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Nearly singular two-dimensional Kolmogorov flows for large Reynolds numbers

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Abstract. We study the convergence of the two-dimensional stationary Kolmogorov flows as the Reynolds number increases to infinity. Since the flows to be considered are stationary solutions of the Navier-Stokes equations, they are smooth whatever the Reynolds number may be. However, in the limit of infinite Reynolds number, they can, at least theoretically, converge to a non-smooth function. Through numerical experiments, we will show that, under a certain condition, some smooth solutions of the Navier-Stokes equations converge to a non-smooth solution of the Euler equations and develop internal layers. Therefore the Navier-Stokes flows are "nearly singular" for large Reynolds numbers. In view of this nearly singular solutions, we propose a possible scenario of turbulence, which is of intermediate nature between Leray's scenario and Ruelle-Takens's one.

§1. Introduction.

We study how stationary solutions to the Navier-Stokes equations depends on the Reynolds number as the number tends to infinity. Specifically, we consider incompressible viscous flows in 2D flat tori. The equations of motion are as follows:

$$\begin{aligned} u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} &= \nu\triangle u - \frac{\partial p}{\partial x} + \gamma\sin(\pi y/b), \\ u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} &= \nu\triangle v - \frac{\partial p}{\partial y}, \\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0, \end{aligned}$$

where (u,v) is the velocity vector, p is the pressure, ν is the kinematic viscosity, and the constant γ is the strength of the external force; $\triangle$ denotes the two dimensional Laplace operator. This system is defined in the following rectangle:

$$(x,y) \in [-a,a] \times [-b,b].$$

We impose the periodic boundary condition on (u,v). In other words, the flow region is a flat torus. The flows of this type are called the Kolmogorov flows ([**5,14**]).

In what follows, we consider a family of solutions of the above equations. The family is parametrized by the viscosity ν and we will study its properties as $\nu \to 0$. From the properties we propose an interpretation of two dimensional turbulent motions.

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In order to describe the solutions in detail, we introduce the stream function and a nondimentional form of the equations. We first define the stream function ψ as

$$u = \frac{\partial \psi}{\partial y}, \qquad v = -\frac{\partial \psi}{\partial x}.$$

Then the equation is written as follows:

$$J(\Delta\psi, \psi) = \nu\Delta^2\psi + \frac{\pi\gamma}{b}\cos(\pi y/b)$$

where J is the bilinear form defined as

$$J(f,g) = \frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x}.$$

We now define the following nondimentional transform:

$$(x,y) \mapsto \frac{b}{\pi}(x',y'), \qquad \psi \mapsto \frac{\gamma b^3}{\nu\pi^3}\psi'.$$

Then, after dropping the primes, we obtain

$$\frac{\Delta^2\psi + \cos y}{R} + J(\psi, \Delta\psi) = 0 \tag{1.1}$$

where R is the Reynolds number defined as

$$R = \frac{\gamma\pi^2}{\nu^2 b^2}.$$

The Grashoff number may be a better name but we employ the name 'Reynolds' number since it is traditionally used in the references (see [**5**] and the references in [**14**]). The equation (1.1) is condidered in the following rectangle

$$[-\pi/\alpha, \pi/\alpha] \times [-\pi, \pi]$$

Note that there are infinitely many ways to get a nondimensional form. Our nondimensionalization is characterized by an external force of order $O(1/R)$. Note also that our nondimensional form (1.1) gives us the following solution:

$$\psi = -\cos y \tag{1.2}$$

which is independent of R.

It is known (see, for instance, [**2,18**]) that all the solutions of (1.1) are real analytic. Hence, in particular, they are infinitely many times differentiable. Suppose now that there is a family of solutions ψ_R $(0 < R < \infty)$. Then we would like to know the behavior of ψ_R as $R \to \infty$. Formally, the equation (1.1) becomes the following

$$J(\psi, \Delta\psi) = 0. \tag{1.3}$$

Since this partial differential equation is not elliptic, it can have nonsmooth solutions (see the next section for detail). So we would like to ask:

(1) Does ψ_R converge when $R \to \infty$?
(2) If they converge, then what properties does the limit function have ?
(3) If they converge, in what topology do they converge ?

When one considers these problems, he might well notice a similarity with a question discussed in DiPerna and Majda [4]. Concentrations in flows approximating to the Euler flow are fully discussed there. In §3 we give an example of numerical solutions which seems to shed light on above questions. We are interested in these questions, since we have found numerically certain ψ_R which do NOT converge in C^∞ topology although all of them are infinitely many times differentiable. Their convergence, however, is uniform in (x, y), thus they do not show bizarre phenomena like concentrations. So our example shows a certain convergence between C^∞ and C^0.

§2. The 2D stationary Euler equation.

Before showing the numerical solutions in the next section, we give in this section some remarks on the 2D Euler equation (1.3). It is important to note that (1.3) can have non-smooth solutions, while all the solutions of the Navier-Stokes equation (1.1) are real analytic. This is most easily understood in the following way. Since (1.3) implies there is a functional relation between $\Delta\psi$ and ψ, the relation

$$-\Delta\psi = F(\psi), \tag{2.1}$$

where F is any function of one variable, implies (1.3). If F is infinitely differentiable, then all the bounded solutions of (2.1) are infinitely differentiable, too. On the other hand, if F is non-smooth, then ψ can be non-smooth: if, for instance, $F \in C^0 \setminus C^1$, then the solutions of (2.1) can not be in C^3 and are generalized solutions of (1.3). Here C^k denotes the set of all the functions which are k times continuously differentiable.

Also, note that

$$\psi(x, y) = g(x) \qquad \text{or} \qquad h(y), \tag{2.2}$$

where g and h are functions of one variable, satisfies (1.3). So, non-smooth g and h give us generalized solutions.

In view of these examples, we must be careful about how many times the solutions of (1.3) are differentiable. Also important is to note that (1.3) has an infinite number of solutions. In fact, the arbitrariness of F, g, and h in (2.1) and (2.2) show that (1.3) has uncountably many solutions. On the other hand, the number of solutions of (1.1) for a fixed R, is generically finite. So, among the uncountable solutions of (1.3), at most countably many solutions can be a limit of solutions of (1.1) as $R \to \infty$. So overwhelming majority of the solutions of (1.3) have no connections with those of (1.1).

The following question would naturally arise:

> What conditions on the solutions of (1.3) makes them be approachable from solutions of (1.1)?

The following theorem, which gives us a necessary but not sufficient condition for a solution of (1.3) to be a limit of those of (1.1), was proved in [15]:

THEOREM. *If a solutions of (2.1) is a limit in $H^2(T)$ of (1.1) as $R \to \infty$, then it satisfies the following condition.*

$$\int_T [F'(\psi)\Phi'(\psi) + \cos y\Phi(\psi)]\, dxdy = 0$$

and

$$\int_T [\nabla\psi\nabla\phi - F(\psi)\phi]\, dxdy = 0$$

for all $\phi \in C^\infty(T)$ *and* $\Phi : \mathbb{R} \to \mathbb{R}$. *Here* T *denote the torus and* $H^2(T)$ *is the Sobolev space of index* 2.

