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STEADY CONFIGURATIONS OF A VORTEX FILAMENT IN BACKGROUND FLOWS

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The three-dimensional configurations of a thin vortex filament embedded in background flows are investigated, based on the localized induction approximation (*LIA*). An analogy is found between trajectories of a charged particle in a magnetic field and steady configurations of a vortex filament in a flow. This analogy allows us to use the Lagrangian or Hamiltonian formalism of classical mechanics for calculating the equilibrium shape of a vortex filament. As an example, an analogy associated with the Kida class is discussed in detail. With this, the left-right duality, that is, the duality between the spatial and body descriptions, becomes a reality for a heavy symmetrical top. It is revealed that a geometric phase enters into a formula for the rotation angle of the top axis.

1 Introduction

A vortex filament is an individual entity exhibiting a vigorous bending and twisting motion owing to self-induction as well as the influence of neighboring vortices and the background flow. A knowledge of such a behavior may help to gain some insight into the complicated structure of high-Reynolds-number flows.

In this paper, we turn our attention to the three-dimensional long bending motion of a concentrated vortex filament. The fluid is assumed to be incompressible and inviscid. For the sake of simplicity, we employ a certain asymptotic theory for self-induction, called "the *localized induction approximation (LIA)*"; the size δ of the vortex-core is so thin that the self-induced velocity at a point $\boldsymbol{X}$ on the filament is dominated by the contribution from the neighboring segment of length L:[1,2]

$$\boldsymbol{X}_t = A\boldsymbol{X}_s \times \boldsymbol{X}_{ss} + \boldsymbol{V}(\boldsymbol{X}, t) \quad , \tag{1}$$

$$A = \frac{\Gamma}{4\pi} \log\left(\frac{L}{\delta}\right) \quad , \tag{2}$$

where $\boldsymbol{X} = \boldsymbol{X}(s,t)$ denotes the filament curve as a function of the arclength s and the time t, $\boldsymbol{V}$ is the velocity field of a prescribed background flow and Γ is the circulation. The subscripts indicate the partial differentiation of indicated variables.

We present a brief review of the known results in the absence of the external flow: $\boldsymbol{V} = \mathbf{o}$. Da Rios[1] transformed (1) into a coupled system of intrinsic equations for the curvature κ and torsion τ of the filament curve:[1,3]

$$\kappa_t = -(\kappa\tau)_s - \kappa_s\tau \quad , \tag{3a}$$

$$\tau_t = \left[(\kappa_{ss} - \kappa\tau^2)/\kappa\right]_s + \kappa_s\kappa \quad . \tag{3b}$$

In the above, the constant A has been absorbed into a rescaling of the time variable. Hasimoto[4] found that (3a) and (3b) combine to yield the nonlinear Schrödinger equation

$$i\psi_t + \psi_{ss} + \frac{1}{2}|\psi|^2\psi = 0 \quad , \tag{4}$$

$$\psi(s,t) = \kappa \exp\{i \int^s \tau ds\} \quad . \tag{5}$$

He obtained a localized twist-wave solution of (1), now called the Hasimoto soliton, via the one-soliton solution of (4).

There are a variety of ways to generalize these findings, among which are attempts to include axial velocity in the core[5,6] and to allow for vortex-line stretching.[7] Despite its importance, the question as to the influence of external flows on the motion of a vortex filament is addressed by only a few authors[8–11] and does not seem to be fully explored.

Our aim is to present a simple recipe for calculating the equlibrium shapes of a vortex filament embedded in background flows. Especially we spotlight the mathematical structure of the equation by pursuing analogies with classical mechanics, electromagnetism, elasticity and geometrical acoustics. Although the steady solutions are not sufficient to have much practical bearings, we believe that they may capture the leading-order behavior of some of slow bending motions of a vortex filament.

