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Navier-Stokes Flow Down a Vertical Column : an Axisymmetric Case

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1 Introduction

The Navier-Stokes equations describing an axially symmetric flow down a vertical column can be written in the following form:

$$\begin{aligned}(1.1)\qquad &\frac{\partial v_r}{\partial t}+v_r\frac{\partial v_r}{\partial r}+v_z\frac{\partial v_r}{\partial z}\\ &=-\frac{\partial p}{\partial r}+\frac{1}{\mathcal{R}}\left(\left(\frac{\partial}{\partial r}\right)^2 v_r++\frac{1}{r}\frac{\partial v_r}{\partial r}+\left(\frac{\partial}{\partial z}\right)^2 v_r-\frac{1}{r^2}v_r\right),\end{aligned}$$

$$\begin{aligned}(1.2)\qquad &\frac{\partial v_z}{\partial t}+v_r\frac{\partial v_z}{\partial r}+v_z\frac{\partial v_z}{\partial z}\\ &=-\frac{\partial p}{\partial z}+\frac{1}{\mathcal{R}}\left(\left(\frac{\partial}{\partial r}\right)^2 v_z++\frac{1}{r}\frac{\partial v_z}{\partial r}+\left(\frac{\partial}{\partial z}\right)^2 v_z\right)-\frac{2}{\mathcal{R}},\end{aligned}$$

$$(1.3)\qquad \frac{\partial}{\partial r}(rv_r)\ +\ \frac{\partial}{\partial z}(rv_z)\ =\ 0$$

The flow takes place in the asymmetric layer whose cross-section along the z -axis is given by

$$\{\ (r,z)\ ;\ a<r<a+1+h(z,t)\ \}$$

at time $t>0$. For the equations themselves, see [1, 8, 4].

Here we have used the dimensionless variables adopted in [8]. The positive constant a is the nondimensional radius of column and $r=a+1+h(z,t)$ is the unknown free surface. We take the mean thickness of the fluid layer as the unit of length. v_r and v_z are the radial component and the z component of the velocity vector, respectively. p is the scalar pressure. The condition at the solid surface is the adhesion condition

$$(1.4)\qquad v_r = v_z = 0 \quad \text{at} \quad r = a\,.$$

The stress balance conditions at the outer surface are the followings :

$$(1.5)\qquad \left(\frac{\partial v_r}{\partial z}+\frac{\partial v_z}{\partial r}\right)\left(1-\left(\frac{\partial h}{\partial z}\right)^2\right)+2\left(\frac{\partial v_r}{\partial r}-\frac{\partial v_z}{\partial z}\right)\frac{\partial h}{\partial z}=0,$$

$$(1.6)\qquad -p+\frac{2}{\mathcal{R}}\left\{\frac{\partial v_r}{\partial r}+\frac{\partial v_z}{\partial z}\left(\frac{\partial h}{\partial z}\right)^2 - \left(\frac{\partial v_r}{\partial z}+\frac{\partial v_z}{\partial r}\right)\left(\frac{\partial h}{\partial z}\right)\right\}\left(1+\left(\frac{\partial h}{\partial z}\right)^2\right)^{-1}$$
$$= -p_0-\gamma\left\{\frac{1}{r}\left(1+\left(\frac{\partial h}{\partial z}\right)^2\right)^{-\frac{1}{2}} - \frac{\partial^2 h}{\partial z^2}\left(1+\left(\frac{\partial h}{\partial z}\right)^2\right)^{-\frac{3}{2}}\right\}$$
$$\text{at } r=a+1+h(z,t),\ z\in \boldsymbol{R},\ t>0.$$

Here p_0 is the atmospheric pressure and γ is the surface tension coefficient. The kinematic impermeability condition at $r = a+1+h(z,t),\ z \in \boldsymbol{R}$ is written as follows

$$(1.7)\qquad \frac{\partial h}{\partial t} - v_r + v_z\frac{\partial h}{\partial z} = 0.$$

A stationary solution to this problem is the following:

$$(1.8)\qquad \left(v_r^{(0)}, v_z^{(0)}\right) = \left(0\ ,\ \frac{1}{2}(r^2-a^2)-(1+a)^2\log\frac{r}{a}\right),$$

$$(1.9)\qquad p^{(0)} = p_0 + \frac{\gamma}{1+a}\,,\quad h^{(0)} \equiv 0\,.$$

In this article we consider the evolution problem for downward periodic disturbances from this equilibrium state and announce some existence results.