Thus the Euler flows which are approachable from the Navier Stokes flows must satisfy a combined variational principle for ψ and F. Although this theorem gives us a necessary condition for an Euler solution to be a limit of Navier-Stokes solutions, we do not know a necessary and sufficient condition. Anyway this theorem imposes a great constraint on F for the classical Euler flow to be an inviscid limit of the Navier-Stokes flows.

We finally remark that, since the domain of the flow is the two dimensional flat torus, no boundary layer is possible in the Kolmogorov flows. There, however, may appear what is called a internal layer. In the next section we show that the Kolmogorov flows have internal layers when the Reynolds number is large.

§3. The Kolmogorov flows of large Reynolds numbers.

Some results of [**5,14**] are recalled here. When $\alpha \geq 1$, then the trivial flow (1.2) is unique for all the Reynolds number. This very remarkable result is proved by Iudovich [**5**]. So the questions in §1 are trivial when $\alpha \geq 1$. On the other hand, when $\alpha < 1$ there are bifurcations of stationary solutions. For instance, if $\alpha = 0.7$, then the bifurcation occurs at $R = R^* \equiv 3.01119\cdots$. This bifurcation is a supercritical pitchfork (Figure 1). We now consider the bifurcating solutions ψ_R for $R^* < R < \infty$. This family is one side of the pitchfork and the other side is given by $\psi_R(\cdot + \pi/\alpha)$. Figure 2 shows the streamlines of the solutions. The solutions are obtained by discretizing (1.1) by the spectral method, see [**14**]. We notice that the vortex near the center of the rectangle is oblate for small Reynolds numbers but prolate for large Reynolds numbers. It is observed that the patterns of the streamlines converge.

In order to show that $\{\psi_R\}$ converge to a solution of (1.3), we plot the points

$$\Big\{(\psi(x,y), -\Delta\psi(x,y)) \;\Big|\; (x,y) = \Big(\frac{m\pi}{\alpha M}, \frac{n\pi}{N}\Big), \quad m = -M, \cdots, 0, 1, \cdots, M,$$
$$n = -N, \cdots, 0, 1, \cdots, N\Big\}.$$

Here M and N are the number of the trancation in the Fourier spectral method used for the computation of (1.1). They ranges from 16 to 40. When $R = 10000$, we used $M = N = 40$. We computed also by $(M, N) = (60, 40)$ and observed almost the same result.

Figure 3 shows the case of $R = 10.0$. Obviously, there is no functional relation. Figure 4 shows the case of $R = 10000.0$ We may say that the points are almost on a continuous curve, which implies that the solution is very close to a 2D Euler solution. We hasten to the following claim: *As* $R \to \infty$, *the stream functions* ψ_R *converge uniformly to a solution of the 2D Euler equation (1.3).*

If we look at Figure 4 carefully, we notice that the curve is continuous but looks like having a straight line segment in it. So it may happen that the function F is not infinitely differentiable. The function might be only piecewise continuous at worst, although it is too early to conclude that, since we are uncertain about the accuracy of our numerical solutions. We will later see, by the energy spectrum, to what extent F is smooth or nonsmooth. Before that we show Figure 5, which is the graph of the function $(-\triangle)^{1/2}\omega$. It seems to suggest that the function, in the limit of $R \to \infty$, is NOT smooth along two curves and is smoother elsewhere. These curves represents internal layers, which may be characterized as discontinuities of the fourth order derivatives of the stream function. This contrasts the fact that the boundary layer is characterized by sharp slopes (or quick changes) of the velocity field. At present, we have little information about the width of the internal layer or its accurate position.

We now define the energy spectrum. We begin with the Fourier expansion of ψ as follows:

$$\psi(x,y) = \sum_{(m,n)\in\mathbb{Z}^2} a(m,n)\exp\left(\sqrt{-1}(m\alpha x + ny)\right).$$

The energy spectrum is, by definition, the sequence $\{e(k)\}_{k=1}^{\infty}$ which is defined through the following equality.

$$E = \frac{1}{2}\int_{-\pi}^{\pi}\int_{-\pi/\alpha}^{\pi/\alpha}(\psi_x^2 + \psi_y^2)\,dxdy = \frac{2\pi^2}{\alpha}\sum_{m,n}(m^2\alpha^2 + n^2)a(m,n)^2 = \sum_{k=1}^{\infty} e(k)$$

Namely,

$$e(k) = \sum_{k^2 \le m^2+n^2 < (k+1)^2} \frac{2\pi^2}{\alpha}(m^2\alpha^2 + n^2)a(m,n)^2.$$

Figure 6 shows the energy spectrum of the solution when $\alpha = 0.7$ and $R = 10000$. The almost linear relation between $\log k$ and $\log e(k)$ implies that

$$e(k) \approx k^{-7}.$$

So one of the derivative of ψ of fourth order may be discontinuous. For the sake of comparison, we draw the energy spectrum when $R = 200.0$ (Figure 7). In this case, the Reynolds number is moderately large and the spectrum shows the exponential decrease. We conclude that the limit function

$$\lim_{R\to\infty} \psi_R$$

exhibits a weak singularity.

In the numerical experiments (see, for instance, [**1**,**7**,**11**]), spectrum of k^{-3} to k^{-4}, or slightly steeper ones, are observed. We have theories such as the Batchelor-Kraichnan power k^{-3} ([**8**]), Saffman's k^{-4} ([**17**]), or somewhere between them ([**6**]). So the singularity of our solutions is much weaker than the observed ones. Krasny [**9**] suggests that the vortex sheet can be an inviscid limit of the Navier-Stokes flows. These evidence shows that turbulent motions can be attributed to solutions whose singularity is much stronger than that of our nearly singular solution (see also [**13**]). However, our solutions can play a part of turbulent processes. This viewpoint is discussed in the next section.

We summarize the result as follows. The theorem in §2 imposes a strong constraint on the Euler flows to be inviscid limits. But it does not make everything smooth. Some Euler flows of finite differentiability can be invicid limit from the Navier-Stokes flows.

We now consider other solutions of (1.1). When $\alpha = 0.8$, we have a pitchfork similar to the one with $\alpha = 0.7$. The solution with $R = 10000$ have the energy spectrum as in Figure 8. It shows almost the same slope as in the case of $\alpha = 0.7$. If we let α increase toward 1.0, we see that the near singularity disappears. In fact, it is numerically shown in [**14**] that the solution converges to

$$\frac{\cos y + \cos x}{-2}$$

as $R \to \infty$ and $\alpha \to 1$. Therefore the algebraic decay of the energy spectrum disappears at $\alpha = 1$. We have little information about the way of disappearance of k^{-7} spectrum as $\alpha \to 1$. Figure 9 shows the case of $\alpha = 0.9999$ and $R = 10000$. We can see two asymptotic lines, one of which has slope -6 and the other -4. It is interesting that the spectrum has two asymptotic lines.