2 Equilibrium Shapes of a Vortex Filament in Background Flows

Consider the equilibrium shape of a filament in a steady flow $\boldsymbol{V}(\boldsymbol{x})$. We assume that the flow field is unaffected by the presence of the vortex. For an equilibrium filament, only the slipping motion along itself is allowed; $\boldsymbol{X}_t = -c(s,t)\boldsymbol{t}$, where $\boldsymbol{t} = \boldsymbol{X}_s$ is the tangential vector of the filament curve and $c(s,t)$ is an arbitrary function. Taking the vector product with $\boldsymbol{t}$, (1) is cast into[12]

$$A\boldsymbol{X}_{ss} = \boldsymbol{X}_s \times \boldsymbol{V}(\boldsymbol{X}) \quad . \tag{6}$$

If we think of s as the time t, $\boldsymbol{V}(\boldsymbol{x})$ as the magnetic field and A as m/q, then (6) is identifiable as the equation governing the motion of a charged particle, with mass m and charge q, in a magnetic field $\boldsymbol{V}$. That is to say, the static shape of a vortex filament in a flow is equivalent to the trajectory of a charged particle moving subject to the Lorentz force. Otherwise expressed, the static balance of a vortex filament has a similarity with the diamagnetism, meaning that the flow field induced by the vortex filament opposes the external field. The simpler form (6) has much in its favor, leading us to the Lagrangian formalism, a powerful tool in classical mechanics, with its Lagrangian $\hat{L}$ given by

$$\hat{L} = \frac{A}{2}\dot{\boldsymbol{X}}^2 + \dot{\boldsymbol{X}} \cdot \hat{\boldsymbol{A}} \quad , \tag{7}$$

where $\hat{\boldsymbol{A}}$ is the vector potential associated with the magnetic field: $\boldsymbol{V} = \mathrm{rot}\,\hat{\boldsymbol{A}}$.

In passing, we note that the propagation of sound rays in a steady low-Mach-number flow $\boldsymbol{U}(\boldsymbol{X})$ obeys the same equation as (6):[13]

$$\frac{d\boldsymbol{t}}{ds} = -\boldsymbol{t} \times \mathrm{rot}\,\boldsymbol{U}(\boldsymbol{X})/c \quad , \tag{8}$$

where $\boldsymbol{t}$ is the unit tangential vector along the ray and c is the sound speed. Fermat's principle for sound rays in a steadily moving medium is written, up to first order in Mach number, as[13]

$$\begin{aligned} &\delta \oint \left\{ \sqrt{(c^2 - U^2)ds^2 + (\boldsymbol{U} \cdot d\boldsymbol{s})^2} - \boldsymbol{U} \cdot d\boldsymbol{s} \right\} / (c^2 - \boldsymbol{U}^2) \\ \approx\ &\frac{1}{c}\delta \oint \left(1 - \frac{\boldsymbol{U} \cdot \boldsymbol{t}}{c}\right) ds = 0 \quad . \end{aligned} \tag{9}$$

The origin of the second term in (9) is traced to the fact that the angular frequency of sound is shifted in a moving medium to $\omega = ck + \boldsymbol{U} \cdot \boldsymbol{k}$ with $\boldsymbol{k}$ being the wavenumber vector. The additional term $\dot{\boldsymbol{X}} \cdot \hat{\boldsymbol{A}}$ in the Lagrangian (7) is reminiscent of the Doppler effect of sound propagation.

3 Kida Class

A proper example that illustrates the benefit from the use of this analogy is steady configurations of a vortex filament moving in a fluid at rest.[14,15] Kida[15] reasoned that such a motion is composed of a translation with velocity V in a certain direction, say z, a rotation with angular velocity Ω about the same axis, and a slipping motion with speed c_0:

$$A\boldsymbol{X}_s \times \boldsymbol{X}_{ss} = -c_0\boldsymbol{X}_s + \Omega\boldsymbol{e}_z \times \boldsymbol{X} + V\boldsymbol{e}_z \quad , \tag{10}$$

where $\boldsymbol{e}_z$ is the unit vector in the z-direction, and c_0, Ω, V are all constants. Comparing (6) with (10) and denoting the differentiation in s by a dot, we have

$$\begin{aligned} A\ddot{\boldsymbol{X}} &= \dot{\boldsymbol{X}} \times \boldsymbol{B}(\boldsymbol{X}) \quad , & (11\text{a}) \\ \boldsymbol{B} &= -\Omega\boldsymbol{e}_z \times \boldsymbol{X} - V\boldsymbol{e}_z \quad . & (11\text{b}) \end{aligned}$$