2 Formulation

We first set

$$(v_r, v_z) = \left(u_r + v_r^{(0)}, u_z + v_z^{(0)}\right) \ , \ p = q + p^{(0)}.$$

To reduce the problem to the one in a fixed domain, we use the method employed in [2]. Let Ω be the equilibrium domain

$$\{ \ (r', z') \ ; \ a < r' < 1 + a, 0 < z' < \ell \ \} \ .$$

Here ℓ denotes the periodicity in $z - axis$. Let $\tilde{h}$ be an extension of h to $\{a < r' < a + 1\}$. For each $t \geq 0$ we define diffeomorphisms

$$\Theta : \Omega \to \Omega(t) = \{(r, z) \, ; a < r < a + 1 + h(z, t), 0 < z < \ell\}$$

by

$$r = (r' - a)\left(1 + \tilde{h}\,(r', z', t)\right) + a \ , \quad z = z' \ .$$

For (u_1, u_2) on Ω we define (u_r, u_z) on $\Omega\,(t)$ by

$$u_r = \frac{r'}{r} u_1 + \frac{r'(r' - a)}{rJ} \frac{\partial \tilde{h}}{\partial z'} u_2 \tag{2.1}$$

$$u_z = \frac{r'}{rJ} u_2 \ . \tag{2.2}$$

J denotes the jacobian of Θ . Set $q = q\,(\Theta(r', z'))$.
We next introduce the coordinates attached to the unperturbed flow :

$$r' = x_1 \ , \ z' = x_2 + \left\{\frac{1}{2}\left((a+1)^2 - a^2\right) - (1+a)^2 \log \frac{1+a}{a}\right\} t' \ ; \ t = t'.$$

We finally introduce an auxiliary unknown defined by

$$\eta = \frac{1}{2}\left((1 + a + h))^2 - (1+a)^2\right) \ . \tag{2.3}$$

In the same manner as in [Sect. 5 , 2], we can derive the problem for the new unknowns. Collecting linear terms in the left hand sides, we can rewrite our problems as follows

$$\frac{\partial \eta}{\partial t} - (1+a)u_1 = -\frac{\partial}{\partial x_2}\left[\frac{1}{2}\eta^2 - \frac{(1+a)^4}{4}\left(\left(1 + \frac{2\eta}{(1+a)^2}\right)\right.\right. \\ \left.\left. \times \left(\log\left(1 + \frac{2\eta}{(1+a)^2}\right) - 1\right) + 1\right)\right] \quad \text{on } x_1 = 1 + a, \tag{2.4}$$

$$\frac{\partial u_j}{\partial t}+\left(\frac{1}{2}\left(x_1^2-(1+a)^2\right)-(1+a)^2\log\frac{x_1}{1+a}\right)\frac{\partial u_j}{\partial x_2}$$
$$-\frac{1}{\mathcal{R}}\left(\frac{\partial^2 u_j}{\partial x_1^2}+\frac{1}{x_1}\frac{\partial u_j}{\partial x_1}+\frac{\partial u_j}{\partial x_2^2}-\delta_{j1}\frac{u_1}{x_1^2}\right)$$
$$+u_1\delta_{j2}\left(x_1-\frac{(1+a)^2}{x_1}\right)+\frac{\partial p}{\partial x_j}=f_j\ ,j=1,2, \tag{2.5}$$

$$\frac{\partial}{\partial x_1}(x_1u_1)\ +\ \frac{\partial}{\partial x_2}(x_1u_2)=0 \tag{2.6}$$

in $a < x_1 < 1+a$,

$$u_1=u_2=0\quad \text{on } x_1=a\ , \tag{2.7}$$

$$(1+a)\frac{\partial u_1}{\partial x_2}+(1+a)\frac{\partial u_2}{\partial x_1}+2\eta=f_3, \tag{2.8}$$

$$p-\frac{2}{\mathcal{R}}\frac{\partial u_1}{\partial x_1}+\gamma\left(\frac{1}{(1+a)^3}\eta+\frac{1}{(1+a)}\frac{\partial^2\eta}{\partial x_2^2}\right)=f_4\ , \tag{2.9}$$

on $x_1 = 1+a$,

$$\int_0^\ell \eta(x_2,t)dx_2=0\ ,\qquad t>0. \tag{2.10}$$

Each f_j in the right hand sides consists of nonlinear terms.
Suppose that initial data

$$\eta_0(x_2)\quad ,\ 0<x_2<\ell\ , \tag{2.11}$$
$$u_0(x_1,x_2)\ ,\qquad a<x_1<1+a\ ,\ 0<x_2<\ell \tag{2.12}$$

are given with certain compatibility conditions.

3 Results

Let $\Omega=(a,a+1)\times(0,\ell)$ in $\boldsymbol{R}^2$. Let $r\geq 0$. $H^r(\Omega)$ denotes the space of functions which belong to $H^r_{loc}((a,1+a)\times\boldsymbol{R})$ and are periodic in x_2 with period ℓ. Let $S_F=\Omega\cap\{X_1=1+a\}$ and let $S_W=\Omega\cap\{X_1=a\}$. We identify S_F with the interval $(0,\ell)$. Denote $H^r(S_F)$ by the space of functions which belong to $H^r_{loc}(\boldsymbol{R})$ and are periodic with period ℓ .

We now state our results obtained until now.