When $\alpha < 1/2$, we have solutions whose streamlines are topologically different from those we have seen so far. Figure 10 shows a solution in which $\alpha = 0.45$ and $R = 10000$. The plot of the vorticity against the stream function is given in Figure 11. This figure shows that there is no single-valued function F satisfying (2.1)in the whole domain. But there are functions satisfying (2.1) locally in the torus, which is clear since the figure is composed of several curves. Its energy spectrum is shown in Figure 12.

Finally we remark that the internal layer is described by

$$\begin{aligned}
&-\epsilon\Delta\omega + \epsilon\cos y - \frac{\partial\psi}{\partial x}\frac{\partial\omega}{\partial y} + \frac{\partial\psi}{\partial y}\frac{\partial\omega}{\partial x} = 0,\\
&-\Delta\psi = \omega,
\end{aligned}$$

where $\epsilon = 1/R$ is a small positive parameter tending to zero. Mathematical theory of singular perturbation may explain the k^{-7} spectrum of the solution. We hope this will happen in future.

§3. Interpretation.

Let us recall some well-known theories. According to the famous scenario by Ruelle and Takens ([**16**]), the onset of temporal chaos is explained by the emergence of a strange attractor resulting from the instability of tori. As has been criticized, this scenario is hard to comply with the spatially complicated structure appearing in the strong (or developed) turbulence. In the dynamical systems viewpoint, the vector field is identified with a point in the function space. So, the dimension of the attractor seems to be the only clue to the spatial complexity. Since all the elements belonging to the attractor are infinitely differentiable vector fields, the dimension or the largeness of the attractor seems to be the most important issue.

Often compared with the scenario is the conjecture by Leray ([**12**]) which asks whether the spontaneous singularities are present in the nonstationary solutions of three dimensional Navier-Stokes equations. While this is a long-unsolved problem, every solutions of two dimensional Navier-Stokes equations are smooth ([**2,18**]). In addition, the two-dimensional Navier-Stokes equations have the finite dimensional global attractor ([**2,19**]) and finite dimensional inertial form ([**10**]). Thus the notion of singularity plays no role in the dynamical systems theory of the two-dimensional Navier-Stokes equations. Leray's attempt to explain turbulence by the singularities are restricted to the three dimensional case. Since the Ruelle-Takens scenario lacks the notion of spatial structure of the individual vector field and since the dynamical systems theory assumes the existence and uniqueness of solutions in the whole space time, it is hard to reconcile their scenario with Leray's one. In particular, co-existence is next to impossible in the case of two dimensional flows.

We now consider the two-dimensional case. The existence of nearly singular solutions seems to support the following understanding of the flow structures: Suppose that the global attractor of the Navier-Stokes equations contains (unstable) stationary solutions. If the Reynolds number is sufficiently large, then the attractor may have nearly singular solutions. This means that the solution is infinitely differentiable but its C^knorm is very large for some k. While the orbit in the phase space wanders around the attractor, it may approach the nearly singular solutions again and again. If we look at this in the physical space, we can see nearly singular distribution of the vorticity. This could be reflected by concerted oscillations of high-frequency modes. The Ruelle-Takens scenario plus nearly singular solutions seems to answer some questions on the two dimensional turbulence. For instance, nearly singular solutions permit a power law in the energy spectrum. We would like to emphasize that algebraic decay of the energy spectrum is possible even when the dimension of the attractor is small. So, we may say that the knowledge of nearly singular solutions, which are present in the Navier-Stokes equations may widen the applicability of the dynamical systems theory to turbulence

Our idea, which is somewhere between Ruelle-Takens and Leray, seems to worth checking in other typical flows. Stronger singularities such as k^{-4} are observed in many numerical simulations. At this stage, weaker singularities are hard to be seen. However, since nearly singular solutions are contained in the global attractor, weaker singularities like ours are expected to be observed at the last stage of the turbulent motion.

Figure Captions

Figure 1 : Bifurcation diagram when $\alpha = 0.7$. There is one and only one pitchfork.

Figure 2 : Streamlines of the bifurcation solutions in Figure 1: a) $R = 3.6$, b) $R = 4.6$, c) $R = 10.0$, d) $R = 10000.0$.

Figure 3 : (x, y) plot of vorticiti (y) versus stream function (x). $R = 10.0$

Figure 4 : (x, y) plot of vorticiti (y) versus stream function (x). $R = 10000.0$

Figure 5 : Graph of $z = (-\Delta)^{1/2}\omega(x, y)$. Four curves are observed, along which the function's derivative changes sharply.

Figure 6 : Energy spectrum. $\alpha = 0.7, R = 10000.0$. The dotted line is a line $[\log e(k)] = -7[\log k]$.

Figure 7 : Energy spectrum. $\alpha = 0.7, R = 200.0$. It decreases exponentially.

Figure 8 : Energy spectrum. $\alpha = 0.8, R = 10000.0$. The dotted line is a line $[\log e(k)] = -7[\log k]$.

Figure 9-a : Energy spectrum. $\alpha = 0.9999, R = 10000.0$. Two dotted lines are drawn: one has a slope of -6 and the other has -4.

Figure 9-a : Energy spectrum in mean. This shows $\hat{e}(k) = (e(k) + e(k + 1) + e(k + 2))/3$. $\alpha = 0.9999, R = 10000.0$. The dotted lines has a slope of -5.

Figure 10 : Streamlines of the flow. $\alpha = 0.45, R = 10000$.

Figure 11 : (x, y) plot of vorticiti (y) versus stream function (x) of the solution in Figure 10. $\alpha = 0.45, R = 10000.0$

Figure 12 : Energy spectrum. $\alpha = 0.45, R = 10000.0$. The dotted line has a slope -7.

