.

(2) $T_{N_n}^{N_{n+p}}$ and $T_{M_n}^{M_{n+p}}$ are primitive.

CONDITION II (Lower Boundedness): There exists a constant $d > 0$ such that $(F_{N_n}^{M_n})_{ii} \geq d$ for all n and i.

It is clear that Condition II follows from Condition I.

Here we put $M = (\cup_n M_n)''$ and $N = (\cup_n N_n)''$.

LEMMA 2.3. *Let $\{M_n\}_{n\in\mathbb{N}}$ and $\{N_n\}_{n\in\mathbb{N}}$ be increasing sequences of finite direct sums of II_1 factors such that for any $n \in \mathbb{N}$ the diagram (2.1) is a commuting square.*

(1) If Condition I holds, M and N are II_1 factors.

(2) If Condition II holds, and M and N are II_1 factors, then the index $[M : N]$ is finite.

Here we give an index formula which is one of our main results of this note.

THEOREM 2.4. *Let $\{M_n\}_{n\in\mathbb{N}}$ and $\{N_n\}_{n\in\mathbb{N}}$ be increasing sequences of finite direct sums of II_1 factors such that for any $n \in \mathbb{N}$ the diagram (2.1) is a commuting square. Set $M = (\cup_n M_n)''$ and $N = (\cup_n N_n)''$.*

(1) Assume M and N are II_1 factors, and $[M : N] < \infty$. Then

$$[M : N] = \lim_n < \vec{t}_n, \vec{f}_n >,$$

where $\vec{f}_n = ((F_{N_n}^{M_n})_{ii}^{-1})_i$ and $\vec{t}_n$ is the trace vector of N_n and $< \cdot, \cdot >$ is the standard inner product.

(2) If Condition I holds, then for all $n \geq n_0$,

$$[M : N] = < \vec{t}_n, \vec{f}_n > = [M_n : N_n].$$

REMARK 2.1: In case that M_n and N_n are finite direct sums of full matrix algebras, the same formula holds too. This formula is not exactly the same as Wenzl's index formula, but essentially equivalent.

Similarly as in [4], we get the next proposition concerned with the relative commutant.

THEOREM 2.5. *Let $\{M_n\}_{n\in\mathbb{N}}$ and $\{N_n\}_{n\in\mathbb{N}}$ be increasing sequences of finite direct sums of II_1 factors and $\{p_{n,i}\}_{i=1}^{m_n}$ be the minimal central projections of N_n such that for any $n \in \mathbb{N}$ the diagram (2.1) is a commuting square. Set $M = (\cup_n M_n)''$ and $N = (\cup_n N_n)''$. Suppose that $N \subset M$ is a pair of II_1 factors with finite index and there exists a constant $c > 0$ such that $\mathrm{tr}(p_{n,i}) > c$ for all i and n.*

Then for any nonzero projection $p \in N_n$, the following inequality holds:

$$\dim\,(N' \cap M) \leq \dim\,(N_n' \cap M_n)_p.$$

§3. Examples. In this section, we give examples of $\{M_n\}_{n\in\mathbb{N}}$ and $\{N_n\}_{n\in\mathbb{N}}$ satisfying Condition II.

Let $A_{-1} \subset A_0$ be an irreducible pair of II_1 factors with index λ. If $\lambda < 4$, then there exists $k \in \mathbb{N}$ such that $\lambda = 4\cos^2(\pi/k)$. In case $\lambda \geq 4$

we put $k = \infty$.

By the basic construction we get a sequence of II_1 factors $A_{-1} \subset A_0 \subset A_1 = \langle A_0, e_1 \rangle \subset A_2 = \langle A_1, e_2 \rangle \subset \cdots$, where $e_i = e_{A_{i-2}}$. Now we define

(3.1) $N_0 = A_0$, $N_i = \langle A_{-1}, e_1, \cdots, e_i \rangle$ for $i \geq 1$ and $M_j = A_j$ for $j \geq 0$.

Then $N_n \cong N \otimes \langle e_1, \cdots, e_n \rangle$, so we can see the structure of N_n from the structure of $\langle e_1, \cdots, e_n \rangle$. This fact is important in the sequel.

LEMMA 3.1. *For all n, the diagram*

$$\begin{array}{ccc} M_n & \subset & M_{n+1} \\ \cup & & \cup \\ N_n & \subset & N_{n+1} \end{array} \tag{3.2}$$

is a commuting square.