The Lagrangian $\hat{L}$ producing (11) is

$$\hat{L} = \frac{A}{2}(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2) - (\frac{V}{2}r^2\dot{\theta} - \frac{\Omega}{2}r^2\dot{z}) \quad , \tag{12}$$

where use has been made of cylindrical coordinates $(r,\ \theta,\ z)$. Inspection of (12) convinces us that θ and z are both cyclic and two first integrals are readily available:

$$\begin{aligned} P_z &= \partial\hat{L}/\partial\dot{z} = A\dot{z} + \Omega r^2/2 = \text{const.} \quad , & (13\text{a}) \\ P_\theta &= \partial\hat{L}/\partial\dot{\theta} = Ar^2\dot{\theta} - Vr^2/2 = \text{const.} & (13\text{b}) \end{aligned}$$

These integrals, augmented by $|\dot{\boldsymbol{X}}|^2 = \dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2 = 1$, coincide with the set of equations handled by Kida.[15] The advantage is that our treatment requires less ingenuity.

Hasimoto and Kambe[16] noticed that (10) is interpreted as an equation describing the static balance of the internal stress and its moment of a thin elastic rod $\boldsymbol{X}(s)$ of circular cross-section. Suppose that a local coordinate frame is attached at each point on the central axis of the rod such that the frames are identical with each other when the rod is unstrained and straight. Let $\boldsymbol{\Omega}$ be the rate of rotation of the coordinate axes along the rod. The unit tangential vector $\boldsymbol{t}$ along the rod then varies according to

$$\frac{d\boldsymbol{t}}{ds} = \boldsymbol{\Omega} \times \boldsymbol{t} \quad . \tag{14}$$

Taking the vector product with $\boldsymbol{t}$, (14) becomes

$$\Omega = \boldsymbol{t} \times \boldsymbol{t}_s + (\boldsymbol{t} \cdot \boldsymbol{\Omega})\boldsymbol{t} \quad . \tag{15}$$

Under the stipulation that the cross-section is circular, the moment $\boldsymbol{M}(s)$ of internal stress is given by

$$\boldsymbol{M} = A(\boldsymbol{X}_s \times \boldsymbol{X}_{ss}) + C\gamma\boldsymbol{X}_s \quad , \tag{16}$$

where A and C are the bending stiffness and torsional rigidity, respectively, and $\gamma = \boldsymbol{t} \cdot \boldsymbol{\Omega}$ is the torsion angle of the rod per unit length. Putting $c_0 = C\gamma$ and $\Omega \boldsymbol{e}_z = \boldsymbol{T}$, (10) reads

$$\begin{aligned} \boldsymbol{T}_s &= \mathbf{o} \quad , & (17a) \\ \boldsymbol{M}_s + \boldsymbol{t} \times \boldsymbol{T} &= \mathbf{o} \quad . & (17b) \end{aligned}$$

Regarding $\boldsymbol{T} = \boldsymbol{T}(s)$ as the internal stress of rod, (17a) expresses the static balance of the internal stress in the absence of internal body force. Equation (17b) corresponds to moment balance. It is an easy task to show that γ is a constant, consistently with Kida's treatment.

As promised by Kirchhoff's analogy, the equations for equilibrium shape of an elastic rod of circular cross-section are known to be identifiable with the equations of motion for a heavy rigid body, with axial symmetry, fixed at a stationary point, i.e. Lagrange's top.[17] Below, we give a brief explanation of it.

Let $\boldsymbol{t}(t)$ and $\boldsymbol{\omega}^{(s)}(t)$ be the unit vector along the axis of symmetry and the angular velocity of the body as functions of time t. By definition,

$$\dot{\boldsymbol{t}} = \boldsymbol{\omega}^{(s)} \times \boldsymbol{t} \quad , \tag{18}$$

where a dot denotes the time-derivative. Taking the vector product with $\boldsymbol{t}$, we get

$$\boldsymbol{\omega}^{(s)} = \boldsymbol{t} \times \dot{\boldsymbol{t}} + (\boldsymbol{t} \cdot \boldsymbol{\omega}^{(s)})\, \boldsymbol{t} \quad , \tag{19}$$