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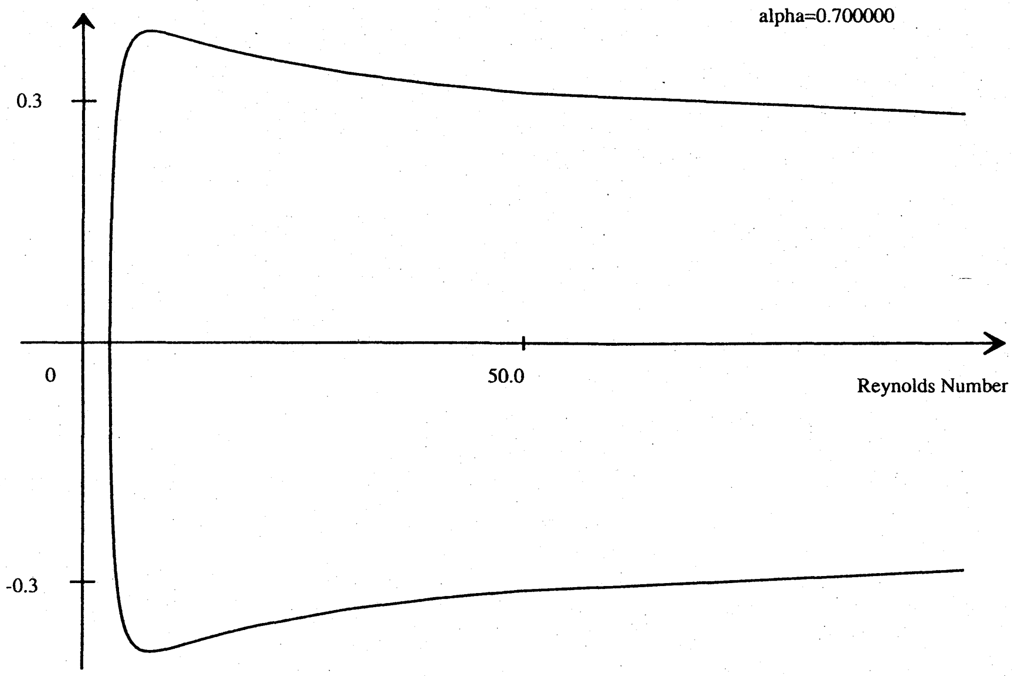


Figure 1

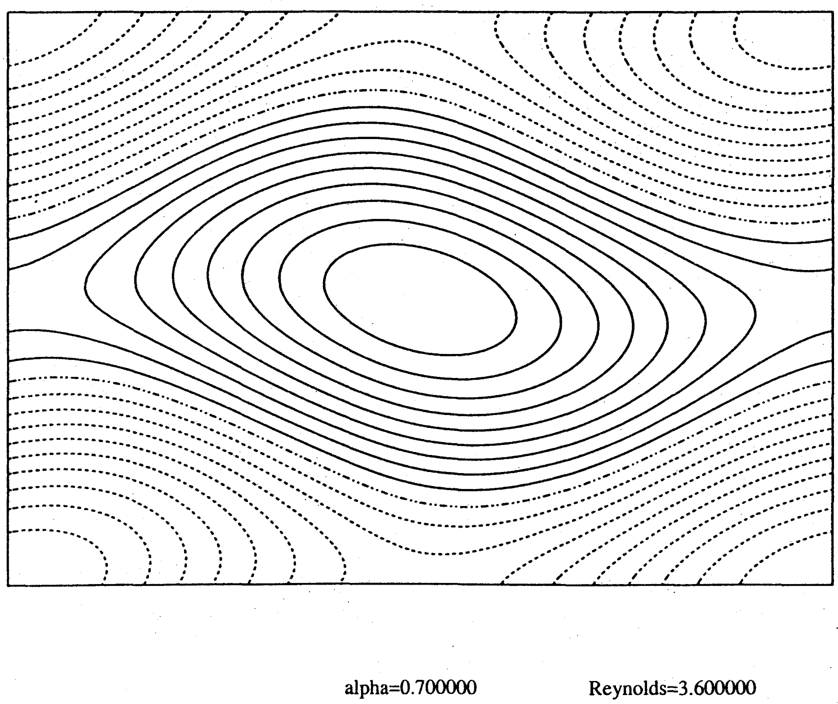
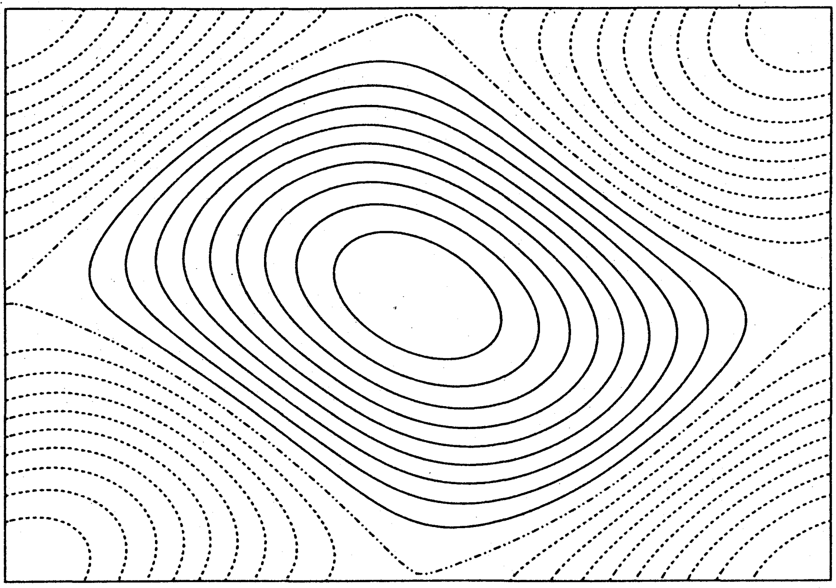
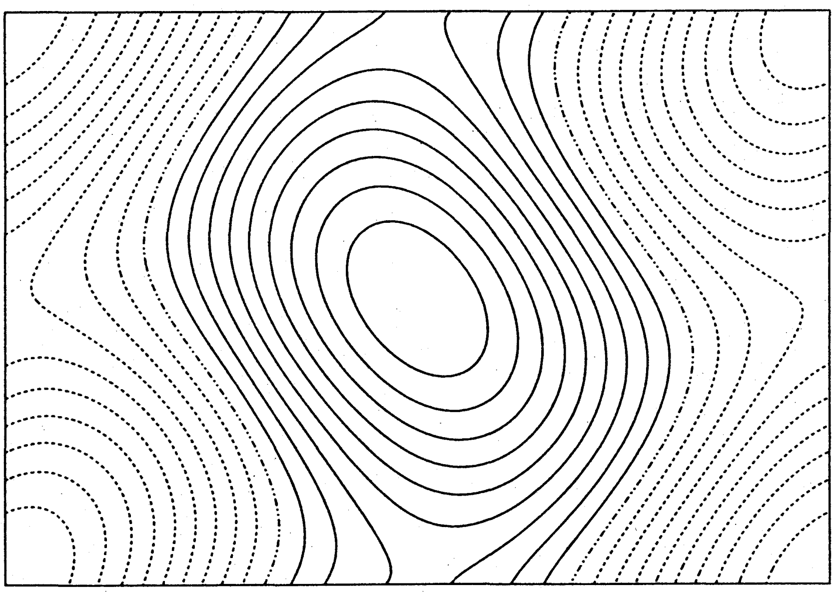