Next we calculate the trace matrices $T_{M_n}^{M_{n+1}}$, $T_{N_n}^{N_{n+1}}$ and $T_{N_n}^{M_n}$.

It is clear that $T_{M_n}^{M_{n+1}} = (1)$, and $T_{N_n}^{N_{n+1}}$ is given in the next proposition.

PROPOSITION 3.2. *Let $\Lambda_{N_n}^{N_{n+1}}$ be the index matrix and $T_{N_n}^{N_{n+1}}$ the trace matrix of the inclusion $N_n \subset N_{n+1}$. Then,*

$$\Lambda_{N_n}^{N_{n+1}} = (d_{i,j}^{(n)})_{ij},\ d_{i,j}^{(n)} = \begin{cases} 1 & j = i, i+1, \\ 0 & \textit{otherwise}, \end{cases}$$

$$T_{N_n}^{N_{n+1}} = (c_{i,j}^{(n)})_{ij},\ c_{i,j}^{(n)} = \begin{cases} \dfrac{\alpha_{n,i}}{\alpha_{n+1,j}} & j = i, i+1, \\ 0 & \textit{otherwise}, \end{cases}$$

where for $n \leq k-3$,

$i = 0, 1, \cdots, [(n+1)/2]$, $j = 0, 1, \cdots, [(n+2)/2]$, $\alpha_{n,i} = \binom{n}{i} - \binom{n}{i-2}$,

and for $n \geq k-2$,

$i = [(n-k+4)/2], \cdots, [(n+1)/2]$, $j = [(n-k+5)/2], \cdots, [(n+2)/2]$, $\alpha_{n,i} = \binom{n}{i} - \binom{n}{i-2} - \binom{n}{i+k-2}$.

PROPOSITION 3.3. *Let* $\Lambda_{N_n}^{M_n}$ *be the index matrix and* $T_{N_n}^{M_n}$ *the trace matrix of the inclusion* $N_n \subset M_n$. *Then,*

$$T_{N_n}^{M_n} = (c_i^{(n)}) \text{ with } c_i^{(n)} = \alpha_{n,i}\lambda^{-i}P_{n+2-2i}(\lambda^{-1})$$

and

$$\Lambda_{N_n}^{M_n} = (d_i^{(n)}) \text{ with } d_i^{(n)} = \lambda^{(n+1-2i)/2}P_{n+2-2i}(\lambda^{-1}),$$

where

$i = 0, \cdots, [(n+1)/2]$ $(n \leq k-3)$;

$i = [(n-k+4)/2], \cdots, [(n+1)/2]$ $(n \geq k-2)$,

and $\alpha_{n,i}$ *is the constant in Proposition 3.2 and* $P_n(t)$ *is Jones polynomial defined by* $P_0(t) = P_1(t) = 1$ *and* $P_n(t) = P_{n-1}(t) - tP_{n-2}(t)$.

Put $M = (\cup_n M_n)''$ and $N = (\cup_n N_n)''$, then M and N are II_1 factors (cf. [**1**]).

THEOREM 3.4. *Let* $A_{-1} \subset A_0$ *be an irreducible pair of* II_1 *factors with index* λ *and construct* $\{M_n\}_n$ *and* $\{N_n\}_n$ *as in (3.1).*

(1) $\{M_n\}_n$ *and* $\{N_n\}_n$ *satisfy Condition II if and only if the index* $\lambda < 4$.

(2) The index $[M : N]$ *is given by*

$$[M : N] = \begin{cases} \dfrac{k}{4\sin^2\frac{\pi}{k}} & \text{if } \lambda < 4, \\ \infty & \text{if } \lambda \geq 4, \end{cases}$$

where k *is an integer such that* $\lambda = 4\cos^2(\pi/k)$.

REMARK 3.1: In case $\lambda < 4$, the pair $N \subset M$ is irreducible, that is, $N' \cap M = \mathbb{C}$, by Theorem 2.5.

REFERENCES

[1] M. Choda, *Index for factors generated by Jones' two sided sequence of projections*, Pacific J. Math., **139** (1989), 1–16.

[2] F. M. Goodman, P. de la Harpe and V. F. R. Jones, "Coxeter Graphs and Towers of Algebras," MSRI publications 14, Springer-Verlag, New York, 1989.

[3] V. F. R. Jones, *Index for subfactors*, Invent. Math., **72** (1983), 1–25.

[4] H. Wenzl, *Hecke algebras of type* A_n *and subfactors*, Invent. Math., **92** (1988), 349–383.

著者　桜本篤司