Because of symmetry, the angular momentum $\boldsymbol{M}(t)$ takes on the form

$$\boldsymbol{M} = A(\boldsymbol{t} \times \dot{\boldsymbol{t}}) + C\omega_3 \boldsymbol{t} \quad , \tag{20}$$

where A and C are the moments of inertia at the stationary point, and $\omega_3 = \boldsymbol{t} \cdot \boldsymbol{\omega}^{(s)}$ is the angular velocity about the axis of symmetry. We can confirm that (20), coupled with (19), is equivalent to (16). Identifying as $\Omega \boldsymbol{e}_z = mgl\boldsymbol{e}_z$, with g being the gravity-acceleration and l being the length of line segment connecting the stationary point with the center of mass, (10) reads

$$\dot{\boldsymbol{M}} = \boldsymbol{t} \times (-mgl\boldsymbol{e}_z) \quad , \tag{21}$$

a formula for the rate of change of the angular momentum. Rewriting (21) and $\dot{\boldsymbol{e}}_z = \mathbf{o}$ (cf.(17a)) in the coordinate system rigidly tied to the body, we reproduce the well-known Euler-Poisson equations for Lagrange's top. It deserves mention that (18) and (21), supplemented by (20), constitute a subset of the system of equations in inertial coordinates.[18] The remaining equation governs the evolution of the inertial tensor itself,[18–20] but the axial symmetry renders it ignorable.

Table 1: Analogy associated with Kida class

	vortex filament	elastic rod	symmetrical top	charged particle
A	$\frac{\Gamma}{4\pi}\log\left(\frac{L}{\delta}\right)$	bending stiffness	moment of inertia with respect to x_1, x_2-*axis*	m/q
C		torsional rigidity	moment of inertia with respect to *axis of symmetry*	
γ_0		torsion angle per unit length	ω_3 (angular velocity about *axis of symmetry*)	
$C\gamma_0$	c_0 (slip velocity)			$-\boldsymbol{B}\cdot\dot{\boldsymbol{X}}$
$\Omega\boldsymbol{e}_z$	angular velocity of rotation	$\boldsymbol{T}_0$ (internal stress)	$mgl\boldsymbol{e}_z$	$-\Omega\boldsymbol{e}_z\times\boldsymbol{X}$ (*azimuthal* component of magnetic field)
$V\boldsymbol{e}_z$	translation velocity	$M_0\boldsymbol{e}_z$ (moment)	$M_z\boldsymbol{e}_z$ (z-component of angular momentum)	$-V\boldsymbol{e}_z$ (z-component of magnetic field)
$\boldsymbol{t}$	$\partial\boldsymbol{X}/\partial s$	$\partial\boldsymbol{X}/\partial s$	$\boldsymbol{k}$ (axis of symmetry)	$\dot{\boldsymbol{X}}$ (velocity)
s	arclength	arclength	t (time)	t (time)

This exemplifies the left-right duality put forward by Arnol'd, that is, the duality between the description in spatial coordinate system and that in body coordinate system.[21,22] The salient feature for a heavy symmetrical top is that the counterpart of the dual pair, the spatial descriptions, is embodied by familiar physical systems. Alternatively speaking, a variety of methods to solve the motion of Lagrange's top are at our disposal and each method admits its own physical interpretation. The description in body coordinates, with the help of Euler angles, is found in standard textbook of classical mechanics. The equation viewed from inertial coordinates was written out and directly integrated by Kida.[15] Its Lagrangian and Hamiltonian formalisms are vividly realized in conjunction with a charged particle or sound rays. In case we rely on the Hamilton-Jacobi method, the action is linked with wave fronts for sound propagation in the geometrical-acoustic approximation. An elastic rod makes visible the relationship between spatial and body frames and thus acts as a mediator for duality.[23] Moreover, there is a purely intrinsic method, independent of choice of any coordinate frame. This is the approach adopted by Levi-Civita[14] and uses (3). Table 1 summarizes the relationships among the physical systems associated with Lagrange's top.

The spinning top whose axis is always vertical and whose angular velocity is constant is called *a sleeping top.* It is likened to a straight-line vortex. The sleeping top loses its stability if the rotation speed is slower than a critical value: $\omega_3^2 < 4Amgl/C^2$. When the rotation speed is reduced below this limit owing to friction, the top wakes up. In the language of a vortex filament, this is what *the Hasimoto soliton* corresponds to.