Figure 2-a



alpha=0.700000 Reynolds=4.600000

Figure 2-b



alpha=0.700000 Reynolds=10.000000

Figure 2-c

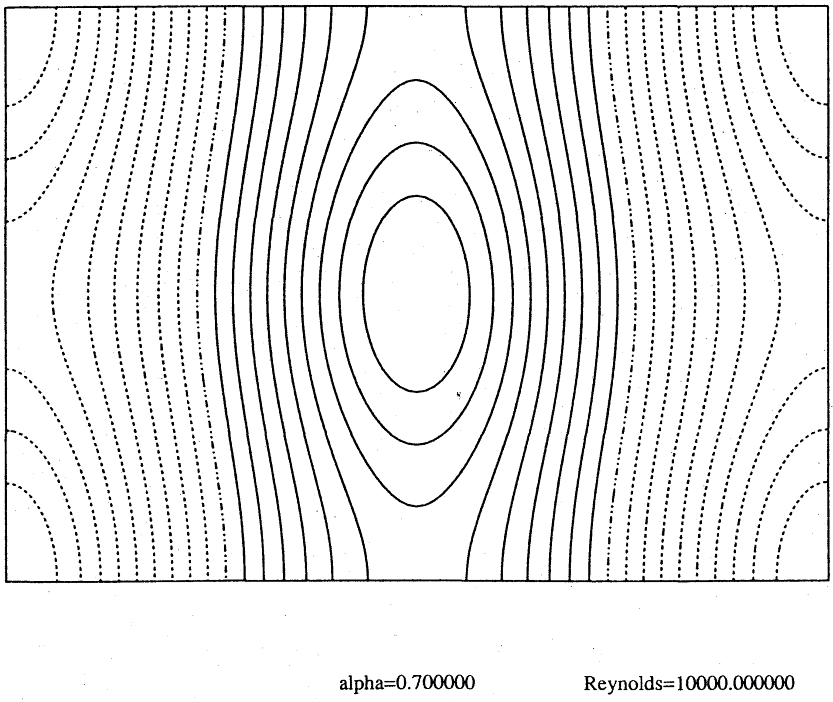


Figure 2-d

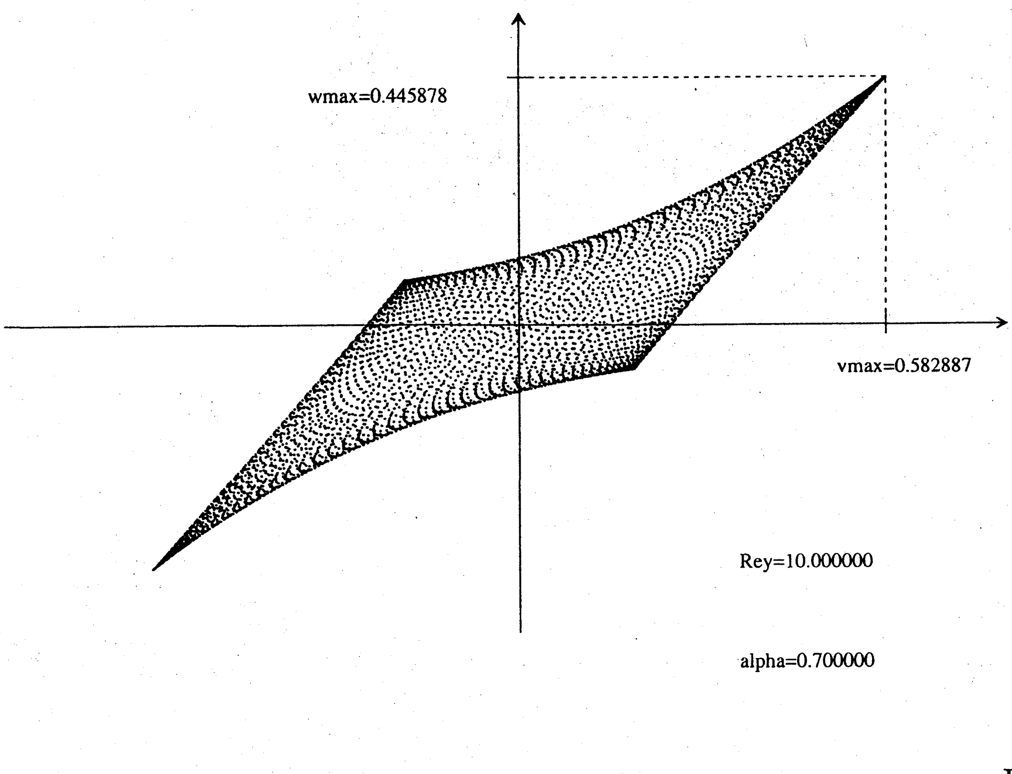


Figure 3

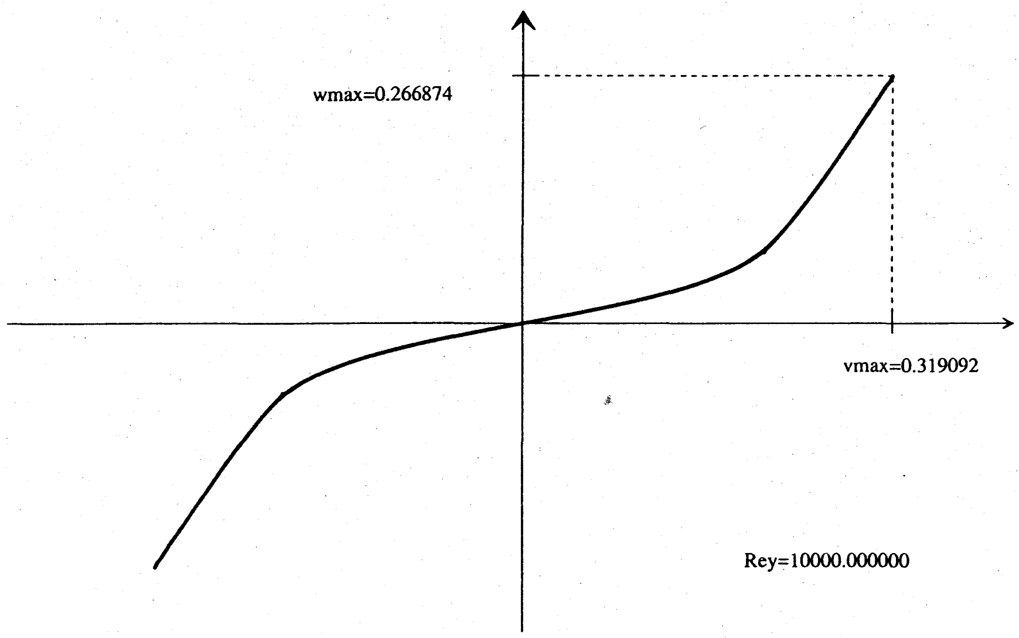
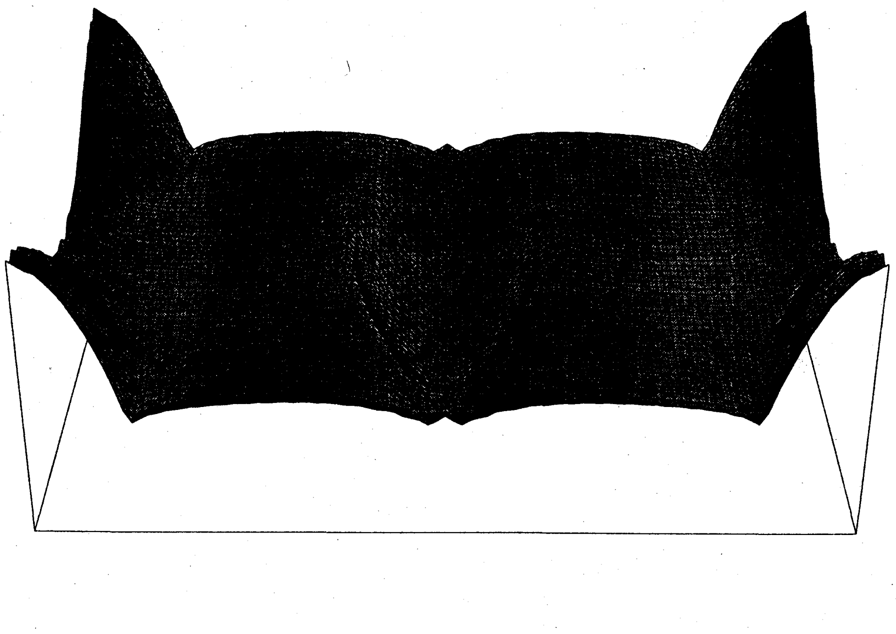