Theorem 3.1 *Let $\mathcal{R} > 0$ and $\gamma > 0$ be arbitrary. Let $0 < \delta < \frac{1}{4}$. Let $u_0 \in H^{1+2\delta}(\Omega)$ and $\eta_0 \in H^{\frac{3}{2}+2\delta}(S_F)$. Assume that*

$$(3.1) \qquad \frac{\partial}{\partial x_1}(x_1 u_{0,1}) + \frac{\partial}{\partial x_2}(x_1 u_{0,2}) = 0 \quad \text{in } \Omega ,$$

$$(3.2) \qquad u_0 = 0 \quad \text{on } S_W,$$

$$(3.3) \qquad \sup\left(|\eta_0| + \left|\frac{\partial \eta_0}{\partial x_2}\right|\right) < 1 \quad \text{on } S_F,$$

$$(3.4) \qquad \int_0^\ell \eta_0(x_2) d_2 = 0 .$$

Then there is $T > 0$, so that the problem (2.4) - (2.12) has a solution (η, u, q) with

$$(3.5) \qquad \eta \in H^0\left(0,T; H^{\frac{5}{2}+2\delta}(S_F)\right) \cap H^{\frac{5}{4}+\delta}\left(0,T; H^0(S_F)\right),$$

$$(3.6) \qquad u \in H^0\left(0,T; H^{2+2\delta}(\Omega)\right) \cap H^{1+\delta}\left(0,T; H^0(\Omega)\right),$$

$$(3.7) \qquad q \in H^0\left(0,T; H^{1+2\delta}(\Omega)\right) \cap H^{\frac{1}{2}+\delta}\left(0,T; H^0(\Omega)\right).$$

Theorem 3.2 *Let $T_1 > 0$ be given. Assume that (3.1)– (3.4). Then there exist $\mathcal{R}_0 > 0, \gamma_0 > 0, a_0 > 0$ and $\varepsilon > 0$ such that , if*

$$\mathcal{R}_0 > \mathcal{R} \quad , \gamma_0 < \gamma \quad , a_0 < a$$

and

$$\|u_0\|_{1+2\delta} + |\eta_0|_{H^{\frac{3}{2}+2\delta}(S_F)} \leq \varepsilon ,$$

then the solution (η, u, q) obtained in Theorem 3.1 can be continued to any $t > 0$ and satisfies

$$u \in C\left(T_1, \infty; H^2(\Omega)\right) ,$$

$$\eta \in C\left(T_1, \infty; H^3(S_F)\right) ,$$

$$\nabla q \in C\left(T_1, \infty; H^0(\Omega)\right).$$

To prove these we use the ideas developed in [2] and [7]. The integral identity below plays an important role. If $v, u \in H^2(\Omega)$, $p \in H^1(\Omega)$ and

$$\frac{\partial}{\partial x_1}(x_1 u_1) + \frac{\partial}{\partial x_2}(x_1 u_2) = 0 \qquad \text{in } \Omega ,$$

then integration by parts gives

$$\int_\Omega v_1 \left[-\frac{1}{\mathcal{R}} \left(\frac{\partial^2 u_1}{\partial x_1^2} + \frac{1}{x_1}\frac{\partial u_1}{\partial x_1} + \frac{\partial^2 u_1}{\partial x_2^2} - \frac{u_1}{x_1^2} \right) + \frac{\partial p}{\partial x_1} \right] x_1 d_1 d_2$$

$$+ \int_\Omega v_2 \left[-\frac{1}{\mathcal{R}} \left(\frac{\partial^2 u_2}{\partial x_1^2} + \frac{1}{x_1}\frac{\partial u_2}{\partial x_1} + \frac{\partial^2 u_2}{\partial x_2^2} \right) + \frac{\partial p}{\partial x_2} \right] x_1 d_1 d_2$$

$$= \int_{\partial\Omega} v_1 \left(p - \frac{2}{\mathcal{R}}\frac{\partial u_1}{\partial x_1} \right) x_1 d_2 + \int_{\partial\Omega} v_2 \left(-\frac{2}{\mathcal{R}} \right) \left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \right) x_1 d_2$$

$$+ \frac{1}{2\mathcal{R}} \langle v \ , \ u \rangle \ ,$$

where

$$\langle v \ , \ u \rangle = \sum_{j,k=1}^{2} \int_\Omega \left(\frac{\partial v_j}{\partial x_k} + \frac{\partial v_k}{\partial x_j} \right) \left(\frac{\partial u_j}{\partial x_k} + \frac{\partial u_k}{\partial x_j} \right) x_1 d_1 d_2$$

$$+ 4 \int_\Omega \frac{v_1}{x_1}\frac{u_1}{x_1} x_1 d_1 d_2 \ .$$

Note that , in Ω , $0 < a < x_1 < 1 + a$. By Korn's inequality (see, for example, [3 , 5]) , we have

Proposition 3.3 *If $v \in H^1(\Omega)$ and $v = 0$ on S_W , then it holds that*

$$\|v\|_1 \leq C \langle v \ , \ u \rangle \ . \tag{3.8}$$

Here C is independent of v .

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