Further, taking the vector product of (21) with $\boldsymbol{t}$, we obtain

$$A\ddot{\boldsymbol{t}} = -mgl[\boldsymbol{e}_z - (\boldsymbol{t}\cdot\boldsymbol{e}_z)\boldsymbol{t}] - A(\dot{\boldsymbol{t}}\cdot\dot{\boldsymbol{t}})\boldsymbol{t} - C\omega_3\dot{\boldsymbol{t}}\times\boldsymbol{t} \quad , \tag{22a}$$

with the constraint that

$$\boldsymbol{t}\cdot\boldsymbol{t} = 1 \tag{22b}$$

This equation is interpreted as an equation describing the motion of a spherical pendulum with charge subjected to a magnetic field. In this case, the gravity-acceleration is mgl/A, being directed in the negative z-axis. The magnetic field $\boldsymbol{B}$ is that produced by a magnetic monopole located at the origin $\boldsymbol{t} = \mathbf{o}$:

$$(q/m)\boldsymbol{B} = -C\omega_3\boldsymbol{t}/|\boldsymbol{t}|^3 \quad , \tag{23}$$

because of (22b). The first term on the RHS of (22a) implies that the particle is exerted by the gravity force but constrained to the surface of a unit sphere. The second one is the centrifugal force produced by the tension of a string connecting the particle with the origin. It is interesting to note that a similar analogy is discovered between Lagrange's top and a spherical pendulum by Berry & Robbins [24] in a different context.

The configuration space for Lagrange's top is all possible rotations, that is, the rotation group $SO(3)$. The parametrization of $SO(3)$ requires three variabls. The traditional way is the use of the Euler angles (θ, φ, ψ) that express the relationship between space and body coordinates. On the other hand, for the motion of a spherical pendulum governed by (22), it is sufficient merely to specify the position of the tip of the vector $\boldsymbol{t}$, that is, to specify θ and φ. It follows that reduction of freedom is achieved in the passage from (21) to (22). The origin of the reduction is traced to the rotational symmetry about the top axis $\boldsymbol{t}$, reflecting the fact that the parameter ψ is irrelevant in (21).The emergence of a Lorentz force in the reduced system (22) is not accidental, but is well captured in the general framework of the cotangent bundle reduction or the Lagrangian reduction, as formulated by Marsden.[25] This fact, in turn, may help to find out the reason why there exists a close analogy between a vortex filament and a charged particle.

As a reverse process, we can reconstruct the solution of (21) from that of the reduced system (22) by quadrature. It is through the reconstruction process that the *anholonomy* enters.[25,26] In the next section, we illustrate that the geometric phase manifest itself in the rotation angle of the axis of Lagrange's top.

4 How much does a heavy symmetrical top rotate ?

The same question was addressed to the rotation of Euler's top, that is, a free rigid body by Montgomery *et al.*[26,27] and Levi.[28] The formula for the spatial rotation during one period T of its body angular momentum vector consists of a geometric and a dynamical part. By the geometric part, we mean that it depends only on the geometry of the closed orbit and is independent of the speed with which the orbit is traversed. In the following, we show that the same is true of the spatial rotation of the axis of Lagrange's top.

Using the Euler angles, the angular velocity ω_3 about the top axis is expressed as

$$\omega_3 = \dot{\psi} + \dot{\phi}\cos\theta \quad . \tag{24}$$

By virtue of the axial symmetry, the axial component M_3 of the angular momentum $\boldsymbol{M}$ is conserved:

$$M_3 = C(\dot{\psi} + \dot{\phi}\cos\theta) = \mu \quad , \tag{25}$$

where μ is a constant. Let us assume that the top axis executes a periodic motion with period T, though this is not always the case. The rotation angle $\Delta\psi$ of the axis in one period is then

$$\begin{aligned} \Delta\psi &= \frac{\mu}{C}T - \int_0^T \dot{\phi}\cos\theta dt \\ &= \frac{\mu}{C}T + \Omega \quad , \end{aligned} \tag{26a}$$

where

$$\Omega = \int \sin\theta d\theta \wedge d\psi \quad , \tag{26b}$$

and use has been made of Stokes' theorem. Since the top axis is expressed as $\boldsymbol{t} = (\sin\theta\sin\phi, -\sin\theta\cos\phi, \cos\theta)$, (26b) stands for the solid angle swept out by the axis, when viewed from the inertial frame. The first term of (26a) is the dynamic phase, and Ω is the geometric phase. The expression (26b) has resemblance with that for the rotation of a free rigid body.[26,27] In the latter case, it occurs for the rotation angle of the angular momentum vector viewed from the body frame. Note that the above procedure is equivalent to the reconstruction of the solution of (21) from that of the reduced system (22). In general, the geometric phase is formulated as the holonomy of some connection on a principal G-bundle. In the following, we give a brief explanation of this from the standpoint of the Lagrangian reduction. Its prescription is detailed in Ref.(25).