Figure 4



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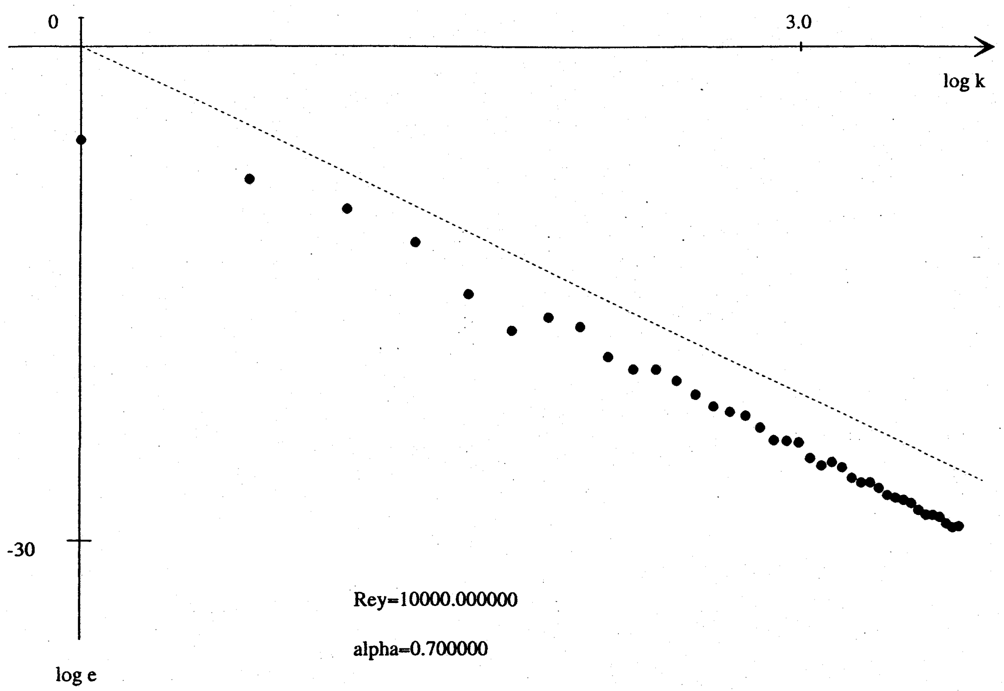

Figure 6

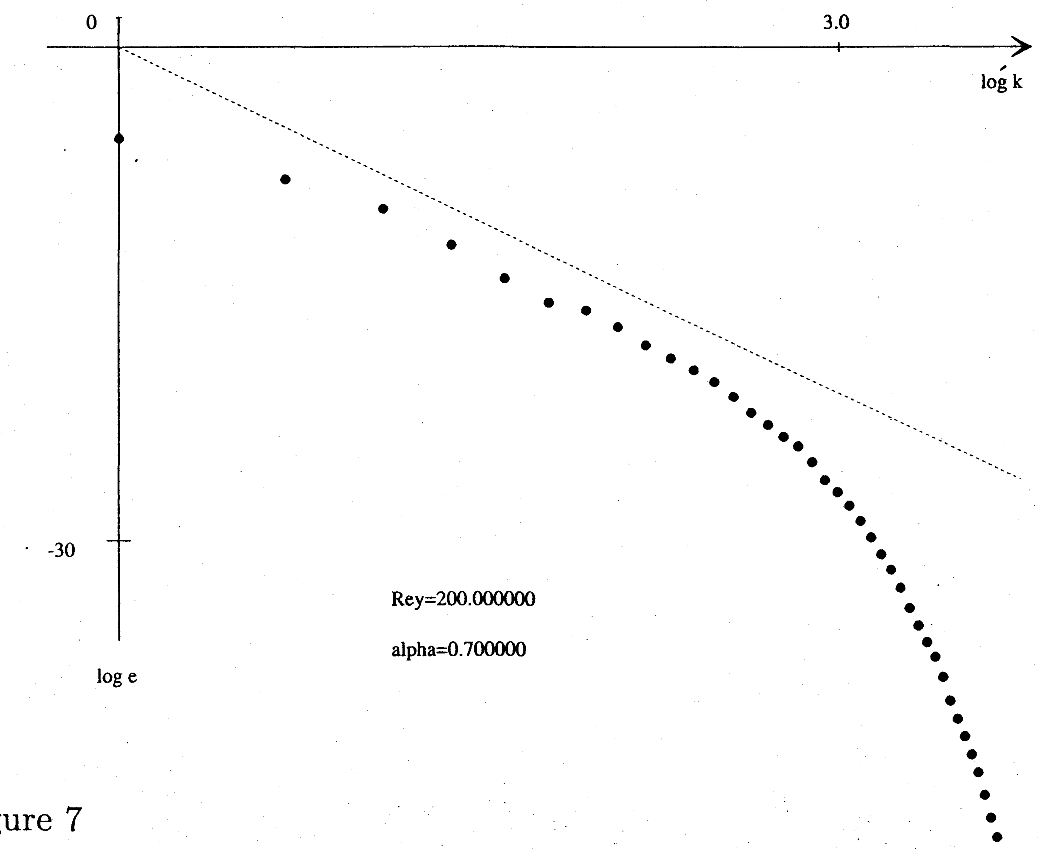

Figure 7

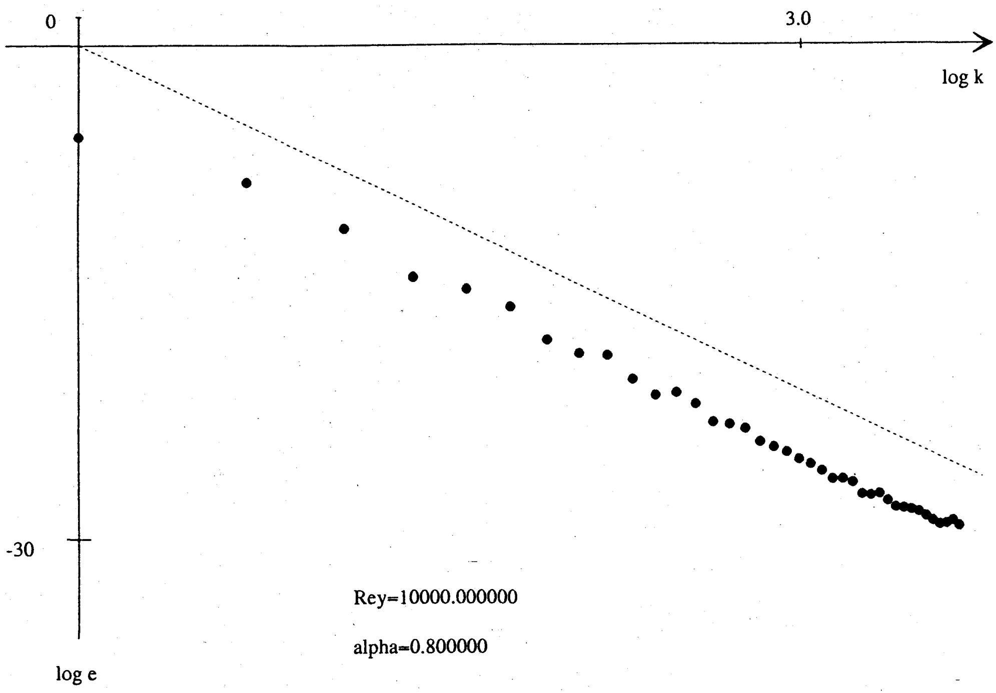

Figure 8

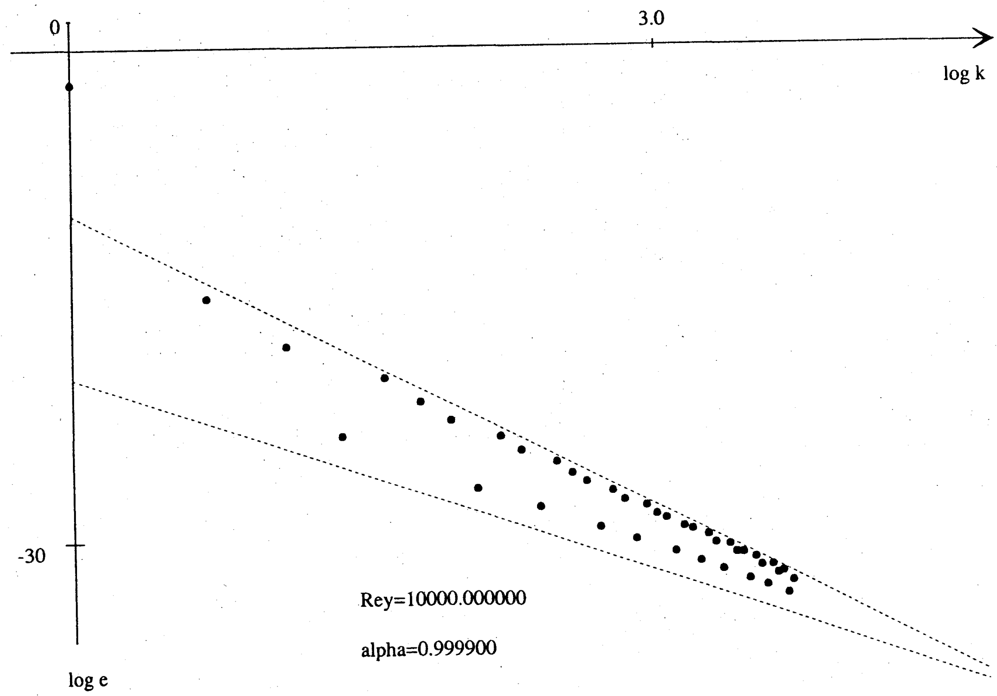

Figure 9-a

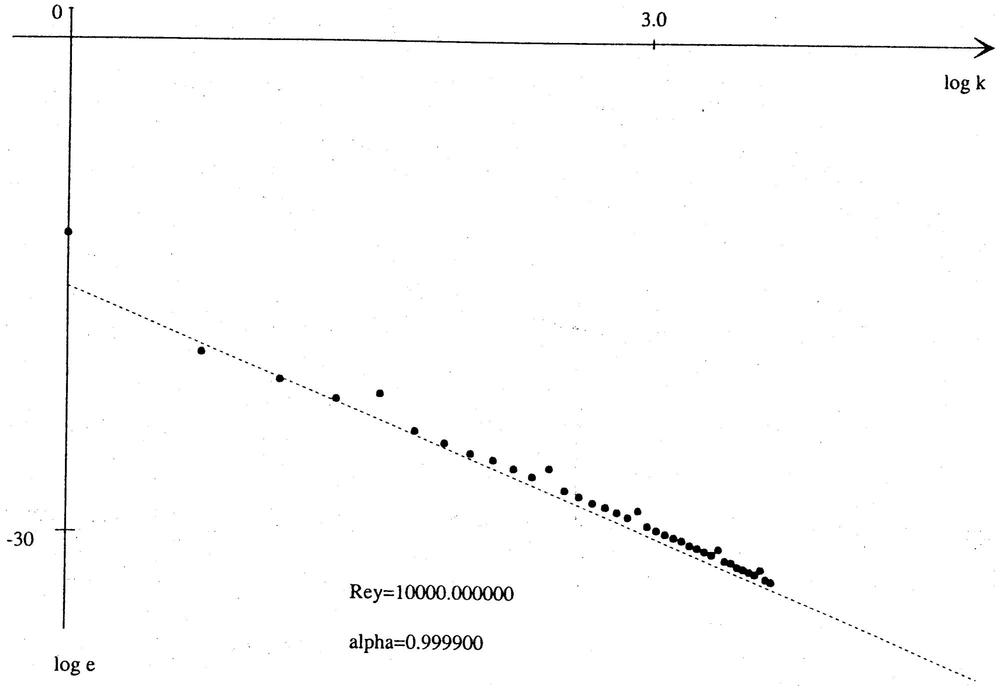

Figure 9-b

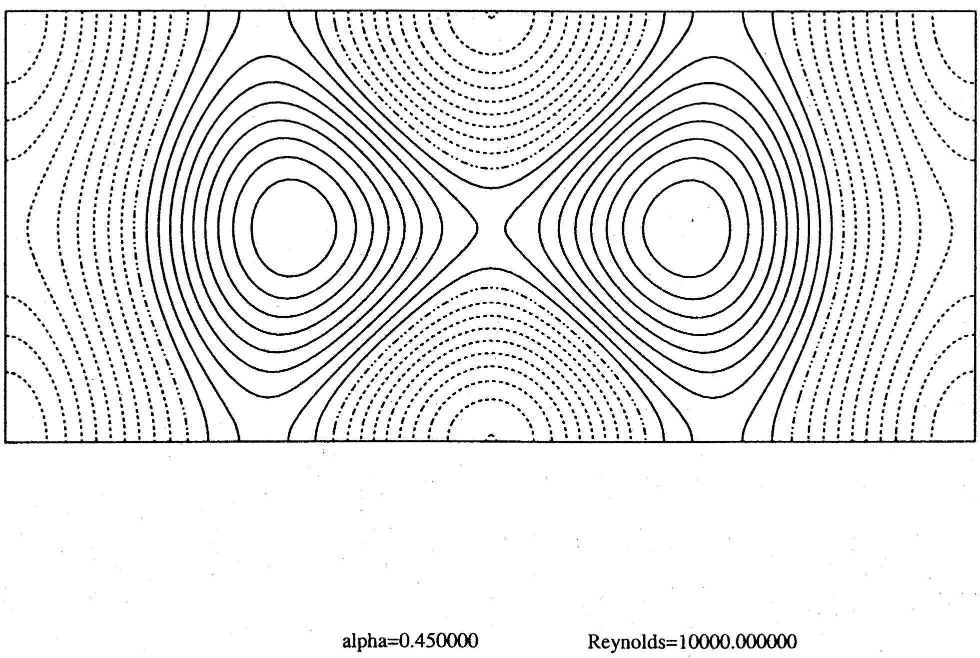

Figure 10

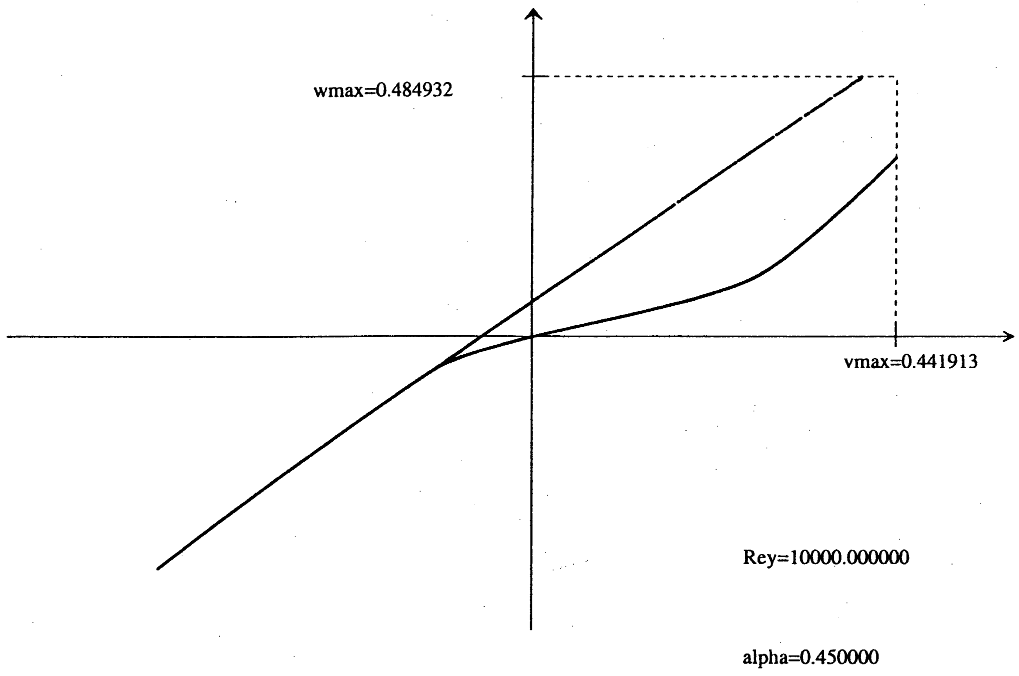

Figure 11

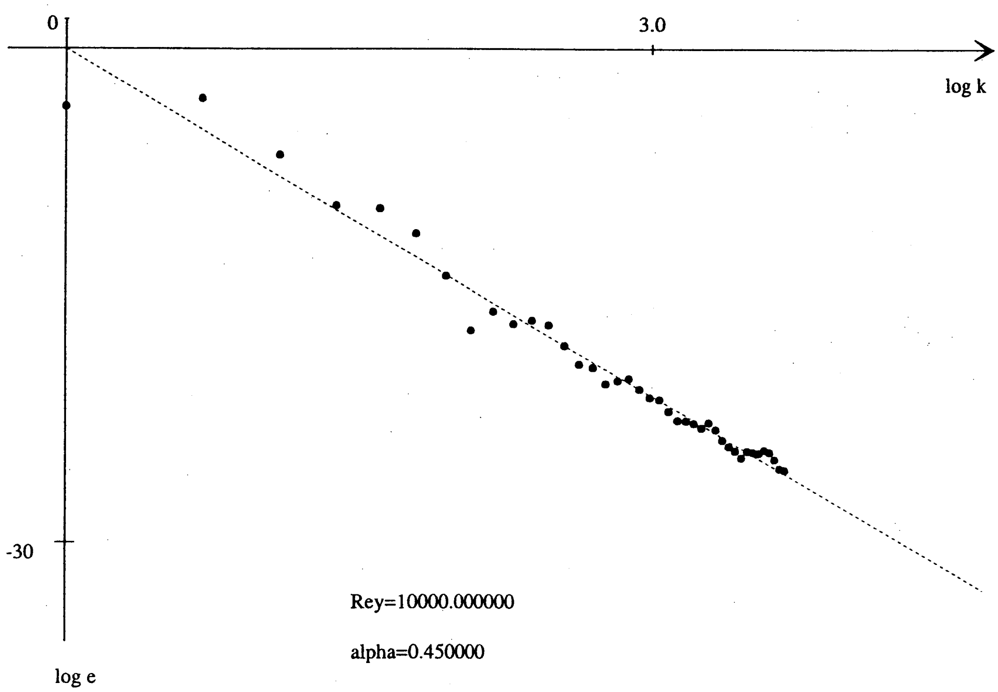

Figure 12