The configuration space is $Q = SO(3)$, and $G = S^1$ acts on Q by rotation about the top axis. The Lagrangian L,

$$L = \frac{A}{2}(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta) + \frac{C}{2}(\dot{\psi} + \dot{\phi}\cos\theta)^2 - mgl\cos\theta \quad , \tag{27}$$

is invariant under the left action by S^1. The Lie algebra $\mathcal{G}$ of S^1 is identified with the axial component ω_3 of the angular velocity and its dual $\mathcal{G}^*$ is identified with the axial component M_3 of the angular momentum. The mechanical connection $\alpha : TQ \to \mathcal{G}$ takes on the form:

$$\alpha = M_3/C = \cos\theta\dot{\phi} + \dot{\psi} \quad . \tag{28}$$

The general statement of the remaining procedure is as follows. For each $\mu \in \mathcal{G}^*$, define 1-form α_μ on Q by

$$< \alpha_\mu(q), v > = < \mu, \alpha(q,v) > \quad , \tag{29}$$

where $(q,v) \in TQ$, and $< \, , \, >$ are natural pairings between spaces and their duals. The Routhian $R^\mu : TQ \to \boldsymbol{R}$ is defined by

$$R^\mu(q,v) = L(q,v) - < \alpha(q,v), \mu > \quad . \tag{30}$$

Notice that this form is slightly different from the well-known one. *The reduced Lagrangian variational principle* states that the solution on Q/G_μ, with G_μ being the isotropy subgroup at $\mu \in \mathcal{G}^*$, satisfies the Euler-Lagrange equations with gyroscopic forcing:

$$\frac{d}{dt}\frac{\partial R^\mu}{\partial \dot{q}^i} - \frac{\partial R^\mu}{\partial q^i} = \dot{q}^j \beta_{ij} \quad . \tag{31}$$

Here β_{ij} are components of a two-form $\beta = \boldsymbol{d}\alpha_\mu$ on Q that drops to Q/G_μ. For $\alpha_\mu = \alpha_i dq^i$,

$$\beta_{ij} = \frac{\partial \alpha_j}{\partial q^i} - \frac{\partial \alpha_i}{\partial q^j} \quad . \tag{32}$$

In the context of Lagrange's top, $G_\mu \cong S^1$ itself, and

$$\alpha_\mu = \mu(\cos\theta d\phi + d\psi) \quad . \tag{33}$$

The Routhian on $T(Q/S^1)$ is,

$$R^\mu = \frac{A}{2}\left(\dot\theta^2 + \dot\phi^2 \sin^2\theta\right) - \left(\frac{\mu^2}{2C} + mgl\cos\theta\right) \tag{34}$$

The Euler-Lagrange equation (31) on the reduced space then reproduces (22). Note that the vector form of α_μ, dropped to $SO(3)/S^1$, is written, in terms of spherical coordinates (r, θ, φ), as

$$\boldsymbol{A} = \left(0, 0, \mu\frac{\cos\theta}{\sin\theta}\right) \quad , \tag{35}$$

where the restriction to the surface $r = 1$ is to be understood. It represents, up to gauge transformations, a vector potential for a monopole field whose source is placed at the origin. We observe that this field is exactly that given by (23).

In accordance with the formula due to Marsden *et al.*,[25,26] the geometrical part Ω of the rotation angle $\Delta\psi$, represented by (26b), is log of the holonomy, that is, the integration of the mechanical connection α along the closed solution curve $c(t)$ on $SO(3)/S^1$:

$$-\oint_{c(t)} \alpha = -\frac{1}{\mu}\int_S \mathrm{rot}\boldsymbol{A}\cdot d\boldsymbol{S} \tag{36}$$

where the last integral is taken over the domain on the surface of a unit sphere $S^2 \cong SO(3)/S^1$ whose boundary is $c(t)$